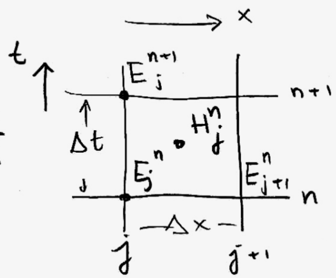
## von Neumann analysis of Yee – 1D scheme: System

(1)




$$\begin{cases} \varepsilon E_t = H_x \\ \mu H_t = E_x \end{cases} \qquad C = \frac{1}{\sqrt{\varepsilon\mu}}, \quad Z = \sqrt{\frac{\mu}{\varepsilon}}$$

Yee:
$$E_j^{n+1} - E_j^n = \overbrace{\frac{\Delta t}{\varepsilon \Delta x}}^{\sigma Z}\left(H_j^n - H_{j-1}^n\right)$$
$$H_j^{n+1} - H_j^n = \underbrace{\frac{\Delta t}{\mu \Delta x}}_{\frac{\sigma}{Z}}\left(E_{j+1}^{n+1} - E_j^{n+1}\right) \qquad \sigma = C\frac{\Delta t}{\Delta x} \text{ ((Courant-\#)}$$

Wave-ansatz (separation of variables, $k$ = wavenumber)

$$E_j^n = \hat E_n e^{ijk\Delta x}, \quad H_j^n = \hat H_n \cdot e^{ijk\Delta x} \cdot \underline{\underline{e^{\frac{ik\Delta x}{2}}}}$$ Staggered grid!

Introduce $\theta = k\Delta x$ = phaseshift per cell

$$\hat E_{n+1} = \hat E_n + \sigma Z\, 2i\sin\frac{\theta}{2}\,\hat H_n$$
$$\hat H_{n+1} = \hat H_n + \frac{\sigma}{Z}\, 2i\sin\frac{\theta}{2}\,\hat E_{(n+1)}$$ note!

So
$$\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_{n+1} = G\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_n, \quad G = \underbrace{\begin{pmatrix} 1 & \sigma Z 2i\sin\frac{\theta}{2} \\ \frac{\sigma}{Z}2i\sin\frac{\theta}{2} & 1-4\sigma^2\sin^2\frac{\theta}{2}\end{pmatrix}}_{\text{von Neumann amplification factor (matrix)}}$$

Scheme is stable, if
$$\left\|\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_n\right\| \le e^{Kt_n}\left\|\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_0\right\| \text{ for all } \theta,$$
and $K$ independent of $\sigma$ if $\sigma < \sigma^*$

- Allows exponential growth with t, but not explosion as $n \to \infty$
- For waves $K = 0$ is desirable ⊛

$$\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_n = G^n\begin{pmatrix}\hat E\\ \hat H\end{pmatrix}_0,$$ so |largest eigenvalue of $G$| $\le 1$ $\Rightarrow$ stability

(2)

Unfortunately expressions of eigenvalues complicated:

$$\lambda_\pm = 1 - 2\sigma^2\sin^2\frac{\theta}{2} \pm 2\sin\frac{\theta}{2}\cdot\sigma\sqrt{\sin^2\frac{\theta}{2}\cdot\sigma^2 - 1}$$

- if $\left|\sin\frac{\theta}{2}\sigma\right| > 1$ $\quad |\lambda_-| > 1$ : Unstable
- if $\left|\sin\frac{\theta}{2}\sigma\right| \le 1$

$$\lambda_\pm = 1 - 2\sigma^2\sin^2\frac{\theta}{2} \pm i\,2\sigma\sin\frac{\theta}{2}\sqrt{1-\sin^2\frac{\theta}{2}\,\sigma^2}$$ stable

and $|\lambda_\pm| = 1$. Indeed, set $\sin\frac{\alpha}{2} = \sin\frac{\theta}{2}\cdot\sigma$

then $\boxed{\lambda_\pm = e^{\pm i\alpha}}$

Now, the waves are computed by diagonalization of $G$, (as for $\underline{u}_t + A\underline{u}_x = 0$)

Right moving wave
$$\begin{pmatrix}E_j^n\\ H_j^n\end{pmatrix} = e^{-in\alpha + ikj\Delta x}\cdot \underline{v}_-$$

$\underline{v}_-$ eigenvector of $G$ corresp. to $e^{-i\alpha}$

Compare with $e^{i(\omega t_n - kx_j)}$ : $\omega n\Delta t = n\alpha$

So $\omega = \frac{\alpha}{\Delta t}$

Numerical wave (phase) speed $\frac{\omega}{k}$

$$= \frac{\alpha}{k\Delta t} = \frac{2\arcsin\left(\frac{C\Delta t}{\Delta x}\sin\frac{k\Delta x}{2}\right)}{k\Delta t}$$

for small $k\Delta x$, $$\approx \frac{2\,\frac{C\Delta t}{\Delta x}\cdot\frac{k\Delta x}{2}}{k\Delta t} \approx C$$