Lab2: TM wave scattering on PEC and dielectric cylinders; Rays and Caustics.

1 DIY MoM Code for scalar Helmholtz wave scattering in 2D

The notes describe the Method of Moments method for the scattering problem for the Helmholtz equation. Your job is to write a matlab code which does the computations, using the pulse basis functions and collocation ("point matching") at the line element midpoints, as described for the Laplace equation in lecture notes and CEMbook. Accuracy can be gained by going to piecewise linear basis functions. If you enforce continuity on the charge distribution, the same number of unknowns obtains, but calculations of integrals are more involved. Discuss with instructor if you want to extend your code in this way.

Matlab functions are available on the home page for the exact solution (a series expansion in Bessel and Hankel functions). The notes give a matlab function

```
field(z,zprime,gamma,sigma,k)
```

which computes the field from the charge distribution on the curve on a grid of points; the helper functions `green` and `greender` compute the Green's function and its derivative. `field` assumes that the 2D geometry is represented as complex numbers. If you want to use it your code should also use complex numbers $z = x + iy$ for (x,y). The notes propose exercises. Never mind the last exercise about memory savings, but *do* work the other two, and think about the jumps at the boundary of the field computed from the integral.

Program the visualization tools first so you can use them to debug the code. It helps to animate the fields; irregularities, waves moving in the wrong direction, discontinuities, etc., are more easily spotted by our vision system when there is motion. Animation is simple because the fields are time harmonic. About 20 snapshots are needed in a period (10 may be too few).

```
% set up grid and field
x = linspace(0,1,20);
[xx,yy]=meshgrid(x,x);
k = 5;
fi = pi/6;
E = exp(i*k*(cos(fi)*xx+sin(fi)*yy));
% animate
n = 20;
c = exp(i*2*pi*linspace(0,1-1/(n-1),n));
for k = 1:5*n
      surf(real(c(mod(k,n-1)+1)*E));
      pause(0.1)
end
```

When the code works, compare the l2 norm of the scattered field on some grid of points, for instance on a circle outside the scatterer, to that computed by the exact solution on some grid.

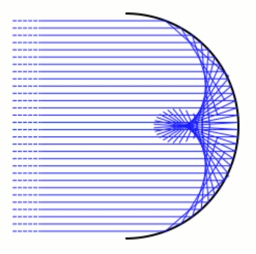
1.1 PEC case

Q1: Plot the field on a dense grid close to the source points, to see the singularity as the field point approaches the source points.

Q2: Plot l2 error vs. number of elements per wavelength for a case with $ka = 5$, where k is the wavenumber, and a is the radius of the circle. What order of convergence do you observe?

Q3: The internal resonance problem
Run a sequence of k-values with fixed a, and plot the l2-norm of the charge distribution (σ) vs. k. What is the smallest k for which the linear system is singular? At this k you should see very large σ, or maybe not: The coefficient matrix becomes singular with a rank-1 null space, but the equations still admit (among many others) the “correct” solution. However, picking the right one requires knowledge of the null space, i.e. the eigensolution, which is the one we try to find!

Q4 Focusing
Replace the circle by a concave parabolic mirror, send in a wave and observe the focusing of the reflected wave. Compute by geometrical optics where the focus should be and check. Then change to a circular mirror, plot the field, and also make a matlab plot of the reflected rays: The curved structure is the caustic. Picture from Wikipedia.



Geometrical optics
works with rays, which are curves normal to the wavefront. In homogeneous media they are straight lines and refract at media discontinuities by Snell’s law. In general there is both a refracted and a reflected ray. We come back to such high-frequency approximations in the last section of the course. The analytical calculation of the caustic to a concave curve finds the envelope to the reflections of a set of parallel rays. For a parabola, the envelope happens to be but a point, for a circle, the shape is familiar from inside your coffee cup.
The envelope of a one-parameter family of curves is tangent at every point to one of the family members. If the family is defined as $f(x,y,C) = 0$, the envelope is the curve (if any) defined by the system $f(x,y,C) = 0$ and $df/dC = 0$

Q5 Far-field transform
Compute the bi-static far-field scattering cross section of your mirror. Evaluate the computed field on a circle outside the scatterer and compute its Fourier coefficients. The large argument asymptotic expansion of Hankel functions can be found in e.g. Abramowitz-Stegun, *Handbook of Mathematical Functions*, or some other collection of formulas.

1.2 Dielectric scatterer

Experience shows that the coding takes great care with signs, especially in the diagonal elements. So, the hard part is getting it completely right. The relative permittivity of the scatterer is ε. Check your code by running with $\varepsilon = 1$ and also an $\varepsilon < 1$ (longer waves inside than outside). Convince yourself by plotting details of the fields at the scatterer surface that the waves inside and outside satisfy the correct interface conditions.

Q6: Repeat the convergence study from the PEC case and $\varepsilon = 4$. Now the waves inside the scatterer are shorter and set the requirement for the element size. How large ε can you compute before matlab runs out of memory or you run out of patience?

Q7: With larger *ka* you will see more clearly the focusing of the transmitted wave. What should ε be in order to put the focus ***on*** the circle? Use rays and Snell's law of refraction to find where the focus is; when it is inside you only need consider *one* refraction. Run the case and check.