Advanced Computation in Fluid Mechanics
Seminar 5

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We now want to derive a posteriori error estimates for the error in output of interest from the computed flow field. To derive such error estimates we introduce a dual (or adjoint) problem linearised at the computed solution, which quantifies how much the computational errors influence the output of interest.

Let $M(w) = ((w, \psi)) = \int_Q w \cdot \psi \; dxdt$ be a mean value output of a velocity w, defined by a smooth weight function $\psi(x,t)$. Let $\hat{u} = (u, p)$ be a weak solution such that

$$((R(\hat{u}), \hat{v})) = 0 \tag{1}$$

for all $\hat{v} \in \hat{V}$, with the residual defined as $R(\hat{w}) \equiv \dot{w} + w \cdot \nabla w + \nabla r$, for $\hat{w} = (w, r)$ (assuming the force $f = 0$). Let $\hat{U} = (U, P)$ be a G2-solution to the Euler equations on a finite element mesh with mesh size h. For simplicity, we assume that $\nabla \cdot u = \nabla \cdot U = 0$.

Let $\hat{\varphi} = (\varphi, \theta)$ be the solution to the *dual linearized problem*

$$-\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta = \psi \qquad \Omega \times I \tag{2}$$

$$\nabla \cdot \varphi = 0 \qquad \Omega \times I \tag{3}$$

$$\varphi = 0 \qquad \partial\Omega \times I \tag{4}$$

$$\hat{\varphi}(\cdot, T) = 0 \qquad \Omega \tag{5}$$

where T denotes transpose, with $(\nabla U^T \varphi)_i = \sum_j (\partial U_j / \partial x_i \varphi_j)$. We assume that the solution to the dual problem $\hat{\varphi} \in \hat{V}$.

For adaptive mesh refinement we want error indicators for each cell K in the finite element mesh, which may change over the time intervals $I_n = (t_{n-1}, t_n)$.

Problems

1. Find a weight function $\psi = (\psi_1, \psi_2, \psi_3)^T$ such that $M(u) = ((u, \psi))$ defines a mean value of the first velocity component u_1 over the subdomain $\omega \times I_m$ with $\omega \subset \Omega$ and $I_m \subset I$, that is:

$$M(u) = \frac{1}{|I_m||\omega|} \int_{I_m} \int_{\omega} u_1 \; dxdt \tag{6}$$

   where $|\omega|$ is the volume of ω and $|I_m|$ is the length of the time interval I_m.

2. Show that

$$((R(\hat{u})-R(\hat{U}),\varphi))-((\nabla\cdot u-\nabla\cdot U,\theta)) = ((u-U,-\dot{\varphi}-(u\cdot\nabla)\varphi+\nabla U^T\varphi+\nabla\theta))$$

3. Show the following output error representation

$$M(u) - M(U) = -\int_Q R(\hat{U})\cdot\varphi \; dxdt \tag{7}$$

4. Show the a posteriori error estimate

$$|M(u) - M(U)| \leq S\|hR(\hat{U}))\|_Q \tag{8}$$

with $S = S(u, U, M) = S(u, U) = C\|\hat{\varphi}\|_{H^1(Q)}$, and $C > 0$ a constant.

5. Show the following local a posteriori error estimate

$$|M(u) - M(U)| \leq \sum_{n=1}^{N}\sum_K \int_{I_n} C\|\hat{\varphi}\|_{H^1(K)} \; \|hR(\hat{U}))\|_K \; dt \tag{9}$$

where the K-index refers to spatial norms (integration in space) over the cells K, that is

$$\|v\|_K = (\int_K |v|^2 \; dx)^{1/2}$$

and

$$\|v\|_{H^1(K)} = (\|\dot{v}\|_K^2 + \|\nabla v\|_K^2 + \|v\|_K^2)^{1/2},$$

and I_n is the time interval $I_n = t_n - t_{n-1}$ with $t_{n-1} = 0$ and $t_N = T$.

Reading

CTIF: chapters 30, 33