Integral equation methods

Book p 154 – 198

Introduction - electrostatics

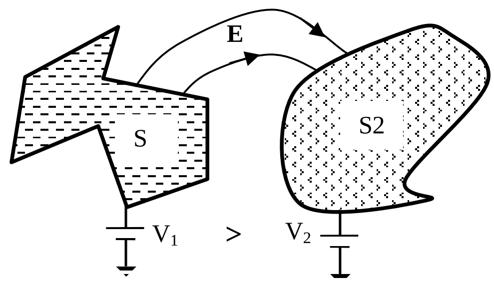
Let us start by discussing a "Newtonian" view of electromagnetics – focusing on the charges, the sources of the fields, and then compare it with the "Laplacian" PDE view. The Coulomb force between two charges Q_i and Q_j at $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\mathbf{F}_{ij} = \frac{1}{4\pi\varepsilon} \cdot \frac{Q_i Q_j}{R_{ij}^3}(\mathbf{x}_i - \mathbf{x}_j), R_{ij} = \left|\mathbf{x}_i - \mathbf{x}_j\right|$$

4π is the solid angle and ε is the dielectric constant of the medium. With n charges, the force on Q_i is

$$\mathbf{F}_i = \frac{Q_i}{4\pi\varepsilon} \cdot \sum_{j \neq i} \frac{Q_j}{R_{ij}^3}(\mathbf{x}_i - \mathbf{x}_j) = Q_i \cdot \underbrace{-\nabla V(\mathbf{x}_i)}_{\mathbf{E}(\mathbf{x}_i)},$$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon} \sum_{j \neq i} \frac{Q_j}{\left|\mathbf{x} - \mathbf{x}_j\right|}$$



The "electrostatic problem" calls for determination of the electric field created by a number of conductors S_i, with given potentials V_i . Only the surfaces carry charges, which will rearrange themselves on the surfaces, driven by the forces. One might try solving the many-body initial value problem

$$m_i \cdot \frac{d^2\mathbf{x}_i}{dt^2} + D\frac{d\mathbf{x}_i}{dt} = Q_i \mathbf{E}_t, i = 1,2,...,n$$

where $\mathbf{E}_t$ is the tangential component of the **E**-field acting on point i, computed from the positions of all the charges. The damping D is necessary; without it, the system would oscillate for ever. Equilibrium obtains when the net force (proportional to **E**) is orthogonal to the conducting surface. Since **E** is the gradient of V, the tangential component of the gradient of V vanishes so V becomes constant on the surface.

The total charge on each body is determined by the initial data. The potential is computable by integration along the **E**-field lines from "infinity", but there is no easy way to determine what the charge on each body should be to produce a desired potential.

In the Laplacian description, one solves the partial differential equation satisfied by V:

$$\Delta V(\mathbf{x}) = 0$$

except at $\mathbf{x} = \mathbf{x}_i$. Surrounding $\mathbf{x}_i$ by a sphere S_δ of radius δ, the formula for the field from a point charge gives

$$\int_{S_\delta} \varepsilon\nabla V \cdot \hat{n}dS = \frac{Q_i}{4\pi} \iint_{\theta,\phi} \frac{\partial}{\partial r}\left(\frac{1}{r}\right)_{r=\delta} \delta d\theta \cdot \delta \sin\theta d\phi = \frac{Q_i}{4\pi} \iint_{\theta,\phi} \frac{1}{\delta^2} \delta d\theta \cdot \delta \sin\theta d\phi = Q_i$$

By superposition, we can write

$$\varepsilon\Delta V(\mathbf{x}) = \sum_{j=1}^{n} Q_j \cdot \delta(\mathbf{x} - \mathbf{x}_j)$$

where δ is the Dirac delta-function. For a continuous charge distribution $\rho(\mathbf{x})$ this becomes

$$\varepsilon\Delta V(\mathbf{x}) = \int \rho(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')dV = \rho(\mathbf{x})$$

whereas the "Newtonian" description is

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{4\pi\varepsilon\left|\mathbf{x} - \mathbf{x}'\right|} dV'$$

The solution G of

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'), G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\left|\mathbf{x} - \mathbf{x}'\right|}$$

(the subscript on the differential operator shows differentiation w.r.t **x**, not **x**') is called the Green's function for the free space Laplace operator with boundary condition $G = 0$ at infinity. Note that:

- the differential equation has constant coefficients, so G depends only on the difference $\mathbf{x} - \mathbf{x}'$;
- the isotropy of the differential operator (actually, rotational invariance) makes G a function only of the distance $|\mathbf{x} - \mathbf{x}'|$

For the electrostatic problem with given conductor potentials, the charge is a surface charge σ and we have

$$V(\mathbf{x}) = \int_S \sigma(x')G(\mathbf{x},\mathbf{x}')dS, \quad S = \bigcup S_i$$

$$V_k = \int_S \sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}')dS, \mathbf{x} \in S_k, k = 1,2,\ldots,n_{cond}.$$

a "first kind Fredholm" integral equation for the unknown σ. We know from potential theory that the Laplace/Poisson problem has a unique solution. Since we can calculate σ from the normal component of the **E**-field,

$$\sigma = \varepsilon \frac{\partial V}{\partial n}$$

we expect the integral equation to have a unique solution too. But First Kind equations are known for their ill-conditioning. Consider solving

$$f(x) = \int_I u(y)K(x,y)dy$$

for u. If the kernel function K is smooth, the integral operator smooths short wavelength variations in u. The converse is that rapid variation in f, such as e.g. measurement noise, is magnified in u. If the magnification is NOT uniformly bounded, such a problem is called ill-posed and requires filtering, *regularization*. However, the point charge kernel function has an integrable $1/r$-type singularity for **x** close to **x**'– not so smoothing, yet nice enough. In 2D, the Green's function is $-\frac{1}{2\pi}\ln(\mathbf{x} - \mathbf{x}')$ and the story is similar.

Discretization, etc., and the plate capacitor example, see the book.

Notes

The $O(h)$ error observed in the computed capacitance is not obvious: The exact solution has an $r^{-1/2}$ type singularity at the plate ends. Thus, the numerical solution cannot converge uniformly pointwise (but it can in l_2 – norm). The convergence is very regular, as the plot shows. One can improve the results by Richardson extrapolation:

$$\left.\begin{aligned} C(h) &= C + Kh + O(h^2) \\ C(2h) &= C + 2Kh + O(h^2) \end{aligned}\right\} \Rightarrow C = C(h) + (C(h) - C(2h)) + O(h^2)$$

The extrapolated value from 10 and 20 elements is 18.71 which has only 0.1% error and is much better than the result with 200 elements.

Scattering of TM_z waves from perfectly conducting objects.

See the notes LectMoM_08.

The development above indicates that any electrostatic field between conductors can be produced by some charge distribution on their surfaces. So maybe *any* solution to the Laplace equation in a closed domain D can, too? The answer is yes, but one needs both a single-layer charge σ and a layer of *dipoles*, say γ. The argument runs as follows: Consider a modified domain D', equal to D excluding a small sphere S_δ around a point **x** and a tube connecting S_δ to the boundary of D. The surface of D' is S'. Let u be a solution to the Laplace equation in D, and $v(\mathbf{x}') = G(\mathbf{x},\mathbf{x}')$.

Then

$$0=\int_{D'}(u\Delta v-v\Delta u)dV=\int_{S'}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS=$$

$$\int_{S'-S_\delta}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS+\underbrace{\int_{S_\delta}u\frac{1}{4\pi\delta^2}\delta^2\sin\theta d\theta d\phi}_{\to u(x)}-\underbrace{\int_{S_\delta}\frac{1}{4\pi\delta}\cdot\frac{\partial u}{\partial n}\delta^2\sin\theta d\theta d\phi}_{\to 0}$$

as δ -> 0. Finally, we obtain the representation

$$u(\mathbf{x})=\int_S\left(u(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}-G(\mathbf{x},\mathbf{x}')\frac{\partial u}{\partial n}(\mathbf{x}')\right)dS$$

The same representation is valid for solutions to the Helmholtz equation, for which the Green's function in 3D is

$$G(\mathbf{x}.\mathbf{x}')=\frac{1}{4\pi}\frac{e^{-ikR}}{R},\, R=|\mathbf{x}-\mathbf{x}'|$$

(see notes for the 2D formula).

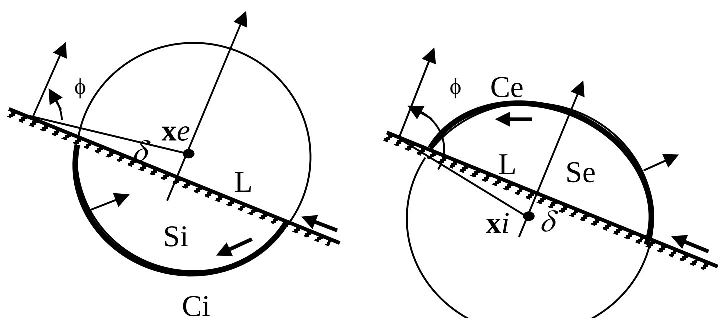
The formula

$$u(\mathbf{x})=\int_S\left(\sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}')-\gamma(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right)dS \qquad (*)$$

defines a solution to the Helmholtz equation both inside and outside *S*. Let the exterior to *S* be *e* and the interior *i*. Due to the singularity of *G* and its derivatives, *u* and its normal derivative jump across *S*:

$$u(e)-u(i)=\gamma, \partial u/\partial n(e)-\partial u/\partial n(i)=\sigma \qquad (**)$$

Here is a proof of the first jump relation, for ease of illustration, for the Laplace operator in 2D. Assume that *u* is continuously differentiable everywhere. Potential theory guarantees that it will be more than that, actually analytic, except possibly on the boundary at corners, etc. The exterior viz. interior points **x**e and **x**i, at distance δ from *S*, are surrounded by circular disks of radius *a*.



As δ vanishes, $\int_S\sigma(x')G(xe,x')dS$ and $\int_S\sigma(x')G(xi,x')dS$ converge to the same limit so σ does not contribute to the jump in *u* (but for *du*/*dn* it does). For the γ-term, we need the expression for *dG*/*dn*:

$$v=\partial G/\partial n'=\nabla G\cdot\mathbf{n}'=dG/dR(\nabla R\cdot\mathbf{n}')=\frac{1}{2\pi R}\cos\phi$$

For **x**e,

$$u(xe)=\int_S v_e(x')u(x')ds=\int_{S-L}v_e(x')u(x')dS+\int_L v_e(x')(u(x')-u(x^*))dS+u(x^*)\int_L v_e(x')ds=$$

$$=\int_{S-L}v_e(x')u(x')dS+\underbrace{\int_L v_e(x')(u(x')-u(x^*))dS}_{Q_e,|Q_e|\le Ka}+u(x^*)\underbrace{\int_{Ci}v_e(x')ds}_{-\alpha_e(a,\delta)/2\pi}$$

where α_e is the subtended angle from **x**e to the intersection of the circle with *S*. The Gauss theorem was used:

$$0=\int_{Si}\Delta GdS=-\int_{Ci}\partial G/\partial n'dS+\int_L\partial G/\partial n'dS;\int_{Ci}v(x')dS=\int_L v(x')dS$$

For **x**i

$$u(xi) = \int_S v_i(x')u(x')dS = \int_{S-L} v_i(x')u(x')dS + \underbrace{\int_L v_i(x')(u(x') - u(x^*))dS}_{Q_i,|Q_i| \le Ka} + u(x^*) \underbrace{\int_{Ce} v_i(x')ds}_{+\alpha_i(a,\delta)/2\pi}$$

Note the different sign on α! There follows

$$u(xe) - u(xi) = \int_S (v_e(x') - v_i(x'))u(x')dS + Q_e - Q_i + (-\alpha_e - \alpha_i), |Q| \le Ka$$

and as δ and *a* vanish, v_e and v_i approach a common limit so the integral vanishes because *u* is bounded, and the sum of α_e and α_i approach 2π, which ends the demonstration. (as usual, modulo the sign +/-)