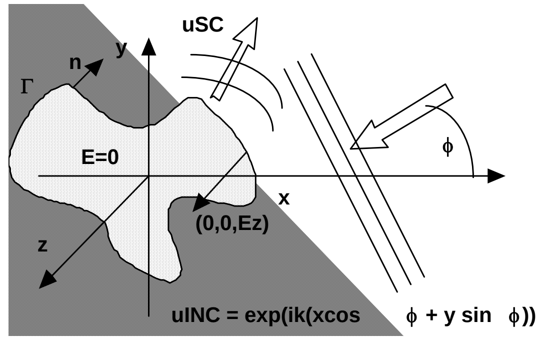
Scattering of plane wave on metallic circular cylinder

We consider time-harmonic TM-waves of angular frequency ω in the plane. The only non-zero E-component is E_z which we call *u*, and it satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\varepsilon\mu}$$

in the exterior domain.



On Γ, the tangential E-field is zero, i.e., the sum of the incoming field and the scattered field vanishes,

$$u^{INC} + u^{SC} = 0$$

Any solution of the Helmholtz equation can be represented by an integral over *Γ*,

$$u^{SC}(x) = \int_\Gamma \left(G(x,x')\sigma(x') - \frac{\partial G(x,x')}{\partial n'}\gamma(x') \right) ds$$

where *G* is the Green's function, satisfying

$$\Delta_x G(y,x) + k^2 G(y,x) = \delta(x-y)$$

with δ the Dirac delta-function. σ and γ are single layer and double layer sources, viz., on Γ. For the exterior wave problem, *G* must be an outgoing wave at infinity, and thus must be the zeroth order Hankel function of the second kind,

$$G(y,x) = \frac{1}{4i} H_0^2\left(k\,|\,y-x\,|\right)$$

Note that *G* is a function only of the difference $y - x$ and that it has a logarithmic singularity at $x - y = 0$.

Consider now the integral evaluated for points e and i just outside and just inside Γ. One can show that the jumps in function value [uSC] and normal derivative [∂uSC/∂n] are

$$uSC(e) – uSC(i) = \gamma,\ \partial uSC/\partial n(e) - \partial uSC/\partial n(i) = \sigma$$

Define the scattered field to be continuous across Γ. This is possible whenever the interior Helmholtz problem with Dirichlet condition has a unique solution. This, in turn, holds whenever $-k^2$ is NOT an eigenvalue of the Laplace operator inside Γ, i.e., for all but a number of discrete values of k^2. Then $\gamma = 0$ and the final integral equation for determining σ becomes

$$u^{INC}(x) = \int_\Gamma \sigma(x')G(x,x')ds$$

This is a "Fredholm integral equation of the first kind" with kernel *G*. First kind equations with smooth kernels are often ill-posed in the sense that short wavelength perturbations to σ are smoothed by the integration. The converse of this statement is that short wavelength components of the LHS are strongly magnified. Such problems have to be regularized by filtering out short wavelength noise.

However, our kernel *G* is (weakly) singular and $\sigma(x)$ contributes strongly to $u^{INC}(x)$. The problem of determining σ from u^{INC} is reasonably well conditioned.

There remains to discretize the integral equation to produce a finite linear system of equations. We will use the collocation method which proceeds by approximating σ by a linear combination of a number of selected basis functions,

$$\sigma(y) \approx \sum_{k=1}^{n} \sigma_k f_k(y)$$

selecting a number of field points z_k just outside Γ and requiring that the equation be satisfied exactly at these points:

$$b_m := u^{INC}(z_m) = \sum_{k=1}^{n} \sigma_k \int_\Gamma f_k(z')G(z_m,z')ds, z_m \in \Gamma(e), m = 1,2,...,M$$

or

$$\mathbf{As} = \mathbf{b}\ ,\ a_{km} = \int_\Gamma f_k(z')G(z_m,z')ds$$

where M is usually chosen = n but may also be taken > n to provide some over-determination in ill-conditioned cases.

The simplest basis functions are constructed by replacing the curve Γ by a polygon with vertices z'_k, edges $\Delta z_k = z'_{k+1} - z'_k$. We take $f_k = 1$ over edge k, 0 elsewhere, the "square pulse" basis functions which give a staircase representation of $\sigma(x)$. z_k are usually chosen as the midpoints of the edges,

$$z_k = 1/2(z'_{k+1} - z'_k)$$

The integrals are evaluated exactly, if possible, or by numerical quadrature. The simplest scheme is to use a one-point rule for all integrals except for $k = m$, the self-contribution of element m, which becomes the diagonal element of the coefficient matrix A.

This is a logarithmic singularity and we choose to use only the first few terms in the expansion of G around z_m: (from e.g. Maple or Mathematics Handbook)

$$H_0^2(z) = J_0(z) - iY_0(z) \approx 1 - i\left(\frac{2}{\pi}\{\ln\left(\frac{1}{2}z\right) + \gamma\}\right) + O(z^2 \ln z), \gamma = 0.5772156649...$$

$$G(z) = \frac{1}{4i}H_0^2(z) \approx \frac{1}{4i} - \frac{1}{2\pi}(\gamma - \ln 2) - \frac{1}{2\pi}\ln z$$

Note the last term: This is the Green's function for the Laplace operator.

Field computation for the Helmholtz equation in 2D

We have seen that the field may be written

$$E(z) = \int_\Gamma \left(G(|z - z'|)\sigma(z') - \gamma(z')\frac{\partial G(|z - z'|)}{\partial n'} \right) ds$$

where the Green's function is $G(r) = \frac{1}{4i}H_0^2(kr)$, the zeroth order Hankel function of the second kind. Primed quantities refer to the curve, **n**' is the normal to the curve Γ and ds is the arc element. The coordinates are represented as complex numbers, $z = x + iy$, etc.

Γ is approximated by a polygon with vertices z'_i, i = 1,…, M, and the field points are z_i, i = 1,…, n. Using the fact that G is a function of k times the length |z - z'| only we may write

$$\gamma \frac{\partial G}{\partial n'} ds = \nabla' G \bullet \mathbf{n}' \gamma ds = -kG'(k|z-z'|)\frac{z-z'}{|z-z'|} \bullet \gamma \frac{i\Delta z'}{|\Delta z'|} ds =$$

$$= kG'(k|z-z'|)\gamma \frac{\mathrm{Re}((z-z')\, i\overline{\Delta z'})}{|z-z'|}$$

where the normal is i times tangent and the scalar product can be expressed
<z1,z2> = Re(z1 conj(z2))

Let the midpoints of the polygon edges be $z'_{i+1/2}$ and $d_i = z - z'_{i+1/2}$., $\Delta_j = z'_{i+1} - z'_i$. The final formula becomes

$$E(z) = \sum_{j=1}^{n} \sigma_j G(k\,|\,d_j\,|)\left|\Delta_j\right| + \gamma_j kG'(k\,|\,d_j\,|)\frac{\mathrm{Re}(id_j\overline{\Delta}_j)}{|\,d_j\,|}$$

This is implemented by the m-file below, vectorized to compute the field on a mesh of *m* x *n* z-points at once. Note that

Exercise

1. Compute σ and γ (as $\partial u/\partial n$ and u) for u^{SC} = a plane wave e^{ikx} on a closed curve of your choice. Plot the field from σ and γ inside and outside the circle. Explain.

2. The polygon edges $\Delta_j = z'_{i+1} - z'_i$ must be small enough to resolve the wavelength. Compare the fields computed in 1. with 5, 10, and 20 elements per wavelength
3. The code is completely vectorized (no loops) but needs memory `ns*m*n` which easily becomes huge. Rewrite the code to use only a given amount of memory by cutting the set of z-points into reasonable size chunks, with a single loop over the chunks. The code is also wasteful in allocating space both to `d`, `G`, and `Gder`. At least one can be discarded without speed penalty. Fix that too.

```
function E = field(z,zprime,gamma,sigma,k)
% computes the field at the points z(1:np) (2D: z(p) = x(p) + i y(p))
% from the
% single layer  sigma(1:ns-1) and
% double layer  gamma(1:ns-1)
% on the curve  zprime(1:ns).
% Greens function for the Helmholtz (Laplace if k = 0) equation
% (delsq + k^2)u = ...
% and all of R2 i.e. H2,0(k|z - zprime|) viz. 1/(2pi) ln (|z-zprime|)
%r
[m,n]  = size(z);
ns     = length(zprime);
np     = m*n;
z      = z(:);                                    % make columns
gamma  = gamma(:);
sigma  = sigma(:);
zprime = zprime(:);

delta  = diff(zprime);                           % edges
zphalf = 0.5*(zprime(1:ns-1) + zprime(2:ns));    % midpoints
d      = z*ones(1,ns-1) - ones(np,1)*(zphalf.'); % all distance vectors
                                                 % at once
G      = Greenfunc(k,d);                         % single layer Green
Gn     = k*Greenfuncder(k,d);                    % double layer Green
sig    = sigma.*abs(delta);                      % length element
E      = reshape(G*sig + (Gn.*real(i*d./abs(d)*diag(conj(delta))))*gamma,m,n);
function G = greenfunc(k,r)
if k == 0 % Laplace
            G = 1/2/pi * log(abs(r));
else
            G = -0.25*i*besselh(0,2,k*abs(r));
end
function G = greenfuncder(k,r)
if k == 0 % Laplace
            G = (1/2/pi)./abs(r);
```

```
else
            G = 0.25*i*besselh(1,2,k*abs(r));
end
```

Exact solutions

The code in the lab will work for any curve defined by the *z'*-points. For a circle of radius R we can compute the exact solution by Fourier expansion.

1. PEC cylinder radius R

The exterior field is

$$u^{SC}(r,\phi) = \sum_{m=-\infty}^{+\infty} c_m e^{im\phi} H_m^2(kr)$$

and the incoming plane wave may be written

$$u^{INC}(R,\phi) = e^{ikR\cos\phi} = \sum_{m=-\infty}^{+\infty} i^m J_m(kR) e^{im\phi}$$

so we obtain from $u^{\mathrm{SC}} + u^{\mathrm{INC}} = 0$ on $r = R$

$$u^{SC}(r,\phi) = -\sum_{m=-\infty}^{+\infty} i^m J_m(kR) e^{im\phi} H_m^2(kr)$$

2. Dielectric cylinder

The fields inside and outside are

$$u_i^{SC} = \sum_{m=-\infty}^{+\infty} a_m e^{im\phi} J_m(k_i r),\ \ u_e^{SC} = \sum_{m=-\infty}^{+\infty} b_m e^{im\phi} H_m^2(k_e r), \quad u_e = u_e^{SC} + e^{ik_e x}$$

(why only J_m inside? and only $H^2{}_m$ outside?). Matching of Fourier components of u and $\partial u/\partial n$ across $r = R$ gives a system of equations for the a_{m}, b_{m}.

Scattering of plane wave on dielectric cylinder

We assume that the cylinder has zero conductivity thus supporting no surface currents. The field inside is no longer zero, and we have continuity of tangential E field

$$[E] = 0.$$

Similarly, the tangential component of **H** must be continuous since there is no surface current. For the TM time harmonic wave

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H} : H_x = \frac{i}{\omega\mu}\frac{\partial E}{\partial y}, H_y = -\frac{i}{\omega\mu}\frac{\partial E}{\partial x}$$

$$\mathbf{n} \times \mathbf{H} = -\frac{i}{\omega\mu}\left(n_x \frac{\partial E}{\partial x} + n_y \frac{\partial E}{\partial y}\right)\mathbf{e}_z = -\frac{i}{\omega\mu}\frac{\partial E}{\partial n}\mathbf{e}_z$$

and thus

$$[1/\mu\ \partial E/\partial n] = 0.$$

The representation of the fields in the exterior and interior are

$$u_e^{SC} = \int_\Gamma \left(G_e \sigma_e - \frac{\partial G_e}{\partial n'} \gamma_e \right) ds$$

$$u_i^{SC} = \int_\Gamma \left(G_i \sigma_i - \frac{\partial G_i}{\partial n'} \gamma_i \right) ds$$

where subscript e refers to exterior, i to interior. The Green's functions differ only in the wave numbers,

$$k_e = \frac{2\pi}{\lambda_e} = \frac{\omega}{c_e} = \omega\sqrt{\varepsilon_e \mu_e}, k_i = \sqrt{\frac{\varepsilon_i \mu_i}{\varepsilon_e \mu_e}} k_e$$

Now define the scattered exterior field to vanish in the interior, and vice versa. It follows from the jump properties across Γ

$$\gamma_e = u_e^{SC}, \sigma_e = \frac{\partial u_e^{SC}}{\partial n'}, -\gamma_i = u_i^{SC}, -\sigma_i = \frac{\partial u_i^{SC}}{\partial n'}$$

and from the continuity of E and ∂E/∂n

$$u_e^{SC}(e) + u^{INC}(e) = u_i^{SC}(i) : \gamma_e + u^{INC}(e) = -\gamma_i$$

$$\frac{\partial u_e^{SC}}{\partial n'}(e) + \frac{\partial u^{INC}}{\partial n'}(e) = \frac{\partial u_i^{SC}}{\partial n'}(i) : \sigma_e + \frac{\partial u^{INC}}{\partial n'}(e) = -\sigma_i$$

The pair of integral equations can now be expressed with σ_i and γ_i as unknowns.

u_i vanishes in the exterior:

$$0 = \int_\Gamma \left(G_i(z,z')\sigma_i - H_i(z,z')\gamma_i \right) ds, z \in \Gamma(e)$$

where H_i has been introduced for ∂/∂n'(G_i). The vanishing of the scattered field in the interior means

$$0 = \int_\Gamma \left[G_e(z,z')(-\sigma_i - \frac{\partial u^{INC}}{\partial n'}(z')) - H_e(z,z')(-\gamma_i - u^{INC}(z')) \right] ds$$

Now u^{INC} satisfies the exterior equation, and hence the terms in u^{INC} become simply $u^{INC}(z)$. There follows

$$u^{INC}(z) = \int_\Gamma \left(G_e(z,z')\sigma_i(z') - H_e(z,z')\gamma_i(z') \right) ds, z \in \Gamma(i)$$

The pair of integral equations are discretized exactly as in the PEC cylinder case. Note that the first has $G(z(e))$ and the second $G(z(i))$ which translates to a sign change of the diagonal elements of the matrices because only the $\partial G/\partial n'$ terms are singular.

3D Electric and magnetic field integral equations.

RGW elements for current density on surface.

Electrically large objects.