

# Fourier integrals and the sampling theorem

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#### **Fourier Integrals**

Read: Strang, Section 4.3.

Fourier series are convenient to describe *periodic* functions (or functions with support on a finite interval). What to do if the function is not periodic?

- Consider  $f : \mathbb{R} \to \mathbb{C}$ . The Fourier transform  $\hat{f} = \mathcal{F}(f)$  is given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad k \in \mathbb{R}.$$

Here, f should be in  $L^1(\mathbb{R})$ .

- Inverse transformation:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

Note: Often, you will se a more symmetric version by using a different scaling.

- Theorem of Plancherel:  $f \in L^2(\mathbb{R}) \Leftrightarrow \hat{f} \in L^2(\mathbb{R})$  and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

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## Fourier Integrals: The Key Rules

$$\widehat{df/dx} = ik\widehat{f}(k) \qquad \int_{-\infty}^{x} f(\xi)d\xi = \widehat{f}(k)/(ik)$$
$$\widehat{f(x-d)} = e^{-ikd}\widehat{f}(k) \qquad \widehat{e^{icx}f} = \widehat{f}(k-c)$$

**Examples of Fourier transforms:** 

$$f(x) = sin(\omega x) \qquad \qquad \hat{f}(k) = -i\pi(\delta(k-\omega) - \delta(k+\omega))$$

$$f(x) = cos(\omega x) \qquad \qquad \hat{f}(k) = \pi(\delta(k-\omega) + \delta(k+\omega))$$

$$f(x) = \delta(x) \qquad \qquad \hat{f}(k) = 1 \text{ for all } k \in \mathbb{R}.$$

$$f(x) = \begin{cases} 1, & \text{if } -L \le x \le L, \\ 0, & \text{if } |x| > L. \end{cases} \qquad \hat{f}(k) = 2\frac{\sin kL}{k} = 2L \text{sinc}(kL),$$

where sinc(t) = sin(t)/t is the *Sinus cardinalis* function.

# Sampling of functions - The Nyquist sampling rate

How many samples do we need to identify  $\sin \omega t$  or  $\cos \omega t$ ? Assume equidistant sampling with step size T. Definition: Nyquist sampling rate:  $T = \pi/\omega$ .

The Nyquist sampling rate, is exactly 2 equidistant samples over a full period of the function. (period is  $2\pi/\omega$ ).

For a given sampling rate T, frequencies higher than the *Nyquist* frequency  $\omega_N = \pi/T$  cannot be detected. A higher frequency harmonic is mapped to a lower frequency one. This effect is called *aliasing*.



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# **Band-limited functions**

- When the continuous signal is a combination of many frequencies  $\omega$ , the largest frequency  $\omega_{max}$  sets the Nyquist sampling rate at  $T = \pi/\omega_{max}$ .
- No aliasing at any faster rate. (Do not set sampling rate equal to the Nyquist sampling rate. Consider  $\sin \omega T$  to see why.)
- A function f(x) is called band-limited if there is a finite W such that its Fourier integral transform  $\hat{f}(k) = 0$  for all  $|k| \ge W$ .

The Shannon-Nyquist sampling theorem states that such a function f(x) can be recovered from the discrete samples with sampling frequency  $T = \pi/W$ .

(Note that relating to above,  $W = \omega_{max} + \varepsilon$ ,  $\varepsilon > 0$ .)

#### The Sampling Theorem

**Theorem:** (Shannon-Nyquist) Assume that f is band-limited by W, i.e.,  $\hat{f}(k) = 0$  for all  $|k| \ge W$ . Let  $T = \pi/W$  be the Nyquist rate. Then it holds

$$f(x) = \sum_{n=-\infty}^{\infty} f(nT) \operatorname{sinc}(\pi(x/T - n))$$

where sinc(t) = sin(t)/t is the *Sinus cardinalis* function. Note: The sinc function is band-limited:

$$\widehat{\mathsf{sinc}}(k) = egin{cases} 1, & ext{if } -\pi \leq k \leq \pi, \ 0, & ext{elsewhere.} \end{cases}$$

Also note: This interpolation procedure is not used in practice. Slow decay, summing an infinite number of terms. Approximations to the sinc function are used, introducing interpolation errors.

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## The Sampling Theorem: Proof

Assume for simplicity  $W = \pi$ . By the inverse Fourier transform,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(k)e^{ikx}dk.$$

Define

$$ilde{f}(k) = egin{cases} \hat{f}(k), & ext{if } -\pi < k < \pi, \\ ext{periodic continuation}, & ext{if } |k| \geq \pi. \end{cases}$$

 $\tilde{f}$  can be represented as a Fourier series in k-space,

$$ilde{f}(k) = \sum_{n=-\infty}^{\infty} \hat{ ilde{f}}_n e^{ink},$$

where

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(k) e^{-ink} dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(k) e^{-ink} dk = f(-n).$$

Hence, for  $-\pi < k < \pi$ ,

$$\hat{f}(k) = \tilde{f}(k) = \sum_{n = -\infty}^{\infty} f(-n) e^{ink}.$$

# The Sampling Theorem: Proof, contd.

Therefore,

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(k) e^{ikx} dk$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} f(-n) e^{ink} \right) e^{ikx} dk$$
$$= \sum_{n=-\infty}^{\infty} f(-n) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik(x+n)} dk$$
$$= \sum_{n=-\infty}^{\infty} f(-n) \frac{\sin \pi (x+n)}{\pi (x+n)}$$
$$= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}$$

If W different from  $\pi$ , rescale x by  $\pi/W$  and k by  $W/\pi$  to complete the proof.