Advanced Computation in Fluid Mechanics
Seminar 1

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1 Inviscid flow

As a model for inviscid fluid flow we consider the Euler equations

$$\dot{\rho} + \nabla \cdot (u\rho) = 0 \qquad \text{(mass conservation)} \tag{1}$$
$$\dot{m} + \nabla \cdot (um) = -\nabla p \qquad \text{(momentum - Newton 2nd law)} \tag{2}$$
$$\dot{\theta} + \nabla \cdot (u\theta) = -p\nabla \cdot u \qquad \text{(internal energy } \theta\text{)} \tag{3}$$

for $m = \rho u$ momentum, ρ density, $u = (u_1, u_2, u_3)$ the velocity vector, θ internal energy, and p pressure, together with an additional law specifying the pressure, for example:

$$p = (\gamma - 1)\theta \qquad \text{(state equation for a perfect gas)} \tag{4}$$
$$\nabla \cdot u = 0 \qquad \text{(incompressible flow)} \tag{5}$$

Assume there exists a solution $\hat{u} = (\rho, m, \theta)$ to the Euler equations for all $(x, t) \in Q = \Omega \times I$, where $\Omega \in \mathbb{R}^3$ is a 3D spatial domain and $I = (0, T]$ is a time interval, with initial conditions $\hat{u}(\cdot, 0) = \hat{u}^0$ and a (slip) boundary condition $u \cdot n = 0$ for all $x \in \Gamma$, with outward unit normal n on the boundary Γ.

Problems

Use (1)-(3) to show that

1. $\dot{m} + \nabla \cdot (um) = \rho(\dot{u} + (u \cdot \nabla)u)$
2. $\dot{k} + \nabla \cdot (uk) = -\nabla p \cdot u$ (with kinetic energy $k = \rho|u|^2/2$)
3. $\dot{e} + \nabla \cdot (ue) = -\nabla \cdot (pu)$ (with total energy $e = k + \theta$)
4. $\displaystyle \frac{d}{dt}\int_\Omega \rho \; dx = \frac{d}{dt}\int_\Omega e \; dx = 0 \qquad \frac{d}{dt}\int_\Omega m \; dx = -\int_{\partial\Omega} pn \; ds$
5. $\dot{K} = -\dot{\Theta} = W$, with $K = \int_\Omega k \; dx,\ \Theta = \int_\Omega \theta \; dx,\ W = \int_\Omega p\nabla \cdot u \; dx$

2 Incompressible flow

For incompressible flow ($\nabla \cdot u = 0$) the work $W = 0$, and thus kinetic energy and internal energy are conserved $\dot{K} = \dot{\Theta} = 0$. Also, for incompressible flow the mass and momentum equations decouple from the energy equation, which for constant density $\rho = 1$ gives the incompressible Euler equations for $\hat{u} = (u, p)$:

$$\dot{u} + (u \cdot \nabla)u + \nabla p = 0 \qquad (x,t) \in Q \tag{6}$$

$$\nabla \cdot u = 0 \qquad (x,t) \in Q \tag{7}$$

with initial conditions $\hat{u}(\cdot, 0) = \hat{u}^0$ and a slip boundary condition $u \cdot n = 0$.

From this equation we can derive the *linearized Euler equations* describing the growth of a perturbation $\hat{v} = (v, q)$, from a perturbation $v^0(x)$ of the velocity $u^0(x)$ at initial time. We can also derive an equation for the *vorticity* $\omega = \nabla \times u$. Both equations are linear convection-reaction systems, where growth of v and ω in time is determined by eigenvalues of the reaction coefficient ∇u.

Problems

1. Derive the linearized Euler equations (*Hint*: Write $u = w + v$)

$$\dot{v} + (u \cdot \nabla)v + (v \cdot \nabla)w + \nabla q = 0 \qquad (x,t) \in Q \tag{8}$$

$$\nabla \cdot v = 0 \qquad (x,t) \in Q \tag{9}$$

$$v \cdot n = 0 \qquad (x,t) \in \partial\Omega \times I \tag{10}$$

$$v(\cdot, 0) = v^0 \qquad x \in \Omega \tag{11}$$

2. Derive the vorticity equations

$$\dot{\omega} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \qquad (x,t) \in Q \tag{12}$$

3. Motivate why ∇u always have eigenvalues with both positive and negative real part for incompressible flow, unless all are zero. (*Hint*: The sum of matrix eigenvalues is equal to the trace of the matrix.)

4. Find solutions to the Euler equations with pure imaginary eigenvalues (zero real parts).

Reading

CTIF: chapters 1-12