

An Introduction to the Modified Nodal Analysis

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1 Introduction

Gilbert Strang provides an introduction to the analysis of electrical circuits in his book *Computational Science and Engineering*. His presentation aims at having the theory as close as possible to his general construction principles for mathematical models which occur throughout the above mentioned book. This notation might be rather unusual for an electrical engineer working in electric circuit analysis. The following explanations are intended to put the derivations into the engineering context.

2 The Principles

The numerical simulation of electric networks is closely related to the network modeling. A well established approach is the description of the network by a graph with branches and nodes. Each branch represents an electric element whose terminals are coupled together at the nodes. (See the simple example in Figure 1.) The simplest network elements are fully described by a relation between a single branch current and the respective branch voltage (e.g., resistors, capacitors, inductors, (independent) voltage and current sources). The describing current-voltage relations are called characteristic equations.

The state of the network at any given time is now completely described by all branch voltages, branch currents, and node potentials. The node potentials are only defined up to a constant. Therefore, one node is assigned the voltage (potential) 0V. This node is called the mass (or, ground) node. The other node potentials are given with respect to this reference node (this way being unique).

In order to complete the network model, the topology of the elements (i.e., their mutual interconnections) has to be taken into account. Assuming the electrical connections between circuit elements to be ideally conducting and the nodes to be ideal, the topology can be described by Kirchhoff's laws.

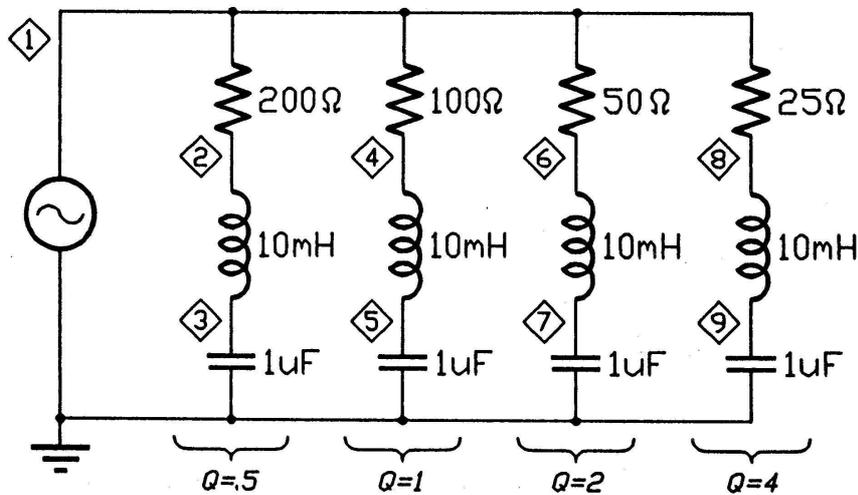


Figure 1: A sample circuit: Schematic of a low-pass filter

Thus, the network modeling consists of two steps:

1. Describe the network elements.
2. Apply Kirchhoff's laws.

3 Network elements

In order to make the presentation as simple as possible, I will only consider uncontrolled, two-terminal elements. Moreover, the elements will be assumed to be linear. The respective branch voltage will be denoted by v while the branch current is i . Note that both can be positive or negative depending on the orientation of the branch.

3.1 Passive Elements

In network modeling one distinguishes three different types of passive elements: resistors, capacitors, and inductors. Their characteristic equations can be described as follows:

Resistors limit, or resist, the flow of electrical current, following the law $v = Ri$. R is the resistance value. Sometimes this relation is written as $i = Gv$ where $G = 1/R$ is called conductivity.

Capacitors store energy in an electrostatic field following the law $q = Cv$ where q is the electric charge. C is the capacitance value. The voltage-current characteristics is given by $i = \frac{d}{dt}q = C\frac{dv}{dt}$ (if C is constant).

Inductors store energy in an electromagnetic field following the law $\Phi = Li$ where Φ is the magnetic flux. L is the self-inductance value. The voltage-current characteristic is given by $v = \frac{d}{dt}\Phi = L\frac{di}{dt}$ (if L is constant).

3.2 Independent Sources

Again for the sake of simplicity, independent sources will be the only active elements we consider. The generalization to controlled elements is straightforward.

Voltage source The current-voltage characteristic is given by $v = E$ with E being the strength of the source. Note that v does not depend on the branch current i .

Current source The current-voltage characteristic is given by $i = I$ with I being the strength of the source. Note that i does not depend on the branch voltage v .

4 Kirchhoff's Laws

The electrical behavior of the network is completely described by Kirchhoff's laws.

Considering one node with branch currents i_1, \dots, i_l entering this node we may describe Kirchhoff's current law (KCL) as $i_1 + \dots + i_l = 0$, that means, the sum of all branch currents entering a node equals zero.

If we consider a loop with the branch voltages v_1, \dots, v_m , then we may formulate Kirchhoff's voltage law (KVL) as $v_1 + \dots + v_m = 0$, that means, the sum of all branch voltages in a loop equals zero.

In a practical network there are very many nodes and loops. In order to describe the topology of the circuit, one must write down KCL and KVL for each of them. It becomes obvious that one needs a systematic way to derive all these equations for a given network. Fortunately, there is a very elegant description of all these individual equations. The magic happens if one uses the (reduced) incidence matrix A of the circuit.

Assume that we have a circuit with $n + 1$ nodes and b branches. Number the nodes and branches accordingly. The ground node is omitted here. For every branch, define an orientation, that is, one node of the branch is assumed to be the “starting” node, the other one is the “end” node. The reduced¹ incidence matrix describes which nodes belong to which branch:

$$a_{kl} = \begin{cases} 1, & \text{if branch } l \text{ has start node } k \\ -1, & \text{if branch } l \text{ has end node } k \\ 0, & \text{else} \end{cases} .$$

It is now convenient to collect all branch currents into one vector² $i = [i_1, i_2, \dots, i_b]'$. Then, the compact notation of KCL is

$$Ai = 0.$$

The incidence matrix allows, additionally, a simple description of the relation between node potentials and branch voltages of the network. If $v = [v_1, v_2, \dots, v_b]'$ is the vector of all branch voltages and $e = [e_1, e_2, \dots, e_n]'$ denotes the vector of all node potentials (excluding the reference (mass) node), then the relation

$$v = A'e$$

is satisfied.

5 Network Analysis

As introduced previously, let i be the vector of all branch currents, v be the vector of all branch voltages, and e be the vector of all node potentials. The first step consists of writing down the network equations and the characteristic equations of all network elements as described in the previous subsections. This gives rise to a system of dimension $2b + n$ for the unknowns i, v, e . The approach leading to this system is called sparse tableau analysis.

The so-called *modified nodal analysis* (MNA) requires a much smaller number of unknowns. In this case, one replaces the branch currents of all current defining elements (conductors, resistors³, current sources) by their characteristic equation, and all branch voltages by node voltages. For a convenient description of this approach, it is appropriate to decompose the incidence matrix according

¹The notion “reduced” comes from the fact that the ground node is left out.

²By convention, all our vectors are assumed to be column vectors. In order to come as close as possible to the MATLAB notation, I will use the apostrophe for denoting the transpose of a matrix or vector: $A^T = A'$. This is valid as long as A is a real matrix.

³via conductance formulation!

to the element types of their branches. By definition of the incidence matrix, every column of A corresponds to one branch. We assume that the branches are enumerated in the following order: first all resistive branches, then all capacitive branches, then all inductive branches, then all voltage source branches, and finally all current source branches. Then A can be decomposed into block form:

$$A = [A_R, A_C, A_L, A_V, A_I],$$

where the index stands for resistive, capacitive, inductive, voltage source, and current source branches, respectively. Using the characteristic equations as indicated above, we obtain the following system:

$$\begin{aligned} A_C \frac{d}{dt} C A'_C e + A_R G A'_R e + A_L i_L + A_V i_V &= -A_I I, \\ \frac{d}{dt} L i_L - A'_L e &= 0, \\ A'_V e &= E. \end{aligned}$$

Here, we have used the following denotations:

- i_V is the vector of all branch currents through voltage sources.
- i_L is the vector of all branch currents of inductive branches.
- I is the vector of the values of all current sources.
- E is the vector of the values of all voltage sources.
- C is the diagonal matrix containing all capacities.
- G is the diagonal matrix containing all conductances (inverses of the resistances).
- L is the diagonal matrix containing all inductivities.

The unknowns in the classical MNA system are the node voltages e , the currents through voltage sources i_V , and the currents through inductors i_L . Thus, the number of unknowns has reduced drastically. In matrix notation, the system can be written down as

$$\begin{bmatrix} A_C C A'_C & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} + \begin{bmatrix} A_R G A'_R & A_L & A_V \\ -A'_L & 0 & 0 \\ A'_V & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ i_L \\ i_V \end{bmatrix} = \begin{bmatrix} -A_I I \\ 0 \\ E \end{bmatrix}.$$

This system consists of differential equations (for e^4 and i_L) and additional equations which do not contain any derivatives of the unknown functions. Such a system is called a differential-algebraic equation. It will play a key role not only in circuit simulation.

5.1 DC Analysis

The abbreviation DC stands for “Direct Current”. When performing a DC analysis we assume that the network consists only of resistors, current and voltage sources. Capacitors and inductors have to be replaced by their DC equivalents: a capacitor is a perfect insulator while an inductor is a perfect conductor. In order to simplify the notation assume that the network does not contain any capacitors or inductors. Therefore, the matrix in front of the derivatives is a zero matrix. Hence the MNA equations reduce to

$$\begin{bmatrix} A_R G A'_R & A_V \\ A'_V & 0 \end{bmatrix} \begin{bmatrix} e \\ i_V \end{bmatrix} = \begin{bmatrix} -A_I I \\ E \end{bmatrix}.$$

This is a linear system of equations. Under some assumptions on the network topology, the system matrix is nonsingular such that the system has a unique solution.

Example Consider the electrical circuit given in Figure 2. According to our convention, we will count the resistive branches first, then the voltage source branch, and finally the current source branch. Let branch number i be the branch through resistor R_i , for $i = 1, 2, 3, 4$. Branch 5 is the voltage source, branch 6 consists of the current source. This provides us with the incidence matrices

$$A_R = \begin{bmatrix} +1 & +1 & 0 & 0 \\ 0 & -1 & +1 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, A_V = \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}, A_I = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

Let for simplicity all resistances being equal to 1Ω . The resulting system is then

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ i_V \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 9 \end{bmatrix}.$$

⁴More precisely, since $A_C C A'_C$ is a singular matrix, not all node voltages are really determined by differential equations. This will need some more detailed investigation during the course.

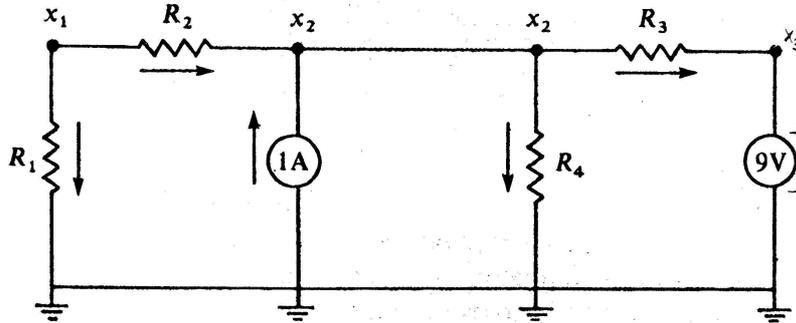


Figure 2: A resistive network

5.2 AC Analysis

The abbreviation AC stands for “Alternating current”. For the purposes of this project, we will assume that all sources are harmonic and have the same frequency f , i.e., they follow the relations

$$v(t) = \hat{V} \sin \omega t, i(t) = \hat{I} \sin \omega t$$

where $\omega = 2\pi f$. Instead of using the amplitudes it is common to work further on with the effective values

$$V = \hat{V} / \sqrt{2}, I = \hat{I} / \sqrt{2}.$$

Under these assumptions it is very convenient to use complex quantities. Instead of the trigonometric functions, complex exponentials are applied. By using Moivre’s law, the time-dependent voltages and currents can be identified as the real (or imaginary) part of an exponential $e^{j\omega t}$ scaled by \hat{V} and \hat{I} , respectively. Since

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t},$$

the differential law describing the capacitors and resistors, respectively, reduce to simple algebraic relations

$$v = Zi,$$

which formally resemble Ohm's law for resistors. The complex equivalent of the resistance is called impedance. The systematic use of this equivalence in analyzing RLC networks is called $j\omega$ -method.

The value of the impedance is easily computed from the technical parameters:

Resistor $Z = R$

Capacitor $Z = \frac{1}{j\omega C}$

Inductor $Z = j\omega L$

Let us now come back to the MNA equations. Inductive and capacitive branches can now be handled in the same way as resistive branches where the impedance replaces the resistance. The inverses of the impedances Z are called admittances Y . Let Y_R, Y_C, Y_L denote diagonal matrices which contain the admittances of the resistive, capacitive and inductive branches, respectively.⁵ Then it holds

$$\begin{bmatrix} A_R Y_R A'_R + A_C Y_C A'_C + A_L Y_L A'_L & A_V \\ & 0 \end{bmatrix} \begin{bmatrix} e \\ i_V \end{bmatrix} = \begin{bmatrix} -A_I I \\ E \end{bmatrix}.$$

The vector of voltages and currents now contain the effective values.

Example Consider the RLC network of Figure 3. Let the branches be ordered according to our convention. Then we obtain the following incidence matrices:

$$A_R = \begin{bmatrix} +1 & 0 \\ 0 & +1 \\ 0 & 0 \end{bmatrix}, A_C = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}, A_L = \begin{bmatrix} 0 \\ +1 \\ -1 \end{bmatrix}, A_V = \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}, A_I = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

and

$$Y_R = \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_4 \end{bmatrix}, Y_C = [j\omega C], Y_L = [1/(j\omega L)].$$

5.3 Transient Analysis

Transient analysis is intended to investigate the time-dependent behavior of electrical networks. The basic question is how the output signal(s) of a circuit depend on variations of the input signal(s). More generally, one is interested to learn how all internal voltages and currents behave in dependence on time-varying input signals. These signals are often modeled as time-varying voltage or current sources. This is what we are doing here. So the time-dependence appears in the right-hand

⁵In particular, $Y_R = G$.

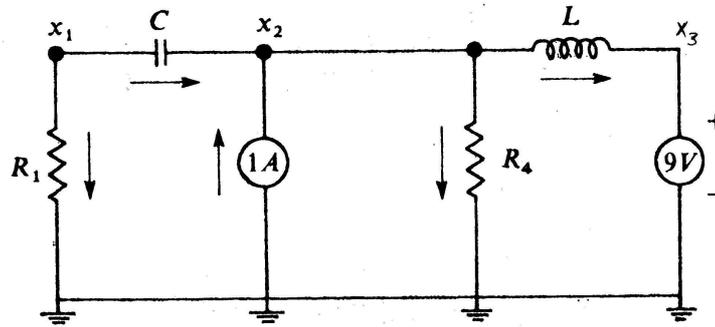


Figure 3: A simple RLC circuit

side, only. In order to solve the problem the complete system of differential-algebraic equations must be solved. For that, an initial state at time $t = 0$ must be known. This is simply an initial condition for the differential-algebraic system, e.g.,

$$e = e^0, \quad i_L = i_L^0, \quad i_V = i_V^0.$$

Note that these initial values must be consistent!⁶

5.4 Computing Currents, Or: How Do We Get Ampère Meters?

It remains to resolve a problem which is connected with the properties of MNA. There are only some currents available for output. Nevertheless, the state of the network can be derived completely from the computed quantities. In particular, we might be interested in knowing the value of the current through a resistive branch. It is not immediately available. But there is a simple trick: put a voltage source with value of 0V in series with the resistor! This voltage source does not change anything in the network. On the other hand, we introduced a device where the current is available. According to KCL, the current through the voltage source

⁶In practice it is possible to consider even nonconsistent initial values. They give rise to impulse responses (mathematically, δ -functions and their derivatives). Often, they represent undesired behavior of the circuit.

is the same as that through the resistor. Hence, voltage sources with 0V are our ampere meters.

6 The Connection to Strang's Book

How does the above calculus fit into Strang's book? Strang uses a notation which is adapted to the general modelling principles he introduces. In our short survey, we used the notation which is commonly used in electrical network simulation. There is an easy translation available. One major difference is the convention for the incidence matrices. Compared to Strang, we used a different sign convention and a "transposed" description of the directed graph. This gives rise to the following table:

| Strang | here |
|--------|---------|
| x | e |
| e | v_R |
| y | i |
| c_i | G |
| A | $-A'_R$ |

Another difference is the handling of the independent sources. While we use the systematic approach by incidence matrices A_V and A_I , Strang eliminates them immediately. Therefore, there is no need to have these incidence matrices.