### Lab 1, FD-TD method for the Maxwell equations

The aim of this exercise is to experiment with the FD-TD method for the Maxwell equations, see e.g. the CEMbook, and

*TafloveFDTD: Computational Electrodynamics: The Finite-Difference Time-Domain Method*, second edition, A. Taflove and S. Hagness, Artech House 2000. This exercise is intended to represent one credit, i.e. a full weeks work.

### 1 The Maxwell equations in 1D

The Maxwell equations in 3D for a plane wave propagating in the *x*-direction with a polarization such that  $E_y = 0$ ,  $E_z \neq 0$  are

$$\begin{cases} Hy_t = \frac{1}{\mu} Ez_x - \frac{\rho}{\mu} Hy\\ Ez_t = \frac{1}{\epsilon} Hy_x - \frac{\sigma}{\epsilon} Ez \end{cases}$$
(1)

where the three components of the electric field  ${\bf E}$  are (Ex, Ey, Ez ) and the three components of the magnetic field  ${\bf H}$  are (Hx, Hy, Hz ).

## 2 General implementation in 1D

Below we give a step by step description for adding functionality to your 1D code. It is highly recommended that you implement all the steps in one Matlab code by using sub-functions or sub-programs. It shall be possible to control the code behavior by control variables or input parameters. Implement the Yee scheme in vacuum for (1) on this grid:



Figure 1: The location and indexing of the discrete variables.

In words: use *n* cells. Place  $E_z$  at  $0 : \Delta x : n\Delta x$  (note, n + 1  $E_z$  components in space) and at  $t = 0 : \Delta t : ...$ 

Place *Hy* at  $\Delta x/2$ : $\Delta x$ : $\Delta x \cdot (n - 1/2)$  (*n Hy* components in space) and at  $t = -\Delta t/2$ : $\Delta t$  : .... Do NOT restrict your code to the case  $\Delta x = 1$ .

 $E_{z}(x,t): E_{z}^{k} = E_{z}((i-1)\Delta x, k\Delta t), i = 1, ..., n+1, k = 0, 1, ...$ 

$$Hy(x,t): Hy_i^k = Hy((i-1/2)\Delta x, (k-1/2)\Delta t), i = 1, ..., n, k = 0, 1, ...$$

 $Ez_i^0 = Ez(t=0)$  and  $Hy_i^0 = Hy(t=-\Delta t/2)$  are initial values and are in most cases put to zero.

(2)

In vacuum the conductivity is equal to zero ( $\sigma = 0$ ), in non-exotic materials the equivalent magnetic resistivity is zero ( $\rho = 0$ ), the permittivity  $\varepsilon \approx 8.8541878 \cdot 10^{-12}$  and the permeability  $\mu = 4\pi 10-7 \approx 1.256637061 \cdot 10^{-6}$ 

The grid size  $\Delta x$  can be arbitrarily chosen but the time step  $\Delta t$  is restricted by the maximum CFL-number which in one dimension is one, ( $\Delta t < 1 \cdot \Delta x/c$ ). In this case our recommendation is to use CFL = 1/2. The number of time steps should be chosen such that the wave will propagate around for a while until the whole picture is clear.

DN2274 Computational Electromagnetics Fall '12 Math p 2 (5)

Choose a pulse form and a pulse length suitable for the problem at hand. Remember that wave-components with a short wave length suffer severely from dispersion. The size of the domain, *n* must be large enough to cover the pulse but small enough in order to get a speedy simulation.

### 2.1 Basic boundary conditions

Use a Perfect Electric Conductor (PEC) as upper boundary, i.e.  $Ez_{n+1}^k = 0, k = 0, 1, 2...,$ and a time dependent source function f(t) at the lower boundary, i.e.

 $Ez_1^k = f(k\Delta t), k = 0, 1, ...$ A suitable function f(t) is e.g. a Gaussian:

 $f(t) = e^{-(t-t_0)^2/t_w^2}, t > 0; = 0, t < 0$ (3)

where  $t_0 = 6t_w$  is a suitable choice. Why? The analytic solution for this boundary condition is  $E_z(x,t) = f(t - x/c)$  where  $c = 1/\sqrt{\mu\varepsilon}$  is the speed of propagation.

### 2.1.1 Assignment

Make a simple stability test by changing the CFL-number to a few different values.

### 2.2 Absorbing Boundary Condition (ABC)

Implement Mur 1st order ABC at the upper boundary in the way recommended in **TafloveFDTD** 

$$Ez_{n+1}^{k+1} = -Ez_n^{k-1} + a_1(Ez_n^{k+1} + Ez_{n+1}^{k-1}) + a_2(Ez_{n+1}^k + Ez_n^k),$$

$$a_1 = \frac{cfl-1}{cfl+1}, a_2 = \frac{2}{cfl+1}, cfl = \frac{c\Delta t}{\Delta x}$$
(4)

You need initial conditions for  $Ez_n^{-1}$  and  $Ez_{n+1}^{-1}$ ; simply put them to zero. Note: Equation (4) is a slightly improved formulation compared to the one Mur originally used,

$$Ez_{n+1}^{k+1} = Ez_n^k + a_1(Ez_n^{k+1} - Ez_{n+1}^k)$$
(5)

#### 2.2.1 Assignment

Make sure that the ABC works correctly. There should be almost no reflections in the ABC. Are there any reflections at all?

### 2.3 Huygens' surface

Implement a Huygens' surface for generation of incident waves. The Huygens' surface divides the domain into two parts. In the left part only the scattered field is calculated and in the right part the total field is calculated.



Figure 2: PEC at the lower boundary and ABC at the upper boundary. A Huygens' surface is introduced to generate incident waves.

Introduce the integer parameter db which stands for distance-to-boundary. The

DN2274 Computational Electromagnetics Fall '12 Math p 3 (5)

parameter *db* describes in which cell the Huygens' surface is located. The updating for the magnetic component (total field) immediately to the right of the Huygens' surface is described by

$$Hy_{db}^{k+1} = \frac{1-\alpha}{1+\alpha} \cdot Hy_{db}^{k} + \frac{\beta}{1+\alpha} \cdot (Ez_{db+1}^{k} - Ez_{db}^{k} - Ezinc_{db}^{k})$$

$$\alpha = \frac{\rho\Delta t}{2\mu}, \beta = \frac{\Delta t}{\mu\Delta x}$$
(6)

The updating for the electric component (scattered field) immediately to the left of the Huygens' surface is described by

$$Ez_{db}^{k+1} = \frac{1-\gamma}{1+\gamma} \cdot Ez_{db}^{k} + \frac{\delta}{1+\alpha} \cdot (Hy_{db}^{k+1} - Hy_{db-1}^{k+1} - Hy_{db}^{k+1})$$

$$\gamma = \frac{\sigma\Delta t}{2\varepsilon}, \delta = \frac{\Delta t}{\varepsilon\Delta x}$$
(7)

Note that  $Hy^{k+1}$  is known from the latest half time step. If  $\sigma = \rho = 0$ ,  $\alpha = \gamma = 0$ . Also note that both (6) and (7) are the usual Yee scheme plus a correction term. Hence, the best way to implement (7) is to first perform the Yee update for all electric fields, and then perform

$$Ez_{db}^{k+1} = Ez_{db}^{k+1} - \frac{\delta}{1+\alpha} \cdot Hyinc_{db}^{k+1}$$
(8)

and similarly for (6).

The relation between the electric field and the magnetic field for a plane wave is described by the impedance Z

$$|E| = Z \cdot |H|, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}$$
(9)

Making sure that these source terms are evaluated with correct *t*-values and taking care of the fact that waves travel with the speed of light gives

$$Hyinc_{db}^{k+1}(=-\frac{1}{Z}f(t-x/c)) = -\frac{1}{Z}f((k+1/2)\Delta t - (db-1/2)\Delta x/c)$$
(10)

 $Ezinc_{db}^{k} (= f(t - x/c)) = f(k\Delta t - (db - 1)\Delta x/c)$ 

For *f*, you can use the Gaussian in (3). Note that f(t - x/c) (almost) = 0 if t < x/c.

#### 2.3.1 Assignment

Make sure that the Huygens' surface generates a right-going wave. The parameter db can be put to 3 when all is working fine, but for debugging purposes db should be set to a higher value to make sure that (almost) no energy is propagating to the left. However, small amounts of high frequency components will propagate to the left due to discretization errors. Verify that you have second-order accuracy.

### 2.4 Absorbing Boundary Condition again

Implement Mur 1st order ABC at the lower boundary and keep the ABC at the upper boundary.

$$Ez_1^{k+1} = -Ez_2^{k-1} + a_1(Ez_2^{k+1} + Ez_1^{k-1}) + a_2(Ez_1^k + Ez_2^k)$$
(11)  
Put a perfect electric conductor in the middle of the domain, at  $x = n/2 \Delta x$  (*n* even):  
 $Ez_{n/2}^m = 0$ , for all *m*. Use the Huygens' surface to generate an incident wave.



#### 2.4.1 Assignment

Validate that there are no reflections in the ABC at the lower boundary. This is done by generating a right going wave at the Huygens' surface and reflecting the wave in the PEC to a left going wave. Does the Huygens' surface interact with the left going wave?

### 2.5 Dielectric material

Remove the perfect electric conductor in the middle of the domain and make the right half a dielectric material with  $\varepsilon r = 4$ . Use the Huygens' surface to generate an incident wave. The amplitude of the reflected wave should be *R* (reflection coefficient) times the amplitude of the incident wave and the transmitted wave should have a amplitude of T (transmission coefficient) times the amplitude of the incident wave where

$$R = \frac{b-1}{b+1}, T = \frac{2}{b+1}, b = \sqrt{\varepsilon_r} \quad (T+R \equiv 1)$$
(12)



Figure 4: A dielectric material in the right half of the domain.

Remember that the speed of light is used in the formulation of the ABC. This means that the ABC at the upper boundary must be adjusted according the value of  $\varepsilon r$ .

#### 2.5.1 Assignment

How good agreement do you get with your code compared to the theoretical reflection/transmission coefficients? Also note the wavelength in the dielectric material.

### 2.6 Second Huygens' surface

To check the accuracy of the Huygen's surfaces, implement a second Huygens' surface at location db2 > db. The second Huygens' surface should *absorb* the wave generated by the first surface if the entire domain is vacuum, but due to numerical errors this will not work perfectly.

This is done with the following correction terms:

$$Ez_{db2}^{k+1} = Ez_{db2}^{k+1} + \frac{\delta}{1+\alpha} \cdot Hyinc_{db2}^{k+1}$$

$$Hy_{db2}^{k+1} = Hy_{db2}^{k+1} + \frac{\delta}{1+\alpha} \cdot Ezinc_{db2}^{k}$$
(14)

Note the change in sign as compared with (6) and (8). Remember to calculate

DN2274 Computational Electromagnetics Fall '12 Math p 5 (5)

f(t - x/c) at this new location. Also note that you must have  $\varepsilon = \varepsilon 0$  in the scattered field region.

### 2.6.1 Assignment

do it.

## 2.7 Simple PML

Change the dielectric material in assignment 2.5 to a material with  $\sigma, \rho \neq 0, \varepsilon = \varepsilon_0$ , and  $\mu = \mu_0 \sigma$ . Choose  $\sigma$  and  $\rho$  to increase as a polynomial from zero at the interface to  $\sigma_{\text{max}}$  and  $\rho_{\text{max}}$  at the upper boundary. Use PEC BC at this boundary. The relation between  $\sigma(x)$  and  $\rho(x)$  should be  $\rho(x) = Z^2 \cdot \sigma(x)$ .

You have now created a Perfectly Matched Layer (PML) in 1D. No splitting of components is needed in 1D!

Experiment with the PML to get a good absorption. Fix the profile to start at  $x = (n - no \ pml \ cells)\Delta x$  where *no pml cells* is the number of PML cells (typically 4 to 12) and calculate  $\sigma$  and  $\rho$  at the location of their corresponding Ez and Hy components. You should calculate the discrete values of  $\sigma$  and  $\rho$  as mean values over the cell.

# 3 Numerical experiment with 2D FDTD software

The software is described in the notes xxx. Set up a simulation in 2D of plane wave propagation along a line, terminated by ABC. Compare the reflections with your results with Mur and, if you did it, 2.7.

# Good Luck !