Reflection & Engquist-Majda non-reflecting conditions

We have derived a non-reflecting condition by considering 1-D waves and the characteristics. The condition admits only waves traveling with outward velocity c (phase speed of light). What happens if we apply it on a multi-dimensional wave problem on a half-space $x < 0$?
Say 2D (x,y).

For $x < 0$, the wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$, on the right half-space $x > 0: u_t + cu_x = 0$

Time-harmonic variation $u = e^{i\omega t}U(x,y)$, and $k = \omega/c = 2\pi/\lambda$. Let the incoming wave be $U_{inc} = \exp(i(k_x x + k_y y))$, the reflected $U_{sc} = R\exp(i(\kappa_x x + \kappa_y y))$ and the transmitted wave $U_{tr} = T\exp(i(ax + by))$.

$k^2 = k_x^2 + k_y^2$ is the dispersion relation for the left half plane, so it follows that

$\kappa_x{}^2 + \kappa_y{}^2 = k^2$ and likewise, $a^2 + b^2 = k^2$

Indeed, $b = 0$ and $k = a$, for the characteristic condition, but we will keep the generality for the moment. This defines the reflection coefficient R and the transmission coefficient T.
At $x = 0$ the waves on the two half-planes must match,

$U_{inc} + U_{sc} = U_{tr}$, and d/dx($U_{inc} + U_{sc}$) = d/dx($U_{tr}$):

$$e^{ik_y y} + Re^{i\kappa_y y} = Te^{iby}, k_x e^{ik_y y} + \kappa_x Re^{i\kappa_y y} = aTe^{iby} \text{ or}$$
$$k_x e^{ik_y y} + \kappa_x Re^{i\kappa_y y} = a\left(e^{ik_y y} + Re^{i\kappa_y y}\right): e^{ik_y y}(k_x - a) = Re^{i\kappa_y y}(a - \kappa_x)$$

and the reflection coefficient becomes

$$R = \frac{k_x - a}{a - \kappa_x} = \frac{k_x - a}{a + k_x} = \frac{k\cos\theta - a}{k\cos\theta + a}$$

It follows, that $\kappa_y = k_y$ and $\kappa_x = -k_x$ because the scattered wave must move away from the interface. Thus, with the characteristic condition, $a = k$ and

$$R = (\cos\theta - 1)/(1 + \cos\theta) = -\tan^2(\theta/2)$$

where θ is the angle between interface normal and wavefront normal.
Example: A source in the center of a square sees a maximal θ of 45° to the corners where R becomes $3 - \sqrt{8} = .17$

The Mur first order condition, discretized as shown, gives a small reflection even for orthogonal waves because the numerical wave speed differs slightly (parts of %) from c. As we just saw, non-orthogonal waves give much larger reflection, so it makes sense to look for continuous models with smaller reflection. This is the subject of the Engquist–Majda family of conditions. The technique is to derive approximations in wave-number space which are transformed back into physical space by $i\omega$ = d/dt, $-ik_x$ = d/dx, etc.

The dispersion relation is $k_x^{\,2} + k_y^{\,2} = k^2$:

$$k_x = \pm\left(k^2 - k_y^2\right)^{1/2} = \pm k\left(1 - \frac{k_y^2}{k^2}\right)^{1/2}$$

$$1st\text{ order}: k_x = -k = -\frac{i\omega}{ic}; -ick_x + i\omega = 0 : cu_x + u_t = 0$$

$$2nd\text{ order}: k_x = -k\left(1 - \frac{1}{2}\frac{k_y^2}{k^2}\right); kk_x = \frac{\omega}{c}k_x = -k^2 + \frac{1}{2}k_y^2 = -k_x^2 - \frac{1}{2}k_y^2 : -u_{tx} = c\left(u_{xx} + \frac{1}{2}u_{yy}\right)$$

Note that the transmitted wave is $U_{tr} = Te^{i(ax+by)}$ where $k^2 = a^2 + b^2$ where a is the x-wave number of the transmitted wave, i.e.

for first order, $a = k$, $R = O(\theta^2)$, and

for second order $a = (k - 1/2k_y^{\,2}/k) = k(1-1/2\sin^2\theta)$, $R = O(\theta^4)$

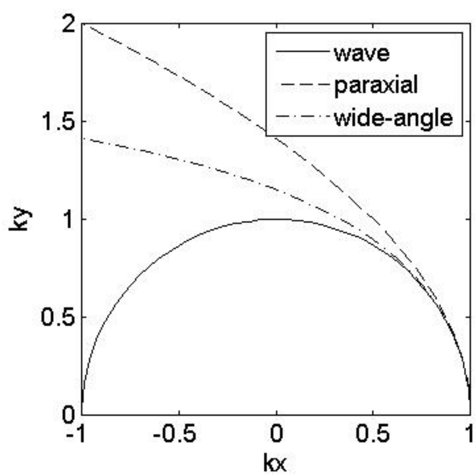
The second order condition is easily implemented on the staggered Yee grid (see copy from Taflove-Hagness)

The dispersion relation may also be used to derive the “paraxial” approximation, which allows a marching type numerical solution at the price of neglecting the back-scatter. It is useful for waveguides (“Beam propagation method”), sound transmission in stratified media, etc.

$$k_x = -k\left(1 - \frac{k_y^2}{2k^2}\right) = -k + \frac{1}{2k}k_y^2 : -iu_x = -ku - \frac{1}{2k}u_{yy}$$

With $u = e^{-ikx}v, iv_x = \frac{1}{2k}v_{yy}$, the Schroedinger equation

The approximations can be illustrated in the (k_x,k_y)-plane: The circle is the wave equation dispersion relation, and the parabola is the paraxial approximation.



The “wide-angle” paraxial approximation comes from a Padé-approximation to the square root:

$$\sqrt{1-x} = \frac{4-3x}{4-x} + O(x^3),$$

giving the PDE

$$k_x = \frac{4 - 3k_y^2/k^2}{4 - k_y^2/k^2}$$

$$-i\left(k^2 u_x + \frac{1}{4}u_{yyx}\right) = \left(k^2 u + \frac{3}{4}u_{yy}\right)$$

Sources

Two kinds, point sources in the computational domain such as current pulses on wires, and waves (cylindrical, plane,…) created by external sources.

Point sources

A point source may be implemented as a prescribed variation of say the E-field in a point. This correctly describes the wave moving away from the source, but also creates reflections from scattered waves hitting the source. In a 1D-case, such a source is a total reflector. In cases where the scattered waves arrive later than the duration of the pulse, one can simply exchange the source for the standard update after the pulse time.
In 2 and 3D, the reflections in the source are much weaker – only in a single gridpoint and we neglect them.

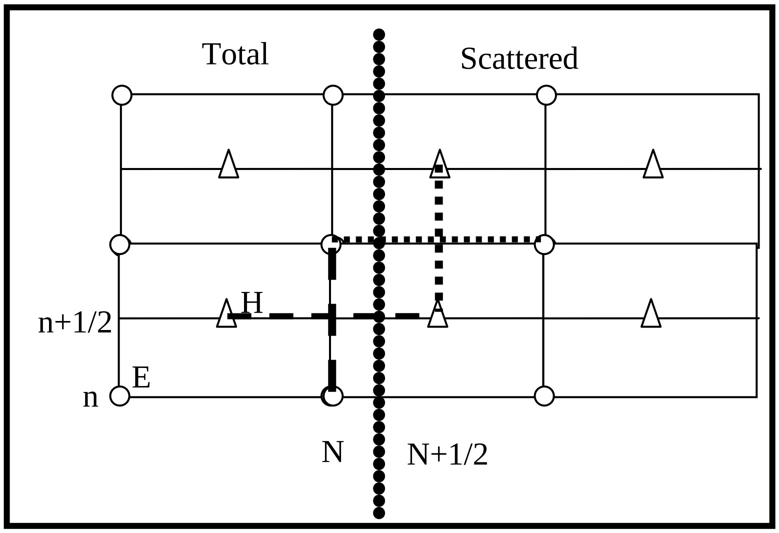
External wave sources

An externally generated wave can be implemented as initial condition, but this is a problem with persistent sources such as a harmonic wave turned on at t = 0. One usually employs "***Huygen's surfaces***" which decompose the domain into a portion outside the scatterer where only the scattered waves U_{sc} are represented on the grid, and a near-field domain where the total field, $U_{tot} = U_{inc} + U_{sc}$ sum of incoming and scattered waves is represented. We illustrate the technique on a 1D-case $\begin{matrix} \varepsilon E_t = H_x \\ \mu H_t = E_x \end{matrix}$ with an incoming wave $\mathbf{U}_{\text{inc}}(x,t) = \begin{pmatrix} E_{inc}(x,t) \\ H_{inc}(x,t) \end{pmatrix}$ and the Yee-scheme. Let the exterior domain be $x > x_{N+1/4}$ (note: the surface is between x_N and $x_{N+1/2}$) so that variables with subscripts >= N+1/2 mean scattered field and with subscripts <= N mean total field.

$$\varepsilon \frac{E_k^{n+1} - E_k^n}{\Delta t} = \frac{H_{k+1/2}^{n+1/2} - H_{k-1/2}^{n+1/2}}{\Delta x}$$

$$\mu \frac{H_{k+1/2}^{n+1/2} - H_{k+1/2}^{n-1/2}}{\Delta t} = \frac{E_{k+1}^n - E_k^n}{\Delta x}$$

is used for k <= N-1/2 and k >= N+1



The update equations for E_N and $H_{N+1/2}$ become:

$$\varepsilon\frac{E_N^{n+1}-E_N^n}{\Delta t}=\frac{H_{N+1/2}^{n+1/2}-H_{N-1/2}^{n+1/2}}{\Delta x}+\frac{1}{\Delta x}H_{inc}(x_{N+1/2},t^{n+1/2})$$

$$\mu\frac{H_{N+1/2}^{n+3/2}-H_{N+1/2}^{n+1/2}}{\Delta t}=\frac{E_{N+1}^{n+1}-E_N^{n+1}}{\Delta x}-\frac{1}{\Delta x}E_{inc}(x_N,t^{n+1})$$