Advanced Computation in Fluid Mechanics
Seminar 3

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1 Weak formulation

First we introduce some definitions from functional analysis to extend the concepts of inner products and norms from linear algebra to function spaces.

The $L_2(Q)$-inner product is defined as $((v,w)) = \int_Q v \cdot w \; dxdt$, with $L_2(Q)$-norm $\|w\|_Q = (w,w)^{1/2}$, and we define the $L_2(\Omega)$-inner product as $(v,w) = \int_\Omega v \cdot w \; dx$ with $L_2(\Omega)$-norm defined by $\|w\|^2 = \int_\Omega w \cdot w \; dx$, the Sobolev norm $\|w\|_{H^1(Q)} = (|w|^2_{H^1(Q)} + \|w\|^2_Q)^{1/2}$ and seminorm $|w|_{H^1(Q)} = (\|\dot{w}\|^2_Q + \|\nabla w\|^2_Q)^{1/2}$, and $\|w\|_{H^1(\Omega)} = (\|\nabla w\|^2 + \|w\|^2)^{1/2}$, $H^1_0(Q) = \{\|w\|_{H^1(Q)} < \infty, \; w(x) = 0 \text{ for all } x \in \Gamma\}$.

We can derive a weak formulation of the incompressible Euler equations with the velocity satisfying a *slip boundary condition* $u \cdot n = 0$: find $\hat{u} = (u,p) \in \hat{V}$ such that

$$((\dot{u}, v)) + (((u \cdot \nabla)u, v)) - ((p, \nabla \cdot v)) + ((q, \nabla \cdot u)) = 0 \tag{1}$$

with $u(x,0) = u^0(x)$, for all test functions $\hat{v} = (v,q) \in \hat{V}$, where $\hat{V} = \{(v,q) \in H^1(Q) \times L_2(Q), v \cdot n = 0 \text{ for all } x \in \Gamma\}$

Problems

1. Derive the weak formulation (1).

2. Show that (assuming that $u \cdot n = 0$ on the boundary):

$$((u \cdot \nabla)u, u) = -\frac{1}{2}((\nabla \cdot u)u, u) \tag{2}$$

3. Under what condition is the following true

$$((u \cdot \nabla)v, w) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v) \tag{3}$$

2 Galerkin finite element method

In a Galerkin method we seek a solution to the weak formulation in a finite dimensional subspace $\hat{V}_h$ of the function space $\hat{V}$: Find $\hat{U} = (U, P) \in \hat{V}_h \subset \hat{V}$ such that

$$((\dot{U}, v)) + (((U \cdot \nabla)U, v)) - ((P, \nabla \cdot v)) + ((q, \nabla \cdot U)) = 0 \tag{4}$$

with $U(x, 0) = U^0(x)$, for all test functions $\hat{v} = (v, q) \in \hat{V}_h$. Different choices of finite element spaces $\hat{V}_h$ correspond to different space-time finite element basis functions.

Problems

1. Assume the test functions (v, q) are piecewise constant in time over time intervals $I_n = (t_{n-1}, t_n]$ of length $k_n = t_n - t_{n-1}$, and that the finite element approximations are continuous piecewise linear over the time intervals I_n. Write out the FEM formulation (4) over one time interval I_n after integration in time, in terms of the solution values $U^n = U(t_n)$, $U^{n-1} = U(t_{n-1})$, $P^n = P(t_n)$, $P^{n-1} = P(t_{n-1})$, for the two cases of using: (i) the midpoint quadrature rule and (ii) the trapezoidal rule for the non-linear term.

2.1 General Galerkin method

We denote with General Galerkin (G2) a Galerkin finite element method with residual based numerical stabilisation. With weighted least squares stabilisation of the residual, the G2 method takes the form: Let $\hat{U} = (U, P) \in \hat{V}_h$ be a finite element approximation of $\hat{u} = (u, p)$, satisfying

$$((\dot{U}, v)) + (((U \cdot \nabla)U, v)) + ((hR(\hat{U}), R(\hat{v}))) - ((P, \nabla \cdot v)) + ((q, \nabla \cdot U)) = 0 \tag{5}$$

with $U(x, 0) = U^0(x)$, for all test functions $\hat{v} = (v, q) \in \hat{V}_h$, with $\hat{V}_h$ a finite element subspace of piecewise polynomial functions defined on a computational mesh in space-time of mesh size $h = h(x, t)$, satisfying a slip boundary condition, with the *Euler residual* $R(\hat{w}) = \dot{w} + U \cdot \nabla w + \nabla r$, for $\hat{w} = (w, r)$.

Problems

1. For the weak formulation (1) show that

$$\frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\|u^0\|^2, \quad t > 0 \tag{6}$$

2. Assume that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$, then show for (4) that

$$\frac{1}{2}\|U(T)\|^2 = \frac{1}{2}\|U^0\|^2 \tag{7}$$

3. Assume that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$, then show for (5) that

$$\frac{1}{2}\|U(T)\|^2 + \|h^{1/2} R(\hat{U})\|_Q^2 = \frac{1}{2}\|U^0\|^2 \tag{8}$$

3 Navier-Stokes equations

To derive a weak form of the Navier-Stokes equations we make a partial integration of the internal force $\nu \Delta u - \nabla p = \nabla \cdot \sigma(\hat{u})$ resulting in a boundary term over Γ:

$$-((\nabla \cdot \sigma(\hat{u}), v)) = ((\sigma(\hat{u}), \nabla \cdot v)) - ((\sigma(\hat{u}) \cdot n, v))_{\Gamma \times I} \tag{9}$$

With Dirichlet boundary conditions the boundary term disappears since then the test function v is zero on the boundary. The weak formulation also admits a natural way to implement Neumann (natural) boundary conditions of the form $\sigma(\hat{u}) \cdot n = g$ with g a given boundary stress.

Problems

1. We sometimes want to give different boundary conditions in the normal and tangent directions on the boundary (defined by the normal vector n and tangent vectors τ_1, τ_2). Rewrite the boundary term $((\sigma(\hat{u}) \cdot n, v))_{\Gamma \times I}$ projected in the normal and tangent directions. (*Hint*: Use that the test function can be written as $v = (v \cdot n)n + (v \cdot \tau_1)\tau_1 + (v \cdot \tau_3)\tau_2$.)

2. In a slip with friction boundary condition we apply a slip velocity condition in the normal direction ($u \cdot n = 0$), and in the tangential direction a friction boundary condition $n^T \sigma(\hat{u})\tau_k = \beta u \cdot \tau_k$ for $k = 1, 2$, where all vectors are column vectors and T denotes the transpose and β is a skin friction parameter. Modify (9) to implement a slip with friction boundary condition.

3. Derive the following energy estimate for the G2 method of the Navier-Stokes equations with slip with friction boundary conditions, using the assumptions that $(U \cdot \nabla U, U) = 0$ and that $(\dot{U}, U) = \frac{1}{2}(\|U(T)\|^2 - \|U^0\|^2)$:

$$\frac{1}{2}\|U(T)\|^2 + \|h^{1/2} R(\hat{U})\|_Q^2 + \sum_{k=1}^{2} \|\beta^{1/2} \hat{U} \cdot \tau_k\|_{\Gamma \times I}^2 = \frac{1}{2}\|U^0\|^2 \tag{10}$$

Reading

CTIF: chapters 28-29