Atmosphere as Air Conditioner

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Abstract

We consider a model for atmospheric circulation based on the Euler equations for a compressible gas. We consider two hydrostatic base solutions depending on height, one with constant temperature and one isentropic with constant temperature gradient or lapse rate. We argue that these solutions represent solutions with maximal and minimal turbulent dissipation with the observed real lapse rate soemwhere in between. We find that the atmosphere acts in a cyclic thermodynmaic process of rising-expanding-cooling and descending-compressing-warming which is similar to that of an air conditioner or refrigerator. We seek solutions as perturbations of the hydrostatic base solutions with the perturbations satisfying a modified form of the incompressible Euler equations.

1 Compressible/Incompressible Euler as Climate Model

As a model of the atmosphere we consider the Euler equations for a compressible prefect gas occupying the cube $Q = (0,1)^3$: Find (ρ, u, e) with ρ density, u velocity and e internal energy depending on x and $t > 0$, such that for $x \in Q$ and $t > 0$:

$$\begin{aligned} D_u\rho + \rho\nabla\cdot u = 0 \\ D_u m + m\nabla\cdot u + \nabla p + g\rho e_3 = 0 \\ D_u e + e\nabla\cdot u + p\nabla\cdot u = 0 \end{aligned} \tag{1}$$

where $m = \rho u$ is momentum, $p = \gamma\rho T$ is pressure with T temperature and $0 < \gamma < 1$ a constant, $e = c_v\rho T$ with c_v the specific heat under constant volume, and $D_u v = \dot{v} + u\cdot\nabla v$ is the material time derivative with respect to the velocity u, together with initial and boundary values, and $e_3 = (0,0,1)$ is the upward direction.

2 Basic Thermodynamics of Atmospheric Circulation

We identify the following hydrostatic base solutions:

$$\begin{aligned} &\bar{u} = 0,\ \bar{T} \sim 1,\ \bar{\rho} = \exp(-x_3),\ \bar{p} \sim \exp(-x_3), \\ &\bar{u} = 0,\ \bar{T} \sim 1 - x_3,\ \bar{\rho} \sim (1-x_3)^{\frac{1}{\gamma}},\ \bar{p} \sim (1-x_3)^{1+\frac{1}{\gamma}}, \end{aligned} \tag{2}$$

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with $\sim$ indicating proportionality. The first solution has constant temperature and exponential drop of density and pressure. The second solution is isentropic defined by equality in the 2nd Law of Thermodynamics as presented in [4]:

$$c_v\,dT + p\,dV \geq 0, \tag{3}$$

which combined with hydrostatic balance $\frac{\partial p}{\partial x_3} = -g\rho$ and the differentiated form $pdV + Vdp = \gamma dT$ of the gas law in the form $pV = \gamma T$, gives

$$(c_v + \gamma)\frac{\partial T}{\partial x_3} = -g. \tag{4}$$

For air $c_v + \gamma \approx 1$ and the isentropic lapse rate is thus -10 Celsius (per kilometer). In reality, turbulent dissipation gives strict inequality in the 2nd Law (3). The observed lapse rate of -6 can thus partly be seen as an effect of turbulent dissipation, with another major effect coming from evaporation and condensation.

The isentropic lapse rate can be seen as being established by a cyclic thermodynamic process with hot light air rising under expansion/cooling and cool air descending under compression/warming, combined with evaporation/condensation. The atmosphere thus acts like an air conditioner or refrigerator transporting heat from the Earth surface (received by insolation) to the top of the atmosphere from where it is radiated to into space, by a cyclic thermodynamic process of expansion/cooling and compression/warming with efficiency boosted by evaporation/condensation. The thermodynamics of a refrigerator is driven by a compressor, which in the case on an atmosphere is taken over by gravitation causing compression of descending air.

3 Joule's Experiment

The basic thermodynamic process of cooling under expansion was experimentally studied by Joule letting a high pressure-density-temperature gas expand from equilibrium in one chamber into another chamber and measuring the temperature difference/gap in the chambers at the new equilibrium. In this process the gas gains kinetic energy by cooling and then comes to rest by turbulent dissipation causing warming. The resulting temperature gap depends on the dynamics of the expansion process, which with maximal turbulent dissipation (or maximal entropy increase) results in zero gap, while isentropic expansion without turbulent dissipation gives maximal gap. In a real process the gap is somewhere in between these extremes as shown in computational simulation in Chapter 166 of [5].

The experience from the Joule experiment suggests to view the real lapse as determined by the amount of turbulent dissipation between the limits of isentropic zero dissipation with maximal lapse rate and maximal dissipation with zero lapse rate (combined with the effect of evaporation/dissipation decreasing the lapse rate).

4 Perturbations of Base Solutions

We seek a solution to (1) on the form $(\bar\rho + \rho, \bar u + u, \bar e + e)$, where $(\bar\rho, \bar u, \bar e)$ is a basic solution, here a static solution with $\bar u = 0$ and $\nabla \bar p = -g\bar\rho$ expressing hydrostatic balance. We assume that the velocity perturbation u satisfies $\nabla \cdot u = 0$ motivated by the fact that the Mach number of atmospheric air flow is small. Inserting this Ansatz into (1), we obtain the following modified form the incompressible Euler equations: Find (ρ, u, p, e) such that

$$\begin{aligned} D_u\rho + u \cdot \nabla\bar{\rho} &= 0 \\ D_u((\bar{\rho} + \rho)u) + \nabla p - g\frac{e}{\overline{T}} &= 0, \\ \nabla \cdot u &= 0, \\ D_u e + u \cdot \nabla\bar{e} &= q, \end{aligned} \tag{5}$$

where q is a heat source modelling evaporation/condensation and we have replaced ρ in the momentum equation by $-\frac{e}{\overline{T}} = -\frac{e\bar{\rho}}{\bar{e}}$, where $\bar{\rho}\overline{T} = \bar{e}$, assuming that $e \approx \bar{\rho}T$ (change of internal energy e primarily by temperature change T) and $\rho\overline{T} + \bar{\rho}T \approx 0$ (total pressure approximately constant).

5 Stability of Base Solutions

We investigate the stability of hydrostatic base solutions $(\bar{\rho}, 0, \bar{e})$ by linearizing the compressible Euler equations at a base solution, to obtain the following system in a perturbation (ρ, u, e)

$$\begin{aligned} \dot{\rho} + u \cdot \nabla\bar{\rho} + \bar{\rho}\nabla \cdot u &= 0 \\ \bar{\rho}\dot{u} + \gamma\nabla e + g\rho e_3 &= 0 \\ \dot{e} + u \cdot \nabla\bar{e} + (1 + \gamma)\bar{e}\nabla \cdot u &= 0 \end{aligned} \tag{6}$$

Assuming $\nabla \cdot u = 0$ and for simplicity $\bar{\rho} = (1 - x_3)$ (with $\gamma = 1$), the system simplifies to (with $g = 1$)

$$\begin{aligned} \dot{\rho} - u_3 &= 0 \\ (1 - x_3)\dot{u} + \nabla p + \rho e_3 &= 0 \\ \dot{e} + u \cdot \nabla\bar{e} &= 0 \end{aligned} \tag{7}$$

where p is a pressure perturbation compatible with $\nabla \cdot u = 0$. Multiplying here the first equation by ρ and the second by u, addding and integrating in space we obtain

$$\frac{d}{dt}\frac{1}{2}\int_Q ((\rho^2 + (1 - x_3)|u|^2)\, dx = 0, \tag{8}$$

showing stability.

6 A Simple Radiation Model

Let $E(x_3)$ be the radiation from an atmospheric layer at height x_3. Subdivide the atmosphere into horisontal layers of width h identified by $h(j - 1) < x_3 < hj$. Balance of incoming and outgoing radiation can, assuming full absorption/emission, be expressed as

$$E(x_3 - h) - 2E(x_3) + E(x_3 + h) = 0 \tag{9}$$

Assuming that $E = cT^4$ according to Stefan-Boltzmann's Radiation Law, this leads to the following differential equation

$$-TT'' = 3(T')^2 \tag{10}$$

with $T' = \frac{dT}{dx_3}$. This equation effectively priviliges a linear temperature profile with $T'' \approx 0$, while the slope or lapse rate T' still is to be determined. The basic problem at hand can be seen as heat conduction modeled by

$$-\epsilon T''(x_3) = 0 \quad \text{for } 0 < x_3 < 1, \quad T(1) = 0, \quad -\epsilon T'(0) = Q \tag{11}$$

with ϵ a coefficient of heat conduction and a Neumann condition at $x_3 = 0$, effectively defining $T'(x_3) = T'(0) = -\frac{Q}{\epsilon}$. In this model the lapse rate of the atmosphere is determined principally by the heat exchange ocean-atmosphere at the ocean surface.

7 G2 Computational Results

We solve the system (5) using the G2 finite element method [1] starting from the above adiabatic base solution with forcing from heating at $x_3 = 0$ and cooling at $x_3 = 1$, with the objective of determining the temperature drop from $x_3 = 0$ to $x_3 = 1$. We find buoyancy driven turbulent solutions, which will be presented shortly. Related results for thermohaline ocean circulation are presented in [2] and [3].

References

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