Integral equation methods – 3D scattering on conductors

Book p 170 ff.

Introduction – vector and scalar potentials and EFIE

The notation is as follows : $S = \partial\Omega_c$ is the boundary of (a number of) perfectly conductive scatterers Ω in a homogeneous medium (vacuum) extending to infinity. A point on a scatterer is a "source point", denoted by a prime **r**' when necessary to set it apart from a "field point" **r** = (*x*,*y*,*z*) outside. *R* is the distance, *R* = |**r**–**r**'|. The scatterers are illuminated by an incoming time-harmonic field, $\mathbf{E}^{\mathrm{i}}(x,y,z)\, e^{i\omega t}$. The scattered field is $\mathbf{E}^{\mathrm{s}}$, and the jump condition on the surfaces is that

$$\mathbf{n} \times (\mathbf{E}^i + \mathbf{E}^s) = 0$$

because the fields vanish inside. Introduce the vector potential **A** for **B** and scalar potential ϕ for **E**,

$$\mathbf{B} = \nabla \times \mathbf{A}, \mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - i\omega\mathbf{A}$$

The arbitrariness in **A** and ϕ is resolved by choosing the Lorentz gauge,

$$i\omega\varepsilon_0\mu_0\phi + \nabla \cdot \mathbf{A} = 0,$$

which simplifies the equations to follow. With free space wavenumber

$$k = \omega / c = \omega\sqrt{\varepsilon_0\mu_0}\ ,$$

current density **J** – which will be confined as a surface current density [A/m] to *S* – and charge density ρ, also confined to *S*, [C/m^2], the components of **A** and ϕ satisfy the Helmholtz equations,

$$-(\Delta + k^2)A_i = \mu_0 J_i, i = 1,2,3$$

$$-(\Delta + k^2)\phi = \frac{1}{\varepsilon_0}\rho$$

The free space Green's function is $G(R) = \dfrac{e^{-ikR}}{4\pi R}$. Since there are no fields inside the conductors, we may choose **A** = 0 and ϕ = 0 inside, as for the 2D PEC scattering of Lab 2, and there follows, from e.g. $\nabla \cdot (\varepsilon_0\mathbf{E}) = \rho$,

$$A_i(\mathbf{r}) = \mu_0 \int_S G(R) J_i(\mathbf{r}')dS, i = 1,2,3$$

$$\phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_S G(R)\rho(\mathbf{r}')dS$$

Now, the charge density ρ satisfies the continuity equation

$$i\omega\rho + \nabla_t \cdot \mathbf{J} = 0,$$

where we use the *t*-subscript to denote differentiation parallel to a surface. More about this below, but let us finish the derivation first:

$$\phi(\mathbf{r}) = -\frac{1}{i\omega\varepsilon_0}\int_S G(R)\nabla'_t \cdot \mathbf{J}(\mathbf{r}')dS$$

and so

$$\mathbf{E}^s(\mathbf{r}) = -i\omega\mu_0 \int_S G(R) J_i(\mathbf{r}')dS + \frac{1}{i\omega\varepsilon_0}\nabla(\int_S G(R)\nabla_t' \cdot \mathbf{J}(\mathbf{r}')dS) \quad (1)$$

and the Electric Field Integral Equation, a First Kind Singular integral equation for the surface current density **J** becomes

$$\mathbf{E}^i(\mathbf{r})_t = i\omega\mu_0 \int_S G(R)\mathbf{J}(\mathbf{r}')_t dS + \frac{i}{\omega\varepsilon_0}(\nabla(\int_S G(R)\nabla_t' \cdot \mathbf{J}(\mathbf{r}')dS))_t \quad (2)$$

Note that **J** is tangential to *S*, and the tangent is taken at **r**. In the second integral, only *G* depends on **r** and is operated on by the gradient. Application of the gradient is no problem, since *G* is defined everywhere. We will use (2) as is for the finite element treatment, but re-write the equation here to see the analogy (and differences) with the 2D TM PEC equation of Lab 2.

$$\nabla G(R) = G'(R)\cdot\nabla R = \frac{G'(R)}{R}(\mathbf{r}-\mathbf{r}') = -\nabla' G(R)$$

where the prime on the gradient means differentiation w.r.t. the source point.

$$\mathbf{E}^i(\mathbf{r})_t = i\omega\mu_0\left[\int_S G(R)\mathbf{J}(\mathbf{r}')_t dS - \frac{1}{k^2}\int_S (\nabla' G)_t \nabla_t' \cdot \mathbf{J}(\mathbf{r}')dS\right]$$

The EFIE is a true vector equation and cannot be solved component by component. In the 2D TMz (or TEz) case, only the gradient (and not the curl) operators are involved:

$$u(\mathbf{r}) = \int_S G(R)\sigma(\mathbf{r}')dS - \int_S \partial G/\partial n'\,\gamma(\mathbf{r}')dS \quad (3)$$

Exercise:
Since each component of E satisfies the Helmholtz equation, why don't we use three copies of (3)?

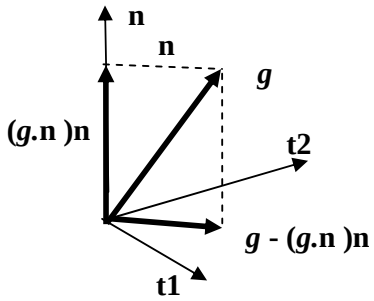
We now return to the tangential gradient (and divergence) operators used above.

The tangential gradient and divergence

Def. On a surface *S* with unit outward normal **n**(**r**), the tangential gradient operator is

$$\nabla_t = \mathbf{t}1(\mathbf{t}1\cdot\nabla) + \mathbf{t}2(\mathbf{t}2\cdot\nabla) = \nabla - \mathbf{n}(\mathbf{n}\cdot\nabla)$$

See the sketch; the gradient is ***g*** and the orthogonal unit surface tangents are **t**1 and **t**2 (**t**2 = **n** x **t**1). The normal component of the gradient can be calculated only for functions defined in a neighborhood of *S*, such as *G* above, but the first form applies also when the function is defined on *S* only.



The tangential divergence satisfies the Surface Divergence Theorem, or the Gauss theorem on the surface. A curve *L* on *S* encloses the surface $A \subset S$. The curve tangent is **t**(*s*), where *s* is the arclength parameter along the curve. Then,

$$\int_A \nabla_t \cdot \mathbf{f}dA = \int_L \mathbf{f}\cdot\mathbf{m}ds, \mathbf{m} = \mathbf{t}\times\mathbf{n} \qquad \text{(SDT)}$$

where **m** is the unit vector parallel to *S* pointing out from *A*.

Example: Tangential divergence on a plane surface.

Choose the coordinate system so the normal direction is *z*, and *x* and *y* span the plane.

$$\nabla_t \cdot \mathbf{f} = \nabla \cdot \mathbf{f} - (\mathbf{n} \cdot \nabla)(\mathbf{f} \cdot \mathbf{n})$$

$$\mathbf{f} = f_1\,\mathbf{e}_x + f_2\,\mathbf{e}_y + f_3\,\mathbf{e}_z, \mathbf{n} = (0,0,1)$$

$$\nabla_t \cdot \mathbf{f} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3 - (\partial_z)(f_3) = \partial_x f_1 + \partial_y f_2$$

so the normal derivative cancels, and we need only the values of **f** on *S*, which is what we expect.

Example: Tangential divergence on a non-planar surface

$$\nabla_t \cdot \mathbf{f} = \nabla \cdot \mathbf{f} - ((\mathbf{n} \cdot \nabla)\mathbf{f}) \cdot \mathbf{n} - \underbrace{\mathbf{f} \cdot ((\mathbf{n} \cdot \nabla)\mathbf{n})}_{\substack{=0 \text{ on plane;} \\ \text{curvature on non-planar}}}$$

Another definition of the surface divergence is

$$\nabla_t \cdot \mathbf{f} = \lim_{|A| \to 0} \frac{1}{|A|} \int_\Gamma \mathbf{f} \cdot \mathbf{m}\, ds \qquad \text{(SDT2)}$$

with definitions as for (SDT) above. The question is of course how to evaluate the normal derivative of the normal, since that is defined a priori only on *S*. But in 2D it is easy:

Example: 2D (*x*,*y*)*-space*

If the curve is smooth, one can introduce a local coordinate system (*p*,*s*) with axes parallel to **n** and **t**, viz. The mapping is 1-1 for sufficiently small distances to the curve *L*, because neighboring normals intersect exactly a distance ρ – the radius of curvature – from *L*. The formula for tangential divergence gives (using def. SDT2 and after some manipulation)

$$\nabla_t \cdot \mathbf{f} = \frac{d}{ds}(\mathbf{f} \cdot \mathbf{t}) \qquad (*)$$

Introducing the "Jacobian" matrix of derivatives of **f**,

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial (x, y)} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

the final result, using (*), is

$$\nabla_t \cdot \mathbf{f} = \mathbf{t} \cdot (\mathbf{J}\mathbf{t}) + (\mathbf{f} \cdot \mathbf{n}) \frac{1}{\rho}$$

Example: The EFIE for TMz waves (Lab2)

For TMz waves, **H** is in the (*x*,*y*)-plane and **E** = *E***e**$_z$. This means that d**E**/dt and curl **H** are along the *z*-axis, and hence **J** = *J***e**$_z$**.** The tangential divergence of **J** becomes zero, because *z*-derivatives vanish. Thus, the second term of the EFIE drops out and

$$E_z^i(\mathbf{r}) = i\omega\mu_0 \int_S G(R) J(\mathbf{r}') dS = i\omega\mu_0 \int_\Gamma \int_{-\infty}^{+\infty} G(\sqrt{r^2 + z^2}) dz J(x, y) ds,$$

$$r = \sqrt{(x - x')^2 + (y - y')^2}$$

and we may recognize (*G* depends also on *k*, but we have suppressed that dependence not to clutter up the notation)

$$\int_{-\infty}^{+\infty} G(\sqrt{r^2+z^2})dz = H_0^2(kr)$$

Numerical solution of the EFIE by weighted residual methods – the Method of Moments.

Define the residual,

$$\mathbf{res}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r})_t - i\omega\mu_0 \int_S G(R)\mathbf{J}(\mathbf{r}')_t\, dS - \frac{i}{\omega\varepsilon_0}\int_S \nabla_t G(R)\nabla_t' \cdot \mathbf{J}(\mathbf{r}')dS$$

for $\mathbf{r}$ on S. The approximate solution is the ansatz $\mathbf{J} = \sum_{k=1}^{N} c_k \mathbf{s}_k$, a linear combination of basis vector functions $\mathbf{s}_k$ – tangential to S. Then choose M test or weight functions $\mathbf{v}_j$, also tangential to S, and require that the moments of the residual vanish,

$$\langle \mathbf{v}_j, \mathbf{res} \rangle = \int_S \mathbf{res}\cdot\mathbf{v}_j dS = 0,\ j = 1,2,...,M$$

The first term is

$$i\omega\mu_0 \int_S \mathbf{v}_j \cdot (\int_S G(R)\mathbf{J}(\mathbf{r}')_t\, dS')dS = i\omega\mu_0 \sum_{k=1}^{N} c_k \underbrace{\int_S \mathbf{v}_j(\mathbf{r})\cdot(\int_S G(R)\mathbf{s}_k(\mathbf{r}')dS')dS}_{a^1_{jk}}$$

Note that since $\mathbf{v}_j$ is tangential, $\mathbf{v}_j(\mathbf{r})\cdot\mathbf{s}_k(\mathbf{r}')_t = \mathbf{v}_j(\mathbf{r})\cdot\mathbf{s}_k(\mathbf{r}')$. The second term is

$$\frac{i}{\omega\varepsilon_0}\int_S \mathbf{v}_j(\mathbf{r})\cdot(\int_S (\nabla_t G(R))(\nabla_{t\cdot}' \cdot \mathbf{J}(\mathbf{r}'))dS')dS = \frac{i}{\omega\varepsilon_0}\sum_{k=1}^{N} c_k \int_S \mathbf{v}_j(\mathbf{r})\cdot\nabla_t(\int_S G(R)(\nabla_{t\cdot}' \cdot \mathbf{s}_k(\mathbf{r}'))dS')dS =$$

$$= -\frac{i}{\omega\varepsilon_0}\sum_{k=1}^{N} c_k \underbrace{\int_S (\nabla_t \cdot \mathbf{v}_j)(\mathbf{r})\int_S G(R)(\nabla_{t\cdot}' \cdot \mathbf{s}_k(\mathbf{r}'))dS')dS}_{a^2_{jk}}$$

where Gauss' theorem for the tangential divergence

$$\int_S \mathbf{g}\cdot\nabla_t f dS = \underbrace{\int_S \nabla_t\cdot(\mathbf{g}f)dS}_{=0} - \int_S f\nabla_t\cdot\mathbf{g}dS$$

was used. The boundary terms vanish since the scatterers are assumed closed.
The coefficients a^1_{jk} and a^2_{jk} must be computed by repeated surface integration, a seemingly very computationally complex process. There remains also to select the basis functions $\mathbf{s}_k$ and weight functions $\mathbf{v}_j$ – and they will be chosen in the finite element scheme so the double surface integrals can be computed with reasonable effort. Before we finish the development we describe simpler finite elements, first, for the scalar Helmholtz and second, for the curl-curl equations.

A finite element solution for the scalar Helmholtz equation

The finite element discretization of a scalar Helmholtz equation is also the subject of Lab 3. See the Book Ch 6. The residual is now

$$res(\mathbf{r}) = -\Delta u - k^2 u - f(\mathbf{r}) \text{ in } \Omega$$

and we let the boundary conditions be $\frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$. The approximate solution is taken as

$$u_h(\mathbf{r}) = \sum_{k=1}^{N} c_k \varphi_k(\mathbf{r}), \varphi_k \in V_h \subseteq H^1(\Omega)$$

The function space $H^1(\Omega)$ consists of functions whose first derivatives are square-integrable over Ω, and V_h is an N-dimensional sub-space spanned by the basis functions φ_k, k= 1,2,…, N

$$\int_\Omega v_j res(\mathbf{r}) d\Omega = \int_\Omega v_j(-\Delta u_h - k^2 u_h - f) d\Omega = (Gauss) =$$

$$= \int_\Omega (\nabla v_j \cdot \nabla u_h - k^2 v_j u_h - v_j f) d\Omega - \int_{\partial\Omega} v_j \underbrace{\frac{\partial u_h}{\partial n}}_{=0} dS$$

which explains why not only v_j and φ_k but also their *gradients* must be square-integrable. The integration by parts allows approximate solutions u_h with only one derivative whereas the differential equation requires two.

The *variational* formulation of the differential equation including boundary conditions is

$$\int_\Omega (\nabla v \cdot \nabla u_h - k^2 v u_h - v f) d\Omega = 0 \text{ for every } v \qquad \text{(V)}$$

and a function u_h which satisfies (V) is called a *weak* solution.

The *Galerkin* recipe chooses $v_j = \varphi_j$, and the discretized equation becomes **Ac** = **b** where the coefficients of **A** and **b** are computed from (V):

$$\sum_{i=1}^{N} c_i \underbrace{\int_\Omega (\nabla \varphi_j \cdot \nabla \varphi_i - k^2 \varphi_j \varphi_i) d\Omega}_{a_{ji}} = \underbrace{\int_\Omega \varphi_j f d\Omega}_{b_j} \quad j = 1,2,...,N$$

Finite elements: P1 over triangles

The simplest finite element basis functions are constructed on a *triangulation*. So Ω is approximated by a polygonal domain, the union of N_e triangles T_k,

$$\Omega \approx \bigcup_k T_k$$

and the basis functions are piecewise linear, with breaks along the edges of the triangles. There is one φ for each vertex (= triangle corner) $\mathbf{r}_k$ of the triangulation, and

$$\varphi_j(\mathbf{r}_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

from which follows

$$u_h(\mathbf{r}_k) = c_k$$

φ_j vanishes over all T_k *except* those which have $\mathbf{r}_j$ as a corner. This property makes the integral of products of basis functions vanish except for a few. Thus, **A** becomes sparse with only a few non-zero elements per row.

The matrix elements of the Helmholtz equation can be computed analytically, but in general must be evaluated by numerical integration, summing the contributions from each triangle.. Here are a few quadrature formulas for $I = \int_T f(\mathbf{r})dS$ for a triangle T with area A and diameter h. $I = \int_T f(\mathbf{r})dS = A\sum_{\alpha=1}^{M} w_\alpha f(\mathbf{r}_\alpha) + O(h^{p+2})$. The points $\mathbf{r}_\alpha$ are called "quadrature points" and the w_α "weights". p is the order of accuracy – the lowest degree polynomial for which the formula is *not* exact.

1. One-point (Centroid). The centroid is $\mathbf{c} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)/3$, $I = A\cdot f(\mathbf{c}) + O(h^4)$, and the formula is exact for polynomials of degree 1.
2. Corner point formula: $I = A/3\cdot(f(\mathbf{r}_1) + f(\mathbf{r}_2) + f(\mathbf{r}_3)) + O(h^4)$, exact for polynomials of degree 1.

The properties of the linear system of equations: **A** is symmetric and sparse: $a_{ij} = 0$ unless corners i and j are neighbors, i.e. appear in a triangle. Sparse Gaussian elimination can solve systems of a million unknowns in a few seconds on a PC. For 3D problems, the situation is much less favorable. There are more unknowns and the matrix is less sparse, and the LU-factors much less sparse. But fast iterative methods are also not known, and one usually resorts to variants of "general" iterative schemes like GMRES, see the end of these notes and the Book.

For $k = 0$, i.e. the Poisson equation, **A** is positive definite, and the solution to the variational problem is also the solution to the minimization problem

$$\mathbf{c} = \arg\min_{\mathbf{x}\in R^N} \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x}$$

This observation shows that the Galerkin solution is optimal, being the best approximant in V_h to the exact solution when measured in the energy norm,

$$u_h = \arg\min_{u_h\in V_h} \int_\Omega |\nabla(u - u_h)|^2 d\Omega$$

Also, very fast iterative methods like the Multi Grid iteration are known, and are actually faster than Gaussian elimination for big 2D problems.

A finite element method for the curl-curl equation

When E_z is available, the magnetic field **H** can be computed by differentiation. But numerical differentiation is sensitive to rounding errors, and the discretization error is one order lower than for E_z. Therefore it makes sense to formulate a problem for the **H**-field in the TMz case. For consistency with the book, we treat instead the TEz-case, solving for $\mathbf{E} = (E_x, E_y)$.
First we derive the curl curl equation for **E**, like in the Book, but we use simpler boundary conditions:

$$\left.\begin{aligned} \nabla\times\mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla\times\mathbf{H} &= \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E} + \mathbf{J}_S \end{aligned}\right\} \Rightarrow \nabla\times(\mu^{-1}\nabla\times\mathbf{E}) - (\omega^2\varepsilon - j\omega\sigma)\mathbf{E} = -j\omega\mathbf{J}_S$$

with PEC, $\mathbf{E}_t = 0$, and PMC, $\mathbf{H}_t = 0$ (perfect magnetic conductor) boundary conditions:

$$\mathbf{E}\times\mathbf{n} = 0, \nabla\times\mathbf{E} = 0 \text{ - the latter because TEz -}$$

These are two (scalar) conditions for the two second order equations.
We need the vector analysis identity $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A}$ leading to the integration by parts formula

$$\int_\Omega \mathbf{w}\cdot(\nabla\times\mathbf{g})d\Omega = \int_\Omega \nabla\cdot(\mathbf{w}\times\mathbf{g})d\Omega + \int_\Omega \mathbf{g}\cdot(\nabla\times\mathbf{w})d\Omega = \int_\Gamma (\mathbf{w}\times\mathbf{g})\cdot\mathbf{n}ds + \int_\Omega \mathbf{g}\cdot(\nabla\times\mathbf{w})d\Omega$$

With the vector weight function **w**, and $\mathbf{g} = \mu^{-1}\nabla\times\mathbf{E}$, the weighted residual equation becomes

$$\int_\Omega \mathbf{w}\cdot(\nabla\times(\mu^{-1}\nabla\times\mathbf{E}) - (\omega^2\varepsilon - j\omega\sigma)\mathbf{E} + j\omega\mathbf{J}_S))d\Omega = 0:$$

$$\int_\Omega \left[\mu^{-1}(\nabla\times\mathbf{w})\cdot(\nabla\times\mathbf{E}) - (\omega^2\varepsilon - j\omega\sigma)\mathbf{w}\cdot\mathbf{E}\right]d\Omega + \int_\Gamma \mathbf{w}\times(\underbrace{\mu^{-1}\nabla\times\mathbf{E}}_{=0})\cdot\mathbf{n}ds = -\int_\Omega j\omega\mathbf{w}\cdot\mathbf{J}_S d\Omega$$

The weighted residual formulation is the variational, or weak, form symmetric in **w** and **E.** An approximate solution $\mathbf{E}_h$ is sought in the function subspace spanned by the vector basis functions $\mathbf{N}_m$

$$\mathbf{E}_h = \sum_{m=1}^{N} E_m \mathbf{N}_m(\mathbf{r})$$

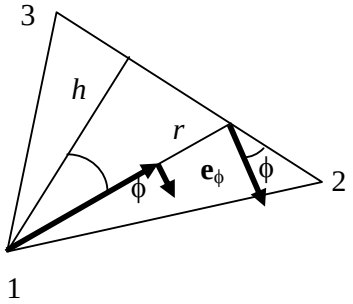
and in the Galerkin method the weight functions are taken to be the $\mathbf{N}_m$, too. The discretized equation becomes **AE = b:** for $i = 1,2,\ldots,N$

$$\sum_{m=1}^{N} E_m \underbrace{\int_\Omega \left[\mu^{-1}(\nabla\times\mathbf{N}_i)\cdot(\nabla\times\mathbf{N}_m) - (\omega^2\varepsilon - j\omega\sigma)\mathbf{N}_i\cdot\mathbf{N}_m\right]d\Omega}_{a_{im}} = \underbrace{-\int_\Omega j\omega\mathbf{N}_i\cdot\mathbf{J}_S d\Omega}_{b_i}$$

The basis functions must be such that the integrals can be evaluated, and we proceed to construct the *Edge* elements on triangles.

Linear Edge elements on triangles

The basis functions will be polynomials inside each triangle. curl $\mathbf{E}_h$ must have no delta-functions anywhere, i.e. across the triangle edges, and this means that the *tangential* component along the edges must be continuous, but the normal component can be discontinuous. Also, the basis functions should have small support. The edges can



provide directions, so we associate a basis function with each triangle edge, non-zero over the two triangles which share the edge. Here is the construction:
Take 1 to be the origin of a polar coordinate system.

and choose $\mathbf{N} = \frac{r}{h}\mathbf{e}_\phi$. Then, the tangential component along 3-2 becomes $\frac{r}{h}\cos\phi = \frac{r}{h}\cdot\frac{h}{r} = 1$.
The same construction over the neighbor triangle across 2-3 gives the same, only we need to keep track of the orientation of the edges, i.e., the triangles must be oriented. Outside the two triangles, **N** = 0, so has tangential component 0 along edges 1-2 and 1-3. That gives the correct continuity because $\mathbf{e}_\phi$ is orthogonal to radius vector. **N** is a first degree polynomial in (*x*,*y*):

$$\mathbf{N} = \frac{r}{h}\mathbf{e}_\phi = \frac{r}{h}\cdot\frac{(-y,x)}{r} = \frac{(-y,x)}{h}; r = \sqrt{x^2+y^2} \quad \text{so} \quad \nabla\times\mathbf{N} = \frac{2}{h}\mathbf{e}_z$$

The $\mathbf{N}_k$ can be constructed from the nodal P1 elements above: Label the edge basis function *i* if associated with the edge across from corner *i*. Then

$$\mathbf{N}_1 = \varphi_2\nabla\varphi_3 - \varphi_3\nabla\varphi_2$$
$$\mathbf{N}_2 = \varphi_3\nabla\varphi_1 - \varphi_1\nabla\varphi_3$$
$$\mathbf{N}_3 = \varphi_1\nabla\varphi_2 - \varphi_2\nabla\varphi_1$$

see book p. xxx (another labeling scheme).

The Null-space of **A**.

The book solves an eigenvalue problem for the interior of a metallic cylinder: To find the eigenfrequencies ω (wave numbers $k = \omega/c$) for

$$\nabla\times(\nabla\times\mathbf{E}) - k^2\mathbf{E} = 0$$

which after application of the Galerkin discretization becomes

$$\sum_{m=1}^{N} E_m \underbrace{\int_\Omega (\nabla\times\mathbf{N}_i)\cdot(\nabla\times\mathbf{N}_m)d\Omega}_{a_{im}} = k^2 \sum_{m=1}^{N} E_m \underbrace{\int_\Omega \mathbf{N}_i\cdot\mathbf{N}_m d\Omega}_{b_{im}}$$

the generalized eigenvalue problem

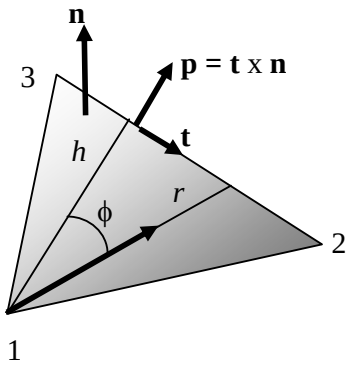
$$\mathbf{A}\cdot\mathbf{E} = k^2\mathbf{B}\cdot\mathbf{E}$$

Note that **A** has a very large null-space: the curl operator annihilates *any* gradient! Contrast this with the Laplace operator, with homogeneous boundary conditions which has only one zero eigenvalue which has a constant as eigenfunction.
For these elements,

$$\dim(\text{nullspace}(\mathbf{A})) = N_{edges} - N_{ele} = N_{nodes} - N_{holes} - 1$$

where N_{edges} is the number of edges, N_{ele} the number of triangles, N_{nodes} is the number of nodes (triangle corners) and N_{holes} is the number of holes in the mesh.

EFIE by MoM: Elements and computational complexity



We now return to the Method of Moments solution of the EFIE.

RGW div-conforming elements

The surface S is tessellated by N_T plane triangles T_k, $S = \widetilde{S} \approx \bigcup_k T_k$. The Rao-Glisson-Wilton elements are piecewise linear vector functions over each *T*. The continuity requirement is that the divergence have no delta-functions, so the normal component must be continuous across triangle edges: such elements are called *divergence*-conforming. The scalar 2D elements are "*gradient*" conforming, and the edge elements are *curl*-conforming. Note that neighboring facets on S are *not* co-planar. The normal is discontinuous across triangle edges, so we are looking for "*surface* divergence"-conforming elements.
Here is the construction:
n is the unit surface normal to the facet, and **t** is the unit tangent vector along the edge 3-2, and **p** = **t** x **n** is the in-surface normal to the edge. The radius vector from 1 is called **r**1. Then, the basis function associated with edge 3-2 is

$$\mathbf{s}_{3\text{-}2} = \mathbf{r}1/h.$$

over the triangle shown. Its in-facet normal component is $|\mathbf{r}1 \cdot \mathbf{p}| = r\cos\phi / h = 1$. The same is obtained for the neighbor element across edge 3-2, and the normal component at 1-2 and 1-3 vanishes. This is as it should, since the basis function is 0 outside the triangle and its neighbor across 3-2.

Complexity

The Galerkin scheme requires double surface integration and can be suspected to be computationally demanding. It is, but with *Ne* elements the complexity, as we see below, is no worse than $O(N_e^2)$ for computing the $N_e x N_e$ full matrix **A**, and this is dominated by the cost of solving the system for sufficiently large *N*.
Choosing the Galerking recipe to take weight functions as the basis functions $\mathbf{s}_k$, the ingredients are (modulo a few factors $\omega\,\varepsilon$ etc.),

$$a^1_{jk} = \int_{S\times S} G(R)\mathbf{s}_j(\mathbf{r})\cdot\mathbf{s}_k(\mathbf{r}')dS'\,dS$$

$$a^2_{jk} = \int_{S\times S} G(R)(\nabla_t\cdot\mathbf{s}_j)(\mathbf{r})(\nabla'_{t.}\cdot\mathbf{s}_k(\mathbf{r}'))dS'\,dS$$

G has an 1/*R* integrable singularity which must be integrated accurately. 1/*R* can be integrated by analytic means over triangles and this can be used together with numerical quadrature.
Assume we use *Q* points $\mathbf{r}_{k\alpha}$, $\alpha = 1,2,\ldots,Q$, weights W_α, over triangle *k*:

$$a^1_{jk} = \sum_{ml\alpha\beta} W_\alpha W_\beta A_m A_l G(|\mathbf{r}_{m\alpha} - \mathbf{r}'_{l\beta}|)\mathbf{s}_j(\mathbf{r}_{m\alpha})\cdot\mathbf{s}_k(\mathbf{r}'_{l\beta})$$

In all, N_e^2 sums, each with $N_e^2 Q^2$ terms, a seemingly $O(N_e^4)$ computation. But the sum is over triangles, and the basis functions will be chosen to have small support so $\mathbf{s}_k$ is non-zero only

for a few *l*. For the RGW elements described below, there is one **s** for each edge. Let *K*(*j*) be set of elements where $\mathbf{s}_j$ is nonzero: for RGW, the two sharing edge *j*. So,

$$a^1_{jk} = \sum_{\substack{m\in K(j),l\in K(k)\\ \alpha,\beta}} W_\alpha W_\beta A_m A_l G(|\mathbf{r}_{m\alpha} - \mathbf{r}'_{l\beta}|)\mathbf{s}_j(\mathbf{r}_{m\alpha})\cdot\mathbf{s}_k(\mathbf{r}'_{l\beta})$$

for each *j*,*k* only 2 x 2 x 3 x 3 = 36 terms.