Advanced Computation in Fluid Mechanics
Written Examination

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1 The incompressible Euler equations

Consider the incompressible Euler equations for $\hat{u} = (u, p)$:

$$\dot{u} + (u \cdot \nabla)u + \nabla p = 0 \qquad (x,t) \in Q \tag{1}$$

$$\nabla \cdot u = 0 \qquad (x,t) \in Q \tag{2}$$

for all $(x,t) \in Q = \Omega \times I$, where $\Omega \in \mathbb{R}^3$ is a 3D spatial domain and $I = (0, T]$ is a time interval, with initial conditions $\hat{u}(\cdot, 0) = \hat{u}^0$ and a slip boundary condition $u \cdot n = 0$.

1.1 Weak formulation

The $L_2(Q)$-inner product is defined as $((v, w)) = \int_Q v \cdot w \; dxdt$, with $L_2(Q)$-norm $\|w\|_Q = (w, w)^{1/2}$, and we define the $L_2(\Omega)$-inner product as $(v, w) = \int_\Omega v \cdot w \; dx$ with $L_2(\Omega)$-norm defined by $\|w\|^2 = \int_\Omega w \cdot w \; dx$, the Sobolev norm $\|w\|_{H^1(Q)} = (|w|^2_{H^1(Q)} + \|w\|^2_Q)^{1/2}$ and seminorm $|w|_{H^1(Q)} = (\|\dot{w}\|^2_Q + \|\nabla w\|^2_Q)^{1/2}$, and $\|w\|_{H^1(\Omega)} = (\|\nabla w\|^2 + \|w\|^2)^{1/2}$.

We can derive a weak formulation of the incompressible Euler equations: find $\hat{u} = (u, p) \in \hat{V}$ such that

$$((\dot{u}, v)) + (((u \cdot \nabla)u, v)) - ((p, \nabla \cdot v)) + ((q, \nabla \cdot u)) = 0 \tag{3}$$

with $u(x, 0) = u^0(x)$, for all test functions $\hat{v} = (v, q) \in \hat{V}$, where $\hat{V} = \{(v, q) \in H^1(Q) \times L_2(Q), v \cdot n = 0 \text{ for all } x \in \Gamma\}$.

1.2 General Galerkin method

With weighted least squares stabilisation of the residual, the G2 method takes the form: Let $\hat{U} = (U, P) \in \hat{V}_h$ be a finite element approximation of $\hat{u} = (u, p)$, satisfying

$$((\dot{U}, v)) + (((U \cdot \nabla)U, v)) + ((hR_U(\hat{U}), R_U(\hat{v}))) - ((P, \nabla \cdot v)) + ((q, \nabla \cdot U)) = 0 \tag{4}$$

with $U(x,0) = U^0(x)$, for all test functions $\hat{v} = (v,q) \in \hat{V}_h$, with $\hat{V}_h$ a finite element subspace of piecewise polynomial functions defined on a computational mesh in space-time of mesh size $h = h(x,t)$, satisfying a slip boundary condition, with the residual $R_U(\hat{w}) = \dot{w} + U \cdot \nabla w + \nabla r$, for $\hat{w} = (w,r)$.

2 Navier-Stokes equations

To derive a weak form of the Navier-Stokes equations we make a partial integration of the internal force $\nu\Delta u - \nabla p = \nabla \cdot \sigma(\hat{u})$ resulting in a boundary term over Γ:

$$-((\nabla \cdot \sigma(\hat{u}), v)) = ((\sigma(\hat{u}), \nabla \cdot v)) - ((\sigma(\hat{u}) \cdot n, v))_{\Gamma \times I} \tag{5}$$

With Dirichlet boundary conditions the boundary term disappears since then the test function v is zero on the boundary. The weak formulation also admits a natural way to implement Neumann (natural) boundary conditions of the form $\sigma(\hat{u}) \cdot n = g$ with g a given boundary stress.

3 A posteriori error estimation

Let $M(w) = ((w,\psi)) = \int_Q w \cdot \psi \; dxdt$ be a mean value output of a velocity w, defined by a smooth weight function $\psi(x,t)$. Let $\hat{u} = (u,p)$ be a weak solution satisfying (3), and let $\hat{U} = (U,P)$ be a G2-solution to the Euler equations on a finite element mesh with mesh size h.

Let $\hat{\varphi} = (\varphi, \theta)$ be the solution to the *dual linearized problem*

$$-\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta = \psi \qquad \Omega \times I \tag{6}$$

$$\nabla \cdot \varphi = 0 \qquad \Omega \times I \tag{7}$$

$$\varphi = 0 \qquad \partial\Omega \times I \tag{8}$$

$$\hat{\varphi}(\cdot, T) = 0 \qquad \Omega \tag{9}$$

where T denotes transpose, with $(\nabla U^T \varphi)_i = \sum_j (\partial U_j / \partial x_i \varphi_j)$. We assume that the solution to the dual problem $\hat{\varphi} \in \hat{V}$.

Problems

1. Derive the linearized Euler equations from (1)-(2) (*Hint*: Write $u = w + v$)

$$\dot{v} + (u \cdot \nabla)v + (v \cdot \nabla)w + \nabla q = 0 \qquad (x,t) \in Q \tag{10}$$

$$\nabla \cdot v = 0 \qquad (x,t) \in Q \tag{11}$$

$$v \cdot n = 0 \qquad (x,t) \in \partial\Omega \times I \tag{12}$$

$$v(\cdot, 0) = v^0 \qquad x \in \Omega \tag{13}$$

2. Motivate why ∇u always have eigenvalues with both positive and negative real part for incompressible flow, unless all are zero. (*Hint*: The sum of matrix eigenvalues is equal to the trace of the matrix.)

3. Derive the weak formulation (3).

4. For the weak formulation (3) show that

$$\frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\|u^0\|^2, \quad t > 0 \tag{14}$$

5. Assume that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$, then show for (4) that

$$\frac{1}{2}\|U(T)\|^2 + \|h^{1/2} R_U(\hat{U})\|_Q^2 = \frac{1}{2}\|U^0\|^2 \tag{15}$$

6. We sometimes want to give different boundary conditions in the normal and tangent directions on the boundary (defined by the normal vector n and tangent vectors τ_1, τ_2). Rewrite the boundary term $((\sigma(\hat{u}) \cdot n, v))_{\Gamma \times I}$ projected in the normal and tangent directions. (*Hint*: Use that the test function can be written as $v = (v \cdot n)n + (v \cdot \tau_1)\tau_1 + (v \cdot \tau_3)\tau_2$.)

7. In a slip with friction boundary condition we apply a slip velocity condition in the normal direction ($u \cdot n = 0$), and in the tangential direction a friction boundary condition $n^T \sigma(\hat{u})\tau_k = \beta u \cdot \tau_k$ for $k = 1, 2$, where all vectors are column vectors and T denotes the transpose and β is a skin friction parameter. Modify (5) to implement a slip with friction boundary condition.

8. Derive the following energy estimate for the G2 method of the Navier-Stokes equations with slip with friction boundary conditions, using the assumptions that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$:

$$\frac{1}{2}\|U(T)\|^2 + \|h^{1/2} R_U(\hat{U})\|_Q^2 + \sum_{k=1}^{2} \|\beta^{1/2} \hat{U} \cdot \tau_k\|_{\Gamma \times I}^2 = \frac{1}{2}\|U^0\|^2 \tag{16}$$

9. Show that

$$((R_u(\hat{u}) - R_U(\hat{U}), \varphi)) - ((\nabla \cdot u - \nabla \cdot U, \theta)) = ((u - U, -\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta))$$

10. Show the following output error representation

$$M(u) - M(U) = -\int_Q R_U(\hat{U}) \cdot \varphi \; dxdt + \int_Q (\nabla \cdot U)\theta \; dxdt \tag{17}$$