Advanced Computation in Fluid Mechanics
Seminar 4

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1 Turbulence

For high Reynolds numbers Re fluid flow is always partly turbulent, typically initiated at flow separation or from transition to turbulence in shear flow (for example in boundary layers). Turbulence is characterised by large energy dissipation, with vortices on a range of scales transporting kinetic energy from the large scale induced by the geometry of the flow domain to the smallest scale determined by the viscosity of the fluid where the kinetic energy is dissipated as heat. Energy transport from large to small scales through vortex stretching is expressed in the vorticity equation by the term $(\omega \cdot \nabla)u$.

The ratio between the largest scale and the smallest scale in turbulent flow (the Kolmogorov scale) gives an estimate on the computational resolution needed to represent all turbulent scales in a Direct Numerical Simulation (DNS), which can be related to the Reynolds number where the number of spatial degrees of freedom scales as $Re^{9/4}$. Since in many applications $Re > 10^6$, DNS is not possible due to the computational cost, instead various methods have been developed where only a subset of the turbulent scales are resolved.

Another feature of turbulence is exponential perturbation growth (chaos), which can be studied in the linearised equations, where we observe that for the linearised Euler equations (as a model for high Re) we always have eigenvalues with real parts of both signs. This puts a restriction on what can be computed in turbulent flow. But some quantities in turbulent flow may be stable and thus computable even if individual particle trajectories are chaotic, for example mean values. Reliable computation of such stable quantities is the goal of computational simulation of turbulence.

2 Turbulence simulation

Reynolds Averaged Navier-Stokes equations (RANS) are derived by taking the average of the Navier-Stokes equations, and describe the evolution of the ensemble mean of the flow field. The challenge of RANS is to determine the so called Reynolds stresses that must be modelled in a turbulence model.

In a Large Eddy Simulation (LES) large scales in the flow are resolved and only the smallest scales are modelled in a subgrid model, and the equations are derived by applying a filter to the Navier-Stokes equations.

Both turbulence models in RANS and subgrid models in LES typically introduce dissipation in the problem, and thus regularises the equations.

A particular challenge for both RANS and LES is to use proper boundary conditions. Turbulent boundary layers are too expensive to resolve for high Reynolds numbers and complex geometry, and instead a cheaper boundary condition is used without full resolution of the boundary layer, for example a wall shear stress model.

The typical form of RANS and LES is a modified momentum equation where the unresolved scales are modelled by a turbulent eddy viscosity $\nu_T = \nu_T(\bar{u})$ which is a function of the regularised flow $\bar{u}$, where the turbulence modeling problem (closure problem) is to determine ν_T:

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} - (\nu + \nu_T)\Delta \bar{u} + \nabla \bar{p} = 0 \qquad (x,t) \in Q \qquad (1)$$

$$\nabla \cdot \bar{u} = 0 \qquad (x,t) \in Q \qquad (2)$$

In the classical Smagorinsky model the eddy viscosity is chosen to be proportional to the square of of the smallest scale in the flow $h = h(x,t)$ and the strain rate tensor $\epsilon(\bar{u})$: $\nu_T = Ch^2|\epsilon(\bar{u})|$ (with $\epsilon_{ij}(\bar{u}) = 0.5(\bar{u}_{i,j} + \bar{u}_{j,i})$).

Problems

1. For homogeneous velocity boundary conditions ($u(x,t) = 0$ for $x \in \Gamma = \partial\Omega$); derive the following energy estimate for $(\bar{u}, \bar{p})$ satisfying (1)-(2):

$$\frac{1}{2}\|\bar{u}(T)\|^2 + D_\nu + D_{\nu_T} = \frac{1}{2}\|\bar{u}^0\|^2 \qquad (3)$$

with the viscous dissipation $D_\nu = \|\nu^{1/2}\nabla\bar{u}\|_Q^2$ and the turbulent dissipation $D_{\nu_T} = \|\nu_T^{1/2}\nabla\bar{u}\|_Q^2$.

2. If the turbulent dissipation $D_{\nu_T} = 1$; show that $\|\nabla\bar{u}\|_Q = \nu_T^{-1/2}$.

3 G2 turbulence simulation

Instead of averaging as in RANS or LES, we can regularise the equations without introducing Reynolds/subgrid stresses. A General Galerkin (G2) method is a computational regularization in the form of a finite element method, which we will use to compute approximations of turbulent flow, here assuming the Reynolds number to be very large so that we neglect the viscous term: Let $\hat{U} = (U, P) \in \hat{V}_h$ be a finite element approximation of $\hat{u} = (u, p)$, satisfying

$$((\dot{U}, v)) + (((U \cdot \nabla)U, v)) + ((hR(\hat{U}), R(\hat{v}))) - ((P, \nabla \cdot v)) + ((q, \nabla \cdot U)) = 0 \quad (4)$$

with $U(x,0) = U^0(x)$, for all test functions $\hat{v} = (v,q) \in \hat{V}_h$, with $\hat{V}_h$ a finite element subspace of piecewise polynomial functions defined on a computational mesh in space-time of mesh size $h = h(x,t)$, satisfying a slip boundary condition, with the *Euler residual* $R(\hat{w}) = \dot{w} + U \cdot \nabla w + \nabla r$, for $\hat{w} = (w,r)$.

An energy estimate shows that G2 regularization introduces a dissipative term

$$D_h(U;t) = \|h^{1/2} R(\hat{U})\|_Q^2 \tag{5}$$

leading to a new equation for the kinetic energy $K(U(t)) = \frac{1}{2}\|U(t)\|^2$ (which is not conserved as in the case of the Euler equations):

$$K(U(t)) + D_h(U;t) = K(U^0). \tag{6}$$

The dissipation from the G2 regularization is directly coupled to the failure to make the residual $R(\hat{U})$ small, which would correspond to a smooth exact solution for which the Euler residual is zero.

Similarly, regularized solutions to the compressible Euler equations modify the energy equations into:

$$\dot{K} - W = -D \qquad \dot{\Theta} + W = D \tag{7}$$

which now represents a combination of the 1st and 2nd Laws of thermodynamics, where work W can be converted back and forth between kinetic and internal energy by compression and expansion, but with the 2nd Law indicating irreversible transfer of kinetic energy into internal energy as $D > 0$ for turbulence and shocks.

Problems

1. We now assume that $\|R(\pi_h \hat{v})\|_0 \leq C\|\hat{v}\|_{H^1(Q)}$ for all functions with $\pi_h \hat{v}$ an interpolant of $\hat{v} = (v,q)$ in the finite element space $\hat{V}_h$ satisfying the interpolation error estimate $\|h^{-1}(v - \pi_h v)\|_Q \leq C\|\hat{v}\|_{H^1(Q)}$. We further recall the Cauchy-Schwarz inequality for $v, w \in L_2(Q)$: $((v,w)) \leq \|v\|_Q \|w\|_Q$

   For turbulent dissipation $D_h(U;t) \sim 1$, assuming $\nabla \cdot U = 0$ and also assuming h is constant, show that

   $$\|R(\hat{U})\|_Q \sim h^{-1/2} \qquad \|R(\hat{U})\|_{-1} \lesssim h^{1/2} \tag{8}$$

   with $f \sim g \Leftrightarrow f = Cg$ and $f \lesssim g \Leftrightarrow f \leq Cg$, with $C > 0$ a constant, and where we define the negative norm as

   $$\|w\|_{-1} = \sup_{v \in H_0^1(Q)} \frac{((w,v))}{\|v\|_{H^1(Q)}}. \tag{9}$$

Reading

CTIF: chapters 13-20 (hard), 25-27,