

ODE, part 3

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Linearization of nonlinear system

Consider an autonomous system of 2 first order equations

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

 $\mathbf{u} = (u_1(t), u_2(t))^T$, $\mathbf{f} = (f_1(\mathbf{u}), f_2(\mathbf{u}))^T$. Assume that $\mathbf{u}^* + \mathbf{v}(t)$ is a solution, then

$$\frac{d}{dt}(\mathbf{u}^* + \mathbf{v}(t)) = \frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{u}^* + \mathbf{v}) = \underbrace{\mathbf{f}(\mathbf{u}^*)}_{=0} + \mathbf{J}(\mathbf{u}^*)\mathbf{v} + O(\|\mathbf{v}\|^2),$$

with the Jacobian $J_{ij} = \frac{\partial f_i(\mathbf{u})}{\partial u_j}$ evaluated at \mathbf{u}^* . $\mathbf{J}(\mathbf{u}^*)$ is a constant matrix, let us denote it by \mathbf{A} .

The dynamics close to the critical point is determined by \mathbf{A} , and we can learn a lot from studying the behavior of solutions of the linear system

$$\frac{d\mathbf{v}}{dt} = \mathbf{A}\mathbf{v}$$

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Stability of critical points

- **1.** Determine the critical points \mathbf{u}^* , where $\mathbf{f}(\mathbf{u}^*) = \mathbf{0}$.
- 2. Compute the Jacobian

$$\mathbf{A} = \mathbf{J}(\mathbf{u}) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix}$$

and evaluate it at $\mathbf{u} = \mathbf{u}^*$.

3. Determine the stability or instability of the linearized system by the eigenvalues of **A**.

If $\mathbf{f}(\mathbf{u}^*) = 0$ then for initial values near \mathbf{u}^* the nonlinear equation $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ imitates the linearized equation with matrix $\mathbf{A} = \mathbf{J}(\mathbf{u}^*)$:

- if **A** is *unstable* at \mathbf{u}^* then so is $\mathbf{u}' = \mathbf{f}(\mathbf{u})$.
- if **A** is *stable* at \mathbf{u}^* then so is $\mathbf{u}' = \mathbf{f}(\mathbf{u})$.

The nonlinear equations has spirals, nodes, and saddle points according to **A**. However, for "borderline" case of a center, the stability of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ is undecided.

Undecided cases

- if **A** has a center (neutral stability with purely imaginary eigenvalues) the stability of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ is undecided. The nonlinear system can have either a center or a spiral.
- if **A** has a borderline node (one double real eigenvalue), the nonlinear system can have either a node or a spiral.

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First example: damped pendulum

Non-dimensionalized equation:

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \sin\theta = 0.$$

With $u_1 = \theta$, $u_2 = d\theta/dt$, get first order system

$$\frac{du_1}{dt} = u_2$$
$$\frac{du_2}{dt} = -\sin(u_1) - cu_2$$

Critical points, $(p\pi, 0)$, p integer. Analysis shows odd multiples of π unstable critical points (mass stationary at top). Stability of critical points with even multiples of π (pendulum hangs down) depends on value of c. [NOTES]



Phase portraits for the pendulum



Undamped pendulum (c = 0): Orbits in the phase plane are contours of constant energy.



Damped pendulum (0 < c < 2): Curves spiral into equilibrium. For larger damping, picture changes again and the stationary points are nodes.

Population dynamics

A simple predator prey model:

$$u_1' = a u_1 - b u_1 u_2 \tag{1}$$

$$u_2' = c u_1 u_2 - d u_2, \tag{2}$$

where u_1 represents the population of the prey and u_2 represents the population of the predator, and a, b, c, d > 0.

Analysis of linearized system yields one critical point that is a saddle point, and one center. Stability of this last point undecided for nonlinear system. [NOTES]



Periodic solutions, i.e. closed curves:

 $a\log u_2 - bu_2 = cu_1 - d\log u_1 + C,$

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where the constant C is determined by $u_1(0)$ and $u_2(0)$.

Periodic solutions - limit cycles.

- A linear system has closed paths only if the eigenvalues of the system matrix are purely imaginary. In this case, every path is closed.
- A nonlinear system can perfectly well have a closed path that is isolated.
- A "limit cycle" is a periodic orbit that trajectories approach.

The Poincaré-Bendixson theorem

Any orbit of a 2D continuous dynamical system which stays in a closed and bounded subset of the phase plane forever must either tend to a critical point or to a periodic orbit.

Hence, if this subset of the phase plane has no critical point, the solution must approach a periodic orbit.

There is no such theorem for systems of larger dimension. It is the special topology of the plane that makes the difference: A closed curve (periodic orbit) divides the plane into two disjoint sets, and orbits cannot cross the boundary.

Famous example: Van der Pol equation

$$rac{d^2 heta}{dt^2}+\mu(heta^2-1)rac{d heta}{dt}+ heta=0, \quad \mu>0$$

- Again, "damped" oscillations, but not certain that the damping is positive.
- For small θ, the "damping" is negative, and the amplitude grows. For large θ, the damping is positive, and solution decays.
- However, not periodic for all initial values.
 When the solution leaves a very small or very large value, it can not get back.
- Instead, all orbits spiral toward a *limit cycle* which is the unique periodic solution to van der Pol's equation.



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Hopf bifurcation

The appearance or the disappearance of a periodic orbit through a local change in the stability properties of a steady point is known as the *Hopf bifurcation*.

Let J0 be the Jacobian of a continuous parametric dynamical system evaluated at a steady point \mathbf{u}^* of it.

Suppose that all eigenvalues of J0 have negative real parts except one pair of conjugate nonzero purely imaginary eigenvalues $\pm i\beta$.

A Hopf bifurcation can arise when these two eigenvalues crosses the imaginary axis because of a variation of the system parameters. (More specific conditions exist).

Remark. For linear systems, we can also talk about a bifurcation as stability is lost, i.e. as a pair of eigenvalues "crosses the imaginary axis". This is however not a Hopf bifurcation (can never get a limit cycle).

The Lorenz attractor - The butterfly effect

Simplified model of weather. A 3-system. The equations that govern the Lorenz oscillator are

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z, \end{aligned}$$



where σ is called the Prandtl number and ρ is the Rayleigh number. All $\sigma, \rho, \beta > 0$, but usually $\sigma = 10$, $\beta = 8/3$, and ρ is varied.

Trajectories that starts close by drift apart after a while, and flip from one leaf of the structure to the other, seemingly at random. The system is deterministic, but very sensitive to perturbations - the butterfly effect.

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