

Linear Algebra, part 1 Linear Systems

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Mathematical Models, Analysis and Simulation Fall semester, 2012

Linear systems of equations

Let **A** be an $m \times n$ matrix (*m* rowns, *n* columns). (**A**)_{*i*,*j*} = a_{ij} , *i* row index, *j* column index. The matrix entries can be real ($a_{ij} \in \mathbb{R}$) or complex ($a_{ij} \in \mathbb{C}$).

Let **x** be a column vector of size $n \times 1$. $\mathbf{x} = (x_1, x_2, \dots, \mathbf{x}_n)^T$.

Matrix vector multiplication: $\mathbf{b} = \mathbf{A}\mathbf{x}$, where \mathbf{b} is a column vector of size $m \times 1$. When \mathbf{b} is given and \mathbf{x} is unknown, want instead to solve (now assume m = n):

$\mathbf{A}\mathbf{x} = \mathbf{b}.$

When does this system have a solution? When is it unique?

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Range, rank, nullspace, nullity

Let $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$, where each \mathbf{a}_i is $m \times 1$. Then $\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n$.

Column space $V = R(\mathbf{A})$ (range of \mathbf{A}) spanned by the columns of \mathbf{A} . rank(\mathbf{A}) = dim(V) = number of linearly independent columns.

If Ax = 0, then x is in the *nullspace* of A. $ker(A) = \{x \in \mathbb{R} : Ax = 0\}.$ The dimension of the nullspace: nullity(A) = dim(ker(A)).If A is $m \times n$, we have that

$$rank(\mathbf{A}) + nullity(\mathbf{A}) = n.$$

Questions

Let **A** be $n \times n$.

- **1.** What do we call A when $nullity(\mathbf{A}) > 0$?
- **2.** When does Ax = b have a unique solution?
- 3. When can we pick vectors b₁ and b₂ such that Ax = b₁ have multiple solutions and Ax = b₂ have no solutions? What can we say about b₁ and b₂?

Large branch of numerical linear algebra

Solve linear system $\mathbf{Ax} = \mathbf{b}$, \mathbf{A} is $n \times n$ where *n* often is a large number.

Where do such systems come from? Many sources, for example:

- Discretization of differential equations by different numerical methods. (Applications to fluid mechanics, electromagnetics, quantum physics, biology, option pricing...)
- Network models and graphs (Electric circuits, mechanical trusses, hydraulic systems).

Solution method: Gaussian elimination

Solution by Gaussian elimination. (In Matlab: x=A\b) Boxed material below taken from Strang, Computational Science and Engineering, 2007.

$$\mathbf{Ku} = \mathbf{f} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \begin{array}{c} 2u_1 - u_2 &= 4 \\ \text{is} & -u_1 + 2u_2 - u_3 = 0 \\ -u_2 + 2u_3 = 0 \end{bmatrix}$$
The first step is to eliminate u_1 from the second equation. Multiply equation 1 by $\frac{1}{2}$ and add to equation 2. The new matrix has a zero in the 2, 1 position—where u_1 is eliminated. I have circled the first two pivots:
$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 \end{bmatrix} \begin{array}{c} 2u_1 - u_2 &= 4 \\ \text{is} & \frac{3}{2} u_2 - u_3 = 2 \\ -u_2 + 2u_3 = 0 \end{bmatrix}$$

Now, multiply Eq 2 by 2/3 and add to Eq 3.

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Gaussian elimination, example continued

This yields:

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + \frac{1}{2} f_1 \\ f_3 + \frac{2}{3} f_2 + \frac{1}{3} f_1 \end{bmatrix} \qquad \begin{array}{c} 2u_1 - u_2 &= 4 \\ \text{is} & \frac{3}{2} u_2 - u_3 = 2 \\ & \frac{4}{3} u_3 = \frac{4}{3} \end{array} \tag{1}$$

Upper triangular matrix **U**. Forward elimination is complete.

Solution by backsubstitution. Last equation determines u_3 . Then the second determines u_2 . With u_3 and u_2 known, easy the find u_1 using the first equation.

LU factorization

Note about the multipliers: When we know the pivot in row j, and we know the entry to be eliminated in row i, the multiplier ℓ_{ij} is their ratio:

Multiplier
$$\ell_{ij} = \frac{\text{entry to eliminate}}{\text{pivot}} \frac{(in \ row i)}{(in \ row j)}$$
 (4)

The convention is to subtract (not add) ℓ_{ij} times one equation from another equation.

Put the multipliers $\ell_{21}, \ell_{31}, \ell_{32}$ etc. into a lower triangular matrix **L**. This yields the LU factorization of **K**:

$$\boldsymbol{K} = \boldsymbol{L}\boldsymbol{U} \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} .$$
(5)

Example 1

Example 1 Add -1's in the corners to get the circulant C. The first pivot is $d_1 = 2$ with multipliers $\ell_{21} = \ell_{31} = -\frac{1}{2}$. The second pivot is $d_2 = \frac{3}{2}$. But there is no third pivot: $C = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} = U.$ In the language of linear algebra, the rows of C are **linearly dependent**. Elimination

In the language of linear algebra, the rows of C are **linearly dependent**. Elimination found a combination of those rows (it was their sum) that produced the last row of all zeros in U. With only two pivots, C is **singular**.

A full set of pivots can not be found. **C** does not have full rank. **C** is singular.

Example 2

Example 2 Suppose a zero appears in the second pivot position but there is a nonzero below it. Then a row exchange produces the second pivot and elimination can continue. This example is **not singular**, even with the zero appearing in the 2, 2 position:

 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ leads to } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \text{ Exchange rows to } U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$ Exchange rows on the right side of the equations too! The pivots become all ones, and elimination succeeds. The original matrix is invertible but not positive definite. (Its determinant is *minus* the product of pivots, so -1, because of the row exchange.)

Permutation matrix **P** swaps the rows. Now, PA = LU. L: Lower triangular matrix with 1s on the diagonal. U: Upper triangular matrix.

A full set of pivots can be found, if the rows are swapped. The matrix does have full rank. The matrix is not singular.

Factorization and determinants

A matrix **A** is non-singular *if and only if* it admits a factorization $\mathbf{PA} = \mathbf{LU}$, where **P** is a row-reordering matrix. ($\mathbf{P} = \mathbf{I}$, the identity matrix if no reordering necessary).

Computation of determinants: $det(\mathbf{PA}) = det(\mathbf{P}) \cdot det(\mathbf{A}) = \pm 1 \cdot det(\mathbf{A}).$ $det(\mathbf{LU}) = det(\mathbf{L}) \cdot det(\mathbf{U}) = det(\mathbf{U}) = product of all pivots.$ Hence, if there are *n* non-zero pivots, then $det(\mathbf{A}) \neq 0$.

Theorem: **A** is non-singular if and only if $det(\mathbf{A}) \neq 0$.

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Symmetric matrices

Assume **A** symmetric, such that $\mathbf{A} = \mathbf{A}^T$. If there is a factorization $\mathbf{A} = \mathbf{L}\mathbf{U}$, it can also be written $\mathbf{A} = \mathbf{L}\mathbf{U} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, where $\mathbf{D} = diag(\mathbf{U})$.

If all pivots are positive, we can write $\mathbf{A} = \mathbf{L}_1 \mathbf{L}_1^T$, where $\mathbf{L}_1 = diag(\sqrt{u_{ii}})\mathbf{L}$. which is the Cholesky factorization.

If the pivots are all positive, the matrix is SPD - symmetric and positive definite.

Definition: A matrix **A** is SPD if it is symmetric and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for all non-zero vectors \mathbf{x} .

Example: The so called normal equations: $\mathbf{A}^T \mathbf{A} \mathbf{x} = \mathbf{A}^T \mathbf{b}$. Here, \mathbf{A} is $m \times n$. If the columns of \mathbf{A} are linearly independent, then $\mathbf{A}^T \mathbf{A}$ is SPD.

Show it! [Notes]

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