



Linear Algebra, part 1

Linear Systems

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Mathematical Models, Analysis and Simulation
Fall semester, 2012

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Linear systems of equations

Let $\mathbf{A}$ be an $m \times n$ matrix (m rows, n columns).

$(\mathbf{A})_{i,j} = a_{ij}$, i row index, j column index.

The matrix entries can be real ($a_{ij} \in \mathbb{R}$) or complex ($a_{ij} \in \mathbb{C}$).

Let $\mathbf{x}$ be a column vector of size $n \times 1$. $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$.

Matrix vector multiplication: $\mathbf{b} = \mathbf{Ax}$,

where $\mathbf{b}$ is a column vector of size $m \times 1$. When $\mathbf{b}$ is given and $\mathbf{x}$ is unknown, want instead to solve (now assume $m = n$):

$$\mathbf{Ax} = \mathbf{b}.$$

When does this system have a solution? When is it unique?

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Range, rank, nullspace, nullity

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$, where each $\mathbf{a}_i$ is $m \times 1$.

Then $\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$.

Column space $V = R(\mathbf{A})$ (*range* of $\mathbf{A}$) spanned by the columns of $\mathbf{A}$.

$\text{rank}(\mathbf{A}) = \dim(V)$ = number of linearly independent columns.

If $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x}$ is in the *nullspace* of $\mathbf{A}$.

$\ker(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R} : \mathbf{Ax} = \mathbf{0}\}$.

The dimension of the nullspace: $\text{nullity}(\mathbf{A}) = \dim(\ker(\mathbf{A}))$.

If $\mathbf{A}$ is $m \times n$, we have that

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n.$$

Questions

Let $\mathbf{A}$ be $n \times n$.

1. What do we call $\mathbf{A}$ when $\text{nullity}(\mathbf{A}) > 0$?
2. When does $\mathbf{Ax} = \mathbf{b}$ have a unique solution?
3. When can we pick vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ such that $\mathbf{Ax} = \mathbf{b}_1$ have multiple solutions and $\mathbf{Ax} = \mathbf{b}_2$ have no solutions? What can we say about $\mathbf{b}_1$ and $\mathbf{b}_2$?

Navigation icons: back, forward, search, etc.

Large branch of numerical linear algebra

Solve linear system

$\mathbf{Ax} = \mathbf{b}$, $\mathbf{A}$ is $n \times n$

where n often is a large number.

Where do such systems come from? Many sources, for example:

- ▶ Discretization of differential equations by different numerical methods. (Applications to fluid mechanics, electromagnetics, quantum physics, biology, option pricing...)
- ▶ Network models and graphs (Electric circuits, mechanical trusses, hydraulic systems).

Navigation icons: back, forward, search, etc.

Solution method: Gaussian elimination

Solution by Gaussian elimination.

(In Matlab: $\mathbf{x}=\mathbf{A} \backslash \mathbf{b}$)

Boxed material below taken from

Strang, Computational Science and Engineering, 2007.

$$\mathbf{Ku} = \mathbf{f} \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 &= 4 \\ -u_1 + 2u_2 - u_3 &= 0 \\ -u_2 + 2u_3 &= 0 \end{aligned}$$

The first step is to eliminate u_1 from the second equation. **Multiply equation 1 by $\frac{1}{2}$ and add to equation 2.** The new matrix has a zero in the 2, 1 position—where u_1 is eliminated. I have circled the **first two pivots**:

$$\begin{bmatrix} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{\frac{3}{2}} & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 &= 4 \\ \frac{3}{2} u_2 - u_3 &= 2 \\ -u_2 + 2u_3 &= 0 \end{aligned}$$

Now, multiply Eq 2 by 2/3 and add to Eq 3.

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Gaussian elimination, example continued

This yields:

$$\begin{bmatrix} \textcircled{2} & -1 & 0 \\ 0 & \textcircled{\frac{3}{2}} & -1 \\ 0 & 0 & \textcircled{\frac{4}{3}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 + \frac{2}{3} f_2 + \frac{1}{3} f_1 \end{bmatrix} \quad \text{is} \quad \begin{aligned} 2u_1 - u_2 &= 4 \\ \frac{3}{2} u_2 - u_3 &= 2 \\ \frac{4}{3} u_3 &= \frac{4}{3} \end{aligned} \quad (1)$$

Upper triangular matrix $\mathbf{U}$.

Forward elimination is complete.

Solution by backsubstitution. Last equation determines u_3 . Then the second determines u_2 . With u_3 and u_2 known, easy to find u_1 using the first equation.

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LU factorization

Note about the multipliers: When we know the pivot in row j , and we know the entry to be eliminated in row i , the multiplier ℓ_{ij} is their ratio:

$$\text{Multiplier } \ell_{ij} = \frac{\text{entry to eliminate (in row } i)}{\text{pivot (in row } j)} \quad (4)$$

The convention is to **subtract** (not add) ℓ_{ij} times one equation from another equation.

Put the multipliers $\ell_{21}, \ell_{31}, \ell_{32}$ etc. into a lower triangular matrix **L**. This yields the LU factorization of **K**:

$$\mathbf{K} = \mathbf{L}\mathbf{U} \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}. \quad (5)$$

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Example 1

Example 1 Add -1 's in the corners to get the circulant C . The first pivot is $d_1 = 2$ with multipliers $\ell_{21} = \ell_{31} = -\frac{1}{2}$. The second pivot is $d_2 = \frac{3}{2}$. But there is no third pivot:

$$C = \begin{bmatrix} \textcircled{2} & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{2} & -1 & -1 \\ 0 & \textcircled{\frac{3}{2}} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{2} & -1 & -1 \\ 0 & \textcircled{\frac{3}{2}} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} = U.$$

In the language of linear algebra, the rows of C are **linearly dependent**. Elimination found a combination of those rows (it was their sum) that produced the last row of all zeros in U . With only two pivots, C is **singular**.

A full set of pivots can not be found. **C** does not have full rank. **C** is singular.

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Symmetric matrices

Assume $\mathbf{A}$ symmetric, such that $\mathbf{A} = \mathbf{A}^T$.

If there is a factorization $\mathbf{A} = \mathbf{L}\mathbf{U}$, it can also be written

$\mathbf{A} = \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, where $\mathbf{D} = \text{diag}(\mathbf{U})$.

If all pivots are positive, we can write

$\mathbf{A} = \mathbf{L}_1\mathbf{L}_1^T$, where $\mathbf{L}_1 = \text{diag}(\sqrt{u_{ii}})\mathbf{L}$.

which is the Cholesky factorization.

If the pivots are all positive, the matrix is SPD - symmetric and positive definite.

Definition: A matrix $\mathbf{A}$ is SPD if it is symmetric and $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$.

Example: The so called normal equations: $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. Here, $\mathbf{A}$ is $m \times n$. If the columns of $\mathbf{A}$ are linearly independent, then $\mathbf{A}^T\mathbf{A}$ is SPD.

Show it! [Notes]