HOMEWORK 2: Equilibrium Problems and Circuit Analysis
for Mathematical Models, Analysis and Simulation (DN2266), Fall 2012
Report due Mon Oct 1, 2012.
Maximum score 6.0 pts.

Read Strang's book, Chapter 2, as well as the handout *Introduction to the modified nodal analysis*.

Note: in the lecture notes, the MNA has been introduced with Strang's notation. In the handout, the notation is different (signs and transpose of matrices in addition to different letters to denote the variables).

1. Problems from the book (Strang) (Score: 2.0)

Chapter 2.2: 1,7

Chapter 2.4: 9,15

2. Least squares problem and Lagrange multipliers (Score: 1.0) Given a least squares problem with linear constraints

$$\min_{Cx=c} \frac{1}{2}\|Ax-a\|_2^2 \tag{1}$$

where A is a $m \times n$, $m > n$, $\text{rank}(A) = n$ and C is $p \times n$, $p < n$, $\text{rank}(C) = p$. The x-value for which the minimum is taken is denoted by $\hat{x}$.

(a) Show that $\hat{x}$ satisfies the following linear system of equations

$$\begin{pmatrix} A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^Ta \\ c \end{pmatrix} \tag{2}$$

where λ is the Lagrange multiplier vector connected to this constrained problem.

(b) The matrix in the linear system in (a) is obviously symmetric. Is it also positive definite? Give a proof or give a counter example.

3. Linear Electrical Networks: DC Analysis (Score: 2.0)

1. Write a MATLAB function `dc_anal` which implements the MNA analysis. Consult sections 5.1 and 5.4 of the handout.

   It is convenient to use the following calling sequence:

   ```
   [e,iV] = dc_anal(AR,AV,AI,R,V,I).
   ```

   The meaning of the parameters is as follows:

- `AR`, `AV`, `AI` are the incidence matrices of the resistive, voltage source, and current source branches, respectively.
- `R`, `V`, `I` are the vectors of the values of the resistors, voltage sources, and current sources, respectively.
- `e` is the vector of node voltages (excluding the ground node), and `iV` is the vector of currents through the voltage source branches.

Let $n+1$ be the number of nodes in the circuit. Then number the nodes from 1 to $n+1$ with the ground node being node number $n+1$. If the number of resistive branches is b_R, the number of voltage source branches b_V, and the number of current source branches b_I, then the matrices and vectors have the following dimensions

matrix	dimension
`AR`	$n \times b_R$
`AV`	$n \times b_V$
`AI`	$n \times b_I$
`R`	b_R
`V`	b_V
`I`	b_I
`e`	n
`iV`	b_V

These dimensions follow the MNA handout. If you follow the notation of Strang, the sizes are transposed, i.e. $b_R \times n$ etc.

What happens if some of the branches are missing? Assume that our circuit does not include current sources. Then, $b_I = 0$. Hence, the matrix `AI` has dimension $n \times 0$. In the MATLAB language, this represents a (dimensioned) empty matrix. This is completely consistent with MATLAB's interpretation of matrices. Similarly, the vector `I` has dimension 0×1. They can be dimensioned in MATLAB using `AI = zeros(`n`,`b_I`)` and `I = zeros(`b_I`,1)`.

2. Consider the electric circuit given in Figure 1. The given quantities are the voltage of the source E, as well as the values of the resistors R_s, R_a, R_b, R_c, R_d and R_g. The problem consists of computing the current through the resistor R_g as a function of R_g.

   (a) The values of the elements are as follows:

   $$E = 10V, R_s = 0.001\Omega, R_a = 100\Omega, R_b = R_c = R_d = 10\Omega.$$

   The resistor R_g shall vary between 10Ω and 200Ω.

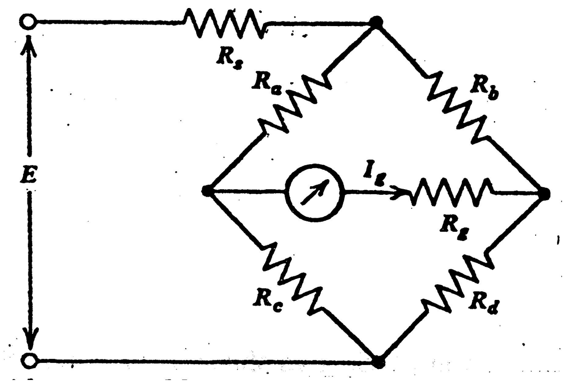

Figure 1: Circuit for the DC analysis problem

(b) Derive the necessary incidence matrices and vectors describing the circuit and compute the current I_g through R_g for sufficiently many values of that resistor by your function `dc_anal`. Plot a graph presenting I_g as a function of R_g. Do not forget to add correct denotations to the axis (values, units etc.)!

3. What is the resistance of a cube?
   We consider a circuit which is constructed as follows:

   (a) Draw the lattice of a cube (that is, all 12 edges).
   (b) Every edge is replaced by a resistor with a resistance of 1Ω.
   (c) The nodes of the circuit are represented by the corners of the cube.

   How large is the resistance between two corners which are located diagonally opposite to each other in the cube lattice?

4. (optional) Another example of this type is provided in Strang, p. 155. Solve problems 2.4.21, 2.4.22 there.

4. Linear Electrical Networks: AC Analysis (Score: 1.0)
In this problem, we are interested in the computation of transfer curves of RLC circuits. The method of choice is the $j\omega$-method. This approach makes the analysis mathematically similar to the DC case. Consult sections 5.2 and 5.4 of the handout.

1. Write a MATLAB function `ac_anal` which implements the $j\omega$-method!

   The appropriate calling sequence for the function might be:

```
[e,iV] = ac_anal(AR,AC,AL,AV,AI,R,C,L,V,I,omega).
```

2. In earlier days of the music business, the black vinyl disks played the role of what the CD is today. For mechanical reasons, the low-frequency electrical signal was not modulated linearly to mechanical vibrations on the vinyl record. Instead, higher frequencies were amplified while lower frequencies were damped. Waves at 1 kHz were neither amplified nor damped. The exact amplification curve (RIAA curve) is defined by a reference circuit. The opposite effect in a phono player is called RIAA equalization. The respective circuit is sketched in Figure 2. The physical parameters are as follows:

$$R1 = 1\Omega, R1A = 212.8m\Omega, C1 = 3.528mF, C1A = 352.8\mu F.$$

The circuit is valid for a frequency f of $E1$ between 20Hz and 20kHz. For simplicity choose an effective voltage of 1V.

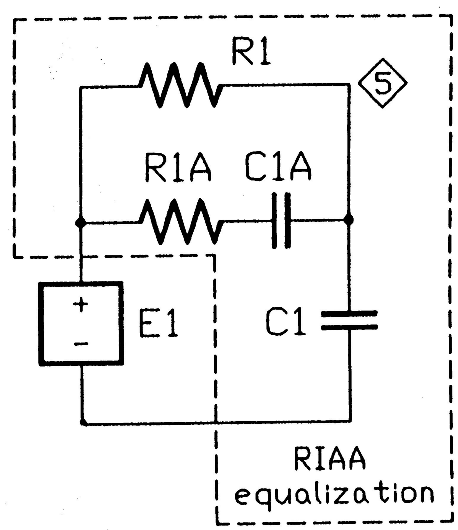

Figure 2: RIAA equalization circuit

Compute and plot the equalization curve! The x-axis shall be the frequency f, while the y-axis is the quotient of the effective voltage $v_5(f)$ at node 5 and $v_5(1kHz)$ (measured in dB, i.e., $y = 20\log_{10}|v_5(f)/v_5(1\text{kHz})|$)! Do not forget to add correct denotations to the plot!