Advanced Computation in Fluid Mechanics
Problem sheet

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February 10, 2011

Consider the Euler equations

$$\dot{\rho} + \nabla \cdot (u\rho) = 0 \qquad \text{(mass conservation)} \tag{1}$$

$$\dot{m} + \nabla \cdot (um) = -\nabla p \qquad \text{(momentum - Newton 2nd law)} \tag{2}$$

$$\dot{\theta} + \nabla \cdot (u\theta) = -p\nabla \cdot u \qquad \text{(internal energy } \theta\text{)} \tag{3}$$

for $m = \rho u$ momentum, ρ density, $u = (u_1, u_2, u_3)$ the velocity vector, θ internal energy, and p pressure, together with an additional law specifying the pressure, for example:

$$p = (\gamma - 1)\theta \qquad \text{(state equation for a perfect gas)} \tag{4}$$

$$\nabla \cdot u = 0 \qquad \text{(incompressible flow)} \tag{5}$$

Assume there exists a solution $\hat{u} = (\rho, m, \theta)$ to the Euler equations for all $(x,t) \in Q = \Omega \times I$, where $\Omega \in \mathbb{R}^3$ and $I = (0,T]$ is a time interval, with initial conditions $\hat{u}(\cdot, 0) = \hat{u}^0$ and a (slip) boundary condition $u \cdot n = 0$ for all $x \in \Gamma$, with outward unit normal n on the boundary Γ.

Problem 1: Without using (4)-(5), show that

$$\dot{m} + \nabla \cdot (um) = \rho(\dot{u} + (u \cdot \nabla)u) \tag{6}$$

$$\dot{k} + \nabla \cdot (uk) = -\nabla p \cdot u \quad \text{(with kinetic energy } k = \rho|u|^2/2\text{)} \tag{7}$$

$$\dot{e} + \nabla \cdot (ue) = -\nabla \cdot (pu) \quad \text{(with total energy } e = k + \theta\text{)} \tag{8}$$

$$\frac{d}{dt}\int_\Omega \rho \, dx = \frac{d}{dt}\int_\Omega e \, dx = 0 \qquad \frac{d}{dt}\int_\Omega m \, dx = -\int_{\partial\Omega} pn \, ds \tag{9}$$

$$\dot{K} = -\dot{\Theta} = W, \quad \text{with } K = \int_\Omega k \, dx, \ \Theta = \int_\Omega \theta \, dx, \ W = \int_\Omega p\nabla \cdot u \, dx \tag{10}$$

For incompressible flow ($\nabla \cdot u = 0$) the work $W = 0$, and thus kinetic energy and internal energy are conserved $\dot{K} = \dot{\Theta} = 0$. Also, for incompressible flow the mass and momentum equations decouple from the energy equation, which for constant density $\rho = 1$ gives the incompressible Euler equations for $\hat{u} = (u, p)$:

$$\dot{u} + (u \cdot \nabla)u + \nabla p = 0 \qquad (x,t) \in Q \tag{11}$$

$$\nabla \cdot u = 0 \qquad (x,t) \in Q \tag{12}$$

with initial conditions $\hat{u}(\cdot, 0) = \hat{u}^0$ and a slip boundary condition $u \cdot n = 0$.

From this equation we can derive the *linearized Euler equations* describing the growth of a perturbation $\hat{v} = (v, q)$, from a perturbation $v^0(x)$ of the velocity $u^0(x)$ at initial time. We can also derive an equation for the *vorticity* $\omega = \nabla \times u$. Both equations are linear convection-reaction systems, where growth of v and ω in time is determined by eigenvalues of the reaction coefficient ∇u.

Problem 2: Derive the linearized Euler equations (with $u = w + v$, and dropping second order terms in v):

$$\dot{v} + (u \cdot \nabla)v + (v \cdot \nabla)w + \nabla q = 0 \qquad (x,t) \in Q \tag{13}$$

$$\nabla \cdot v = 0 \qquad (x,t) \in Q \tag{14}$$

$$v \cdot n = 0 \qquad (x,t) \in \partial\Omega \times I \tag{15}$$

$$v(\cdot, 0) = v^0 \qquad x \in \Omega \tag{16}$$

Problem 3: Derive the vorticity equations

$$\dot{\omega} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \qquad (x,t) \in Q \tag{17}$$

Problem 4: Motivate why ∇u always have eigenvalues with both positive and negative real part for incompressible flow (unless all are zero).

Eigenvalues of both signs correspond to exponential growth of perturbations v in the linearized equations for any exact solution to the Euler equations, characterized by exponential growth of vorticity ω in the vorticity equations; through so called vortex stretching by the term $(\omega \cdot \nabla)u$.

This exponential growth of vorticity connects to what is referred mathematically to *blow up* in Euler, where vorticity becomes unbounded and the Euler solution ceases to exist as an exact solution, which in turn can be connected to transition to turbulence and formation of shocks (in compressible flow) characterized by locally very large (unbounded) gradients of the solution.

A *well-posed problem* should have a solution that (i) exists and (ii) is stable to perturbations (connected to uniqueness). Thus we are lead to the conclusion that the Euler equations are not well-posed, since (i) the solution blows up, and (ii) perturbations grow exponentially. Instead we will consider *regularized solutions* that exist, for which we study *well-posedness* of different output (mean values, forces etc.).

General Galerkin (G2) is a computational regularization of the Euler equations in the form of a finite element method, which we will use to compute approximations of turbulent flow. G2 can be interpreted as a so called Implicit Large Eddy Simulation (ILES). Let $\hat{U} = (U, P) \in \hat{V}_h$ be a finite element approximation of $\hat{u} = (u, p)$, satisfying

$$(\dot{U}, v) + ((U \cdot \nabla)U, v) + (hR(\hat{U}), R(\hat{v})) - (P, \nabla \cdot v) + (q, \nabla \cdot U) = 0 \qquad (18)$$

with $U(x, 0) = U^0(x)$, for all test functions $\hat{v} = (v, q) \in \hat{V}_h$, with $\hat{V}_h$ a finite element subspace of piecewise polynomial functions defined on a computational mesh in space-time of mesh size $h = h(x, t)$, satisfying a slip boundary condition, with the *Euler residual* $R(\hat{w}) = \dot{w} + U \cdot \nabla w + \nabla r$, for $\hat{w} = (w, r)$.

The $L_2(Q)$-inner product is defined as $(v, w) = \int_Q v \cdot w \, dxdt$, with $L_2(Q)$-norm $\|w\|_0 = (w, w)^{1/2}$, and we define the $L_2(\Omega)$-norm by $\|w\|^2 = \int_\Omega w \cdot w \, dx$, the Sobolev norm $\|w\|_{H^1(Q)} = (|w|^2_{H^1(Q)} + \|w\|_0^2)^{1/2}$ and seminorm $|w|_{H^1(Q)} = (\|\dot{w}\|_0^2 + \|\nabla w\|_0^2)^{1/2}$, and $\|w\|_{H^1(\Omega)} = (\|\nabla w\|^2 + \|w\|^2)^{1/2}$, $H^1_0(Q) = \{\|w\|_{H^1(Q)} < \infty, \ w(x) = 0 \text{ for all } x \in \Gamma\}$, and the negative norm

$$\|w\|_{-1} = \sup_{v \in H^1_0(Q)} \frac{(w, v)}{\|v\|_{H^1(Q)}} \qquad (19)$$

Problem 5: For the incompressible Euler equations, show that

$$\frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\|u^0\|^2, \quad t > 0 \qquad (20)$$

Problem 6: Assume that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$, then show for G2 that

$$\frac{1}{2}\|U(T)\|^2 + \|h^{1/2}R(\hat{U})\|_0^2 = \frac{1}{2}\|U^0\|^2 \qquad (21)$$

Thus G2 regularization introduces a dissipative term

$$D_h(U; t) = \|h^{1/2}R(\hat{U})\|_0^2 \qquad (22)$$

leading to a new equation for the kinetic energy $K(U(t)) = \frac{1}{2}\|U(t)\|^2$, which no longer is conserved:

$$K(U(t)) + D_h(U; t) = K(U^0). \qquad (23)$$

The dissipation from the G2 regularization is directly coupled to the failure to make the residual $R(\hat{U})$ small, which would correspond to a smooth exact solution.

Similarly, regularized solutions to the compressible Euler equations modify the energy equations into:

$$\dot{K} - W = -D \qquad \dot{\Theta} + W = D \qquad (24)$$

which now represents a combination of the 1st and 2nd Laws of thermodynamics, where work W can be converted back and forth between kinetic and internal energy by compression and expansion, but with the 2nd Law indicating irreversible transfer of kinetic energy into internal energy as $D > 0$ for turbulence and shocks.

We now assume that $\|R(\pi_h \hat{v})\|_0 \leq C\|\hat{v}\|_{H^1(Q)}$ for all functions with $\pi_h \hat{v}$ an interpolant of $\hat{v} = (v, q)$ in the finite element space $\hat{V}_h$ satisfying the interpolation error estimate $\|h^{-1}(v - \pi_h v)\|_0 \leq C\|\hat{v}\|_{H^1(Q)}$. We further recall the Cauchy-Schwarz inequality for $v, w \in L_2(Q)$: $(v, w) \leq \|v\|_0 \|w\|_0$

Problem 7: For turbulent dissipation $D_h(U; t) \sim 1$, assuming $\nabla \cdot U = 0$ and also assuming h is constant, show that

$$\|R(\hat{U})\|_0 \sim h^{-1/2} \qquad \|R(\hat{U})\|_{-1} \lesssim h^{1/2} \tag{25}$$

with $f \sim g \Leftrightarrow f = Cg$ and $f \lesssim g \Leftrightarrow f \leq Cg$, with $C > 0$ a constant.

Let $M(w) = \int_Q w \cdot \psi \; dxdt$ be a mean value output of a velocity w, defined by a smooth weight function $\psi(x, t)$, and let $\hat{u} = (u, p)$ and $\hat{U} = (U, P)$ be G2-solutions on two meshes with maximal mesh size h. For simplicity, we assume that $\nabla \cdot u = \nabla \cdot U = 0$. Let $\hat{\varphi} = (\varphi, \theta)$ be the solution to the *dual linearized problem*

$$-\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta = \psi \qquad \Omega \times I \tag{26}$$

$$\nabla \cdot \varphi = 0 \qquad \Omega \times I \tag{27}$$

$$\varphi = 0 \qquad \partial\Omega \times I \tag{28}$$

$$\hat{\varphi}(\cdot, T) = 0 \qquad \Omega \tag{29}$$

where T denotes transpose, with $(\nabla U^T \varphi)_i = \sum_j (\partial U_j / \partial x_i \varphi_j)$.

Problem 8: Show the following output error representation

$$M(u) - M(U) = \int_Q (R(\hat{u}) - R(\hat{U})) \cdot \varphi \; dxdt \tag{30}$$

Problem 9: Show the a posteriori error estimate

$$|M(u) - M(U)| \leq S(\|hR(\hat{u})\|_0 + \|hR(\hat{U}))\|_0) \tag{31}$$

with $S = S(u, U, M) = S(u, U) = C\|\hat{\varphi}\|_{H^1(Q)}$, and $C > 0$ a constant.

Thus the effect of a residual perturbation $R(\hat{u}) - R(\hat{U})$ on the output M is determined by $S(u, U)$. For a G2 solution $\hat{U}$ we can test well-posedness by $S(U, U)$.

For adaptive mesh refinement we want error indicators for each cell K in the finite element mesh, which may change over the time intervals $I_n = (t_{n-1}, t_n)$.

Problem 10: Show the following local a posteriori error estimate

$$|M(u) - M(U)| \leq \sum_{n=1}^{N} \sum_{K} \int_{I_n} C\|\hat{\varphi}\|_{H^1(K)} \left(\|hR(\hat{u})\|_K + \|hR(\hat{U}))\|_K\right) dt \quad (32)$$

where the K-index refers to spatial norms (integration in space) over the cells K, that is

$$\|v\|_K = (\int_K |v|^2 \, dx)^{1/2}$$

and

$$\|v\|_{H^1(K)} = (\|\dot{v}\|_K^2 + \|\nabla v\|_K^2 + \|v\|_K^2)^{1/2},$$

and I_n is the time interval $I_n = t_n - t_{n-1}$ with $t_{n-1} = 0$ and $t_N = T$.