ROYAL INSTITUTE
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PDEs, part 3: Hyperbolic PDEs

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Hyperbolic equations

(Sections 6.4 and 6.5 of Strang).
Consider the model problem (the transport or advection equation, or one-way wave equation as Strang calls it):

$$u_t + a u_x = 0 \qquad x \in \mathbb{R}, t > 0$$
$$u(x,0) = g(x)$$

where g is a compactly supported smooth function.
t is time, x is the spatial variable.

Models transport and wave propagation.

For well posedness, we need that:

1. A solution exists.
2. The solution is unique.
3. The solution depend continuously on the data.

First, let us show (1)-(3) for the simplest case when a is constant.
[NOTES]

More general hyperbolic equations in 1D: Variable coefficient

Again, but now with a variable coefficient $a = a(x)$:

$$u_t + au_x = 0 \qquad x \in \mathbb{R}, t > 0$$
$$u(x, 0) = g(x).$$

Models for example variable propagation speed due to variations in the material.

Define the characteristics X as the curves in the (x, t) plane, given by the ODEs:

$$\frac{dX}{dt} = a(X), \quad X(0) = x_0.$$

Then we have that

$$u(X(t), t) = g(x_0)$$

since $\frac{d}{dt}u(X(t), t) = 0$ (show by chain rule) and $u(X(0), 0) = g(x_0)$. (When a is constant, $X = x_0 + at$, and $u(x_0 + at, t) = g(x_0)$, and hence $u(x, t) = g(x - at)$.)

Characteristics and propagation of information

- At time t, the value $g(x_0)$ only affect the value in one point: $u(X(t), t)$.
- Reversed: $u(x, t)$ only depends on the value of the initial data in a specific point.

Information travels along the characteristic curves

- Consider the distance between the initial point and the point being affected,

$$|x_0 - X(x_0, t)| \leq |\int_0^t \frac{dX}{d\tau} d\tau| = |\int_0^t a dt| \leq |a|_\infty \cdot t.$$

We say that:

Information travels with a limited velocity in hyperbolic problems

- Compare to parabolic problems where the solution depends on all of the initial data $g(x)$ for $t > 0$: Information travels infinitely fast.

More general hyperbolic equations in 1D: System with constant coefficients

Consider:

$$\begin{aligned} \mathbf{u}_t + \mathbf{A}\mathbf{u}_x &= 0 \qquad \mathbf{A} \in \mathbb{R}^{p\times p}, \mathbf{u} \in \mathbb{R}^p, \mathbf{g} \in \mathbb{R}^p, \\ \mathbf{u}(x,0) &= \mathbf{g}(x). \end{aligned}$$

If $\mathbf{A}$ is diagonalizable and have real eigenvalues, then this is a hyperbolic equation.
Let $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ where $\mathbf{S} = (\mathbf{e}_1, \ldots, \mathbf{e}_p)$, eigenvectors, and
$\mathbf{\Lambda} = diag(\lambda_1, \ldots, \lambda_p)$, eigenvalues.
Define $\mathbf{v} = \mathbf{S}^{-1}\mathbf{u}$ with $\mathbf{v} = (v_1, \ldots, v_p)$. This yields

$$\mathbf{v}_t + \mathbf{\Lambda}\mathbf{v}_x = 0, \quad \text{or} \quad \frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0, \quad k = 1, \ldots, p.$$

Hence, we have p decoupled scalar equations with constant coefficients.

System with constant coefficients, contd.

We have the solutions

$$v_k(x,t) = v_k(x - \lambda_k t, 0),$$

and hence

$$\mathbf{u}(x,t) = \mathbf{S}\mathbf{v} = \sum_{k=1}^{p} v_k(x - \lambda_k t, 0)\mathbf{e}_k,$$

where $v_k(0,x) = (\mathbf{S}^{-1}\mathbf{g})_k$.

- The eigenvalues λ_k gives the propagation speed for corresponding eigenmode.
- $\mathbf{u}(x,t)$ depend on $\mathbf{g}(x)$ evaluated in p points: $\{x - \lambda_k t\}_{k=1}^{p}$.
- The lines $x - \lambda_k t$ are called characteristics or characteristic curves.
- Information in initial data is spread along characteristics with finite propagation speed.
- As for the scalar case: when $\mathbf{A}$ is variable, $\mathbf{A} = \mathbf{A}(x)$, the characteristics are no longer straight lines.

The wave equation

Consider the scalar wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x). \end{aligned}$$

Let $v = u_t$ and $w = u_x$;

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} v \\ w \end{pmatrix}_x = 0$$

A has eigenvalues $\lambda = \pm c$, i.e. propagation with speed c both to the right and to the left.
The eigenvectors are $(c, 1)^T$ and $(c, -1)^T$.

Solution formula

Finding the solution for the first order system, we can find the general solution formula for the wave equation on the whole line.
The formula is d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[g(x-ct) + g(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy.$$

Boundary conditions

Often, we are interested in the solution on a limited interval.

$$\begin{aligned} u_t + a u_x &= 0 \quad t > 0 \qquad x \in (0,1), \\ u(x,0) &= g(x) \qquad\quad x \in (0,1). \end{aligned}$$

How can we choose boundary conditions at $x = 0$ and $x = 1$?
Especially sensitive for hyperbolic problems.
Define inflow and outflow boundaries! [NOTES]
Rule:

- Boundary condition must be given at inflow boundary.
- Boundary condition can not be given at outflow boundary.

In this example:

$a > 0$: One boundary condition at $x = 0$.

$a < 0$: One boundary condition at $x = 1$.

It is also possible to have periodic boundary conditions, $u(0) = u(1)$.

Boundary conditions, contd.

For systems with p components ($\mathbf{u} \in \mathbb{R}^p$), we have p families of characteristic curves.
Following the same reasoning as above gives the rule

No of positive eigenvalues = No of boundary conditions at $x = 0$.

No of negative eigenvalues = No of boundary conditions at $x = 1$.

Example:
The wave equation. Eigenvalues $\pm c$, i.e. one BC at each boundary.

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \\ u(0) = u(1) &= 0 \qquad \text{or} \qquad u_x(0) = u_x(1) = 0. \end{aligned}$$

Numerical methods for the advection equation

We want to numerically solve

$$u_t + a u_x = 0,$$
$$u(x,0) = g(x).$$

(For the moment, we ignore the boundaries).

- For hyperbolic problem it is impractical to use the route of semi discretizations. (Several common numerical schemes cannot be derived this way and many natural choices yield unstable schemes).
- We will discretize directly in both space and time (a full discretization - FD). Introduce:

  $x_j = j\Delta x$, $j = \ldots, -1, 0, 1, \ldots$
  $t_n = n\Delta t$, $n = 0, 1, \ldots$
  and let $u_j^n \approx u(x_j, t_n)$.
- We also let $\lambda = \Delta t/\Delta x$, which is the "Courant" or "CFL" number for hyperbolic equations.
- Discretizations of hyperbolic problems *do not* become stiff and explicit methods are therefore almost always used.

Examples of common methods

1. Lax-Friedrich's
2. Upwind (Euler scheme)
3. Lax-Wendroff
3. Beam-Warming

(1) and (2) are first order accurate in time and space.
(3) and (4) are second order accurate in time and space.

All three schemes can be shown to be conservative, i.e to yield

$$\sum_j u_j^n = const, \quad \text{independent of } n.$$

This is desirable, since it holds for the exact solution that

$$\int u(x,t)dx = const.$$

[NOTES]

Analysis, convergence of numerical approximation

Let $\tilde{\mathbf{u}}^n = (\tilde{u}_1^n, \ldots, \tilde{u}_N^n)^T$, where $\tilde{u}_j^n = u(x_j, t_n)$, the exact solution.
The solution of the discretized problem is $\mathbf{u}^n = (u_1^n, \ldots, u_N^n)^T$.
Define the discrete L_2-norm:

$$\|g\|_{\Delta x} = \left(\Delta x \sum g_j^2\right)^{1/2}$$

The numerical solution converges to the true solution if

$$\lim_{\Delta x \to 0} \max_{0 \le n \le N_{\Delta t}} \|\mathbf{u}^n - \tilde{\mathbf{u}}^n\|_{\Delta x} = 0$$

when $\Delta t = \mu(\Delta x)^\alpha$ (for some μ, α), and $T = N_{\Delta t} \cdot \Delta t$ is fixed.

For hyperbolic problems, we will naturally have $\Delta t = \mu \Delta x$, i.e. $\mu = \frac{\Delta t}{\Delta x}$.

Lax equivalence theorem

Remember the Lax equivalence theorem, which we have already used for parabolic equations:
The Lax equivalence theorem holds for a general linear well-posed time dependent PDE. For a finite difference method for such a problem it holds:

The method is convergent if it is *consistent* and *stable*.

We will consider a general one-step method

$$\sum_{k=-\gamma}^{\delta} b_k(\mu) u_{j+k}^{n+1} = \sum_{k=-\alpha}^{\beta} c_k(\mu) u_{j+k}^n,$$

where $\mu = \frac{\Delta t}{\Delta x} = const$.
(If the method is explicit, $b_k = 0$ if $k \neq 0$, and LHS is u_j^{n+1}).

Consistency

- Local truncation error: τ_j^n. Residual when inserting the exact solution $\tilde{u}_j^n$ into the numerical scheme.
- If $\tau_j^n = O((\Delta x)^p + (\Delta t)^q)$ we say that the method is of order p in space and q in time.
- The method is consistent if $p \geq 1$ and $q \geq 1$.

Example: The Lax-Friedrich scheme.
Insert the exact solution:

$$\tau_j^n = \frac{1}{\Delta t}\left(u(x_j, t_{N+1}) - \frac{1}{2}\left(u(x_{j+1}, t_N) + u(x_{j-1}, t_N)\right)\right) + \frac{a}{2\Delta x}\left(u(x_{j+1}, t_N) - u(x_{j-1}, t_N)\right)$$

Taylor expand and use the PDE $u_t + au_x = 0$. This yields

$$|\tau_j^n| = O\left(\frac{(\Delta x)^2}{\Delta t} + \Delta t + (\Delta x)^2\right).$$

The Lax-Friedrich scheme is consistent if $\Delta t/\Delta x = \mu = const$.
Then, $|\tau_j^n| = O(\Delta x)$.

Stability

- We say that the method is stable if

$$\|\mathbf{u}^n\|_{\Delta x} \leq C(T)\|\mathbf{u}^0\|_{\Delta x} \quad \text{for } n = 0, \ldots, N_{\Delta t},$$

  where $N_{\Delta t} \cdot \Delta t = T$ and $\Delta t = \mu(\Delta x)^\alpha$ (for some μ, α).
- OBS! $C(T)$ does not depend on Δt or Δx.

How to check stability?

1. The general CFL condition. (Necessary condition).
2. Fourier (von Neumann -) analysis (sufficient condition for linear, constant coefficient PDEs with periodic BC or Cauchy problems), otherwise a necessary condition.

1) The general CFL condition

Introduce the concept of "domain for dependence", both for the continuous problem and the numerical approximation. I.e. what earlier values affect the solution $u(x,t)$ and u_j^n, respectively.

The method is unstable if $D(x_j, t_n)$ is not a subset of $D_{num}(x_j, t_n)$.

Then the numerical method cannot predict what the real value should be.

For an explicit "3 point formula":

$$u_j^{n+1} = C_1 u_{j+1}^n + C_0 u_j^n + C_{-1} u_{j-1}^n,$$

we must have

$$|a| \frac{\Delta t}{\Delta x} \leq 1$$

in order for the method to be stable. [NOTES]
This is a necessary (but not sufficient) condition.

2) von Neumann analysis

Assume a periodic problem, $x \in [0,1)$ and $u(0,t) = u(1,t)$.
Let $x_j = j\Delta x$, $j = 0, \ldots, N-1$, $\Delta x = 1/N$.
Let $\hat{u}_k^n$ be the Fourier coefficients of u, such that

$$u_j^n = \sum_{k=-N/2}^{N/2-1} \hat{u}_k^n e^{ikx_j}$$

Consider the discrete one step method,
$\sum_{\ell=-\gamma}^{\delta} b_k(\mu) u_{j+\ell}^{n+1} = \sum_{\ell=-\alpha}^{\beta} c_k(\mu) u_{j+\ell}^n$, and plug in one component of the solution, $\hat{u}_k^n e^{ikx_j}$,

$$\sum_{\ell=-\gamma}^{\delta} b_\ell(\mu) \hat{u}^{n+1} e^{ik(x_j + \Delta x \ell)} = \sum_{\ell=-\alpha}^{\beta} c_\ell(\mu) \hat{u}^n e^{ik(x_j + \Delta x \ell)}$$

and so

$$\hat{u}^{n+1}(k) = \hat{Q}(k)\hat{u}^n(k), \quad \text{or} \quad \hat{u}^n(k) = (\hat{Q}(k))^n \hat{g}(k),$$

if $\hat{g}(k)$ is the corresponing Fourier coefficient for the initial condition.

2) von Neumann analysis, contd

So, we have

$$\hat{u}^{n+1}(k) = \hat{Q}(k)\hat{u}^n(k), \quad \text{or} \quad \hat{u}^n(k) = (\hat{Q}(k))^n \hat{f}(k),$$

where

$$\hat{Q}(k) = \frac{\sum_{\ell=-\alpha}^{\beta} c_\ell(\mu) e^{ik(\ell\Delta x)}}{\sum_{\ell=-\gamma}^{\delta} b_\ell(\mu) e^{ik(\ell\Delta x)}}.$$

We have

$$\|u^n\|_{\Delta x}^2 = \Delta x \sum (u_j^n)^2 = [\text{Parseval's formula}] = 2\pi\Delta x \sum_{k=-N/2}^{N/2-1} |\hat{u}_k^n|^2 \leq$$

$$\leq 2\pi\Delta x \sum_{k=-N/2}^{N/2-1} |\hat{Q}(k)|^{2n} |\hat{u}_k^n|^2 \leq |\hat{Q}(k)|_\infty^{2n} \cdot 2\pi\Delta x \sum_{k=-N/2}^{N/2-1} |\hat{u}_k^0|^2 =$$

$$= |\hat{Q}(k)|_\infty^{2n} \|u^0\|_{\Delta x}^2$$

For stability, for each wave number k we need

$$\left|\hat{Q}(k)\right| \leq 1 + C \cdot \Delta t.$$

Symbol of exact solution

We have

$$u_j^n = \sum_{k=-N/2}^{N/2-1} \hat{u}_k^n e^{ikx_j} = \sum_{k=-N/2}^{N/2-1} (\hat{Q}(k))^n \hat{g}_k e^{ikx_j}.$$

The exact solution is given by $u(x,t) = g(x-at)$. Hence, the corresponding Fourier expansion should be

$$u(x_j, t_n) = \sum_{k=-N/2}^{N/2-1} \hat{g}_k e^{ik(x_j - an\Delta t)} = \sum_{k=-N/2}^{N/2-1} (e^{-ika\Delta t})^n \hat{g}_k e^{ikx_j}.$$

Hence, for the exact solution we have $\hat{Q}(k)^{exakt} = e^{-ika\Delta t}$.

Errors of different character- dissipation and dispersion

$\hat{Q}(k)^{exakt} = e^{-ika\Delta t}$.

Dissipative errors

We have that $|\hat{Q}(k)^{exakt}| = 1$, but for a numerical method we might have $|\hat{Q}(k)| \leq 1$ and the magnitude might depend on Δt and Δx.

If $|\hat{Q}(k)| < 1$ for some wave number k, artificial damping of this frequency. This is a dissipative, "smearing" error.

Typical for first order methods. The Lax-Friedrich method is more dissipative than the upwind method.

Dispersive errors

We can write $\hat{Q}(k) = |\hat{Q}(k)|e^{-ik\alpha_k \Delta t}$, for some α_k.

For the exact solution $\alpha_k = a, \forall k$, i.e. the propagation speed is the same for all frequencies.

For a numerical scheme, α_k is the numerical propagation speed for frequency k. If $\alpha_k \neq a$, we get a dispersive error, i.e. different frequencies artificially have different speed of propagation.

Typical for second order methods.

Modified equation

It is possible to find a PDE that is exactly satisfied by the numerical approximation u_j^n (by Taylor series expansion). Will have infinite number of terms involving higher powers of Δt and Δx.

Truncate this series at some point, and you have a "modified equation".

This is another way to see the character of the leading order error.

For example: Let $\Delta t/\Delta x$ be fixed. Then the Lax-Friedrich scheme approximates

$$u_t + au_x = 0$$

to first order, but it approximates the equation

$$u_t + au_x = \frac{1}{2}\Delta t \left(\left(\frac{\Delta x}{\Delta t} \right)^2 - a \right) u_{xx}$$

to second order. With $\Delta t < \Delta x/a$, we have an advection diffusion equation

$$u_t + au_x = \varepsilon u_{xx}.$$

Solution: Wave will propagate with speed a, but εu_{xx} term will diffuse and smear solution. ε depends on grid size, and will vanish as grid is refined ($\Delta t/\Delta x$ const). [NOTES].

Modified equation, contd

Similarly, the Beam-Warming scheme approximates

$$u_t + a u_x = 0$$

to second order, but it approximates the equation

$$u_t + a u_x = \frac{a(\Delta x)^2}{6}\left(2 - \frac{3a\Delta t}{\Delta x} + \left(\frac{a\Delta t}{\Delta x}\right)^2\right) u_{xxx}$$

to third order.
What does this mean?