Blowup vs Illposedness of Smooth Solutions of the Incompressible Euler/Navier-Stokes Equations

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Abstract

We present evidence that the problem of breakdown/blowup of smooth solutions of the Euler and Navier-Stokes equations, is closely related to Hadamard's concepts of wellposedness and illposedness. We present a combined criterion for blowup, based on detecting increasing L_2-residuals and stability factors, which can be tested computationally on meshes of finite mesh size.

1 The Clay Navier-Stokes Millennium Problem

The Clay Mathematics Institute Millennium Problem on the *incompressible Navier-Stokes equations* [5, 8] asks for a proof of (I) global existence of smooth solutions for all smooth data, or a proof of the converse (II) non global existence of a smooth solution for some smooth data, referred to as *breakdown* or *blowup*.

The analogous problem for the inviscid *incompressible Euler equations* is mentioned briefly in [8] and in [7] described as "a major open problem in PDE theory, of far greater physical importance than the blowup problem for Navier-Stokes equations, which of course is known to the nonspecialists because it is a Clay Millenium Problem". In the recent survey [3] the problem is described as "one of the most important and challenging open problems in mathematical fluid mechanics". Since the viscosity the Millennium Problem is allowed to be arbitarily small and solutions of the Euler equations are defined as *viscosity solutions* of the Navier-Stokes equations under vanishing viscosity, the Euler equations effectively are included in the Millenium Problem as a limit case.

In [16] we presented evidence that a specific initially smooth solution of the Euler equations, potential flow around a circular cylinder, in finite time exhibits blowup into a turbulent non-smooth solution, that is we presented evidence of (II). More generally, we presented evidence that all (non-trivial) initially smooth Euler solutions exhibit blowup into turbulent solutions. In particular, we argued that blowup can be detected computationally on computational meshes of finite mesh size. This work closely connects to the new resolution of d'Alembert's paradox presented in [15].

2 The Beale-Kato-Majda Non Blowup Criterion

In [8, 6, 7, 12] existence of a smooth velocity-pressure solution (u,p) of the incompressible Euler equations in $\mathbb{R}^3$ over the time interval $[0,T]$, is identified by the following *non-blowup criterion* by Beale-Kato-Madja [1]:

$$C(\omega) = \int_0^T \|\omega(\cdot,t)\|_\infty dt < \infty, \tag{1}$$

where $\omega(x,t) = \nabla \times u(x,t)$ is the vorticity at $(x,t) \in \mathbb{R}^3 \times [0,T]$ and $\|\cdot\|_\infty$ the maximum-norm over $\mathbb{R}^3$. More precisely, it is proved (under zero forcing) that for $s \geq 3$ there is a constant C (depending on $C(\omega)$, T and $\|u(0)\|_s$), such that

$$\max_{0\leq t\leq T} \|u(t)\|_s \leq \exp(CC(\omega))\|u(0)\|_s, \tag{2}$$

where $\|\cdot\|_s$ is the $H^s(\mathbb{R}^3)$-norm. In particular, it is argued that if $C(\omega) < \infty$, then also $\exp(CC(\omega)) < \infty$ and thus that if the inital velocity $u(0) \in H^s(\mathbb{R}^3)$, then $u(t) \in H^s(\mathbb{R}^3)$ for $0 < t \leq T$. In other words, it is argued that initial smoothness is preserved modulo the exponential growth factor $\exp(CC(\omega))$. The non blowup criterion (1) thus states that if $\|\omega(t)\|_\infty$ does not blow up to infinity as required for violation of (1), then neither the stronger norms $\|u(t)\|_s$ for $s \geq 3$ blow up to infinity for $0 \leq t \leq T$.

In [16] we questioned the relevance of this conclusion with the argument that since an exponential factor can be very large even if the exponent is not large (compare googol $= 10^{100}$ with 100), it is not clear that (2) implies smoothness in the sense that $\|u(t)\|_s$ is bounded if $\|u(0)\|_s$ is bounded. A statement about smoothness as boundedness of derivatives, can obviously be questioned under multiplication by googol.

There is another aspect of (1), which is even more cumbersome, namely that it seems in a sense to be void of application: There is strong evidence that (1) is never true, because all Euler solutions blow up for T non small [16]. Thus the implication (1) $\Rightarrow$ (2) seems to be empty. We meet here the same situation as with the computational non-blowup criterion for Navier-Stokes solutions presented in [4], which also seems void of application in the case of small viscosity [16]. Any criterion for non-blowup would be vacous, if there is always blowup. Of course, one could argue that (1) can be turned into a criterion for blowup of the form $C(\omega) = \infty$, which however, would be trivial in the sense that if $\omega = \nabla \times u$ blows up, then so does a derivative of u, namely $\nabla \times u$. Motivated by a blowup condition of the form $C(\omega) = \infty$, considerable effort has gone into finding Euler solutions developing infinite vorticity, however without any definitive results [7, 12].

3 Illposedness and Blowup

In [16] we pointed to the fact that since Hadamard [9], it is well understood that solving differential equations such as the Euler equations, perturbations of data (forcing and initial/boundary values) have to be taken into account. If a vanishingly small perturbation can have a major effect on a solution, then the solution (or problem) is *illposed*, and in this case the solution may not carry any meaningful information and thus may be meaningless from both mathematical and applications points of view. According to Hadamard, only a *wellposed* solution, for which small perturbations have small effects (in some suitable sense), can be meaningful. In this perspective it is remarkable that wellposedness is not an issue in the Millenium Problem formulation [8]. It may be connected to a common misinterpretation of wellposedness as "continuous dependence" with the essential quantitative aspect of requiring small effects of small perturbations being lost.

It is thus natural to view the bound (2) with a very large exponential factor, rather as an expression of illposedness and blowup, than non-blowup: If $\|u(T)\|_s = 10^{100}$ while $\|u(0)\|_s = 100$, then initial data shows blowup by a factor 10^{98}, which expresses (extreme) illposedness in the H^s-norm.

The objective of this note is to substantiate that there is a close connection between blowup and illposedness. This opens to detecting blowup by computationally detecting strong perturbation growth, again as in [16] on meshes of finite mesh size.

On the other hand, seekingas in [12] to detect blowup by computationally showing that $C(\omega) = \infty$, in principle requires an infinitely small mesh size capturing a vorticity tending pointwise to infinity, which is impossible to realize. Accordingly, blowup is not detected in [12], despite that double exponential growth of the vorticity is discovered, which we argue is an indication illposedness and blowup.

4 The Euler Blowup Problem

The Euler equations express conservation of mass, momentum and total energy of a fluid with vanishingly small viscosity (inviscid fluid). In the case of *compressible* flow, it is well known that initially smooth solutions to the Euler equations in general develop into discontinuous *shock solutions* in finite time. Such shock solutions thus exhibit *blowup* in the sense that they have infinitely large derivatives and *Euler residuals* at the shock violating the Euler equations pointwise. The formation of shocks shows *non-existence of pointwise solutions* to the compressible Euler equations. Concepts of *weak solution* have been developed accomodating after-blowup shock solutions with Euler residuals being large in a strong (pointwise) sense and vanishingly small in a weak sense, but both the existence and uniqueness of weak solutions represent open problems since long.

Regularized Euler equations are augmented by a viscous term with small positive viscosity coefficient with the effect that the blowup to infinity of derivates and Euler residuals is curbed. Existence of pointwise solutions to suitably regularized Euler equations, referred to as *viscosity solutions*, follows by standard analytical techniques, see [6, 14]. *Viscosity solutions* have pointwise (strongly) large and weakly small Euler residuals as a reflection of the non-existence of pointwise (strong) solutions to the Euler equations.

Proving convergence of viscosity solutions to weak solution limits of the Euler equations under vanishing viscosity, has remained a main challenge to analytical mathematics since the 1950s, but the progress has been limited to model problems; the difficulty is related to the non-existence of pointwise solutions and lack of viscosity in the Euler equations. Accordingly we have proposed as a possibly more fruitful object of mathematical study *wellposedness of viscosity* *solutions under vanishing viscosity*, that is, what outputs of viscosity solutions are wellposed under vanishing viscosity [13].

Incompressible flow does not form shocks and one may ask if initially smooth solutions of the incompressible Euler equations exhibit blowup or not, which is the Millennium Problem in the case of vanishing viscosity. The existing literature, see [6, 10, 11, 12] and references therein is not decisive and evidence for both blowup and not blowup is presented. The study has further been limited in time to before blowup, discarding the highly relevant question of what happens after blowup.

In [16] we presented evidence of blowup for the incompressible Euler equations drawing from our recent work [13] and references therein, widening the study to both before and after blowup. We computed specific viscosity solutions by a *least squares residul-stabilized finite element method* referred to as Euler General Galerkin or *EG2*. We detected wellposedness of mean-value outputs such as drag and lift (coefficients). We found that the phenomenon of *turbulence* in incompressible flow, plays a similar role in blowup as that of shock formation in compressible flow: Initially smooth solutions of regularized incompressible Euler equations in general in finite time show blowup into *turbulent solutions*, characterized by pointwise large (weakly small) Euler residuals and substantial *turbulent dissipation*. We gave evidence that the blowup into turbulence results from pointwise instability, forcing smooth solutions to develop into turbulent solutions, as a parallel to the inevitable shock formation in compressible flow.

We detected blowup by increasing space-time L_2-norms of EG2 Euler residuals with decreasing mesh size, and gave evidence that blowup can be detected by computation with finite mesh size. We also included the process after blowup. We referred to this approach as *global blowup*, as compared to *local blowup* based on (1) used in [12]. We thus avoided the seemingly impossible task of a providing a precise analysis of the route to local blowup of a smooth exact Euler solution. Instead we observed initial smooth potential flow develop into turbulent flow identified by increasing L_2-residuals under decreasing mesh size. In this approach there is no pointwise unique route to blowup with a unique blowup time, since the transition to turbulence feeds on the mesh-dependent residual perturbations in EG2 computation. We showed that the transition to turbulence in potential flow is driven by exponential perturbation growth in time with corresponding logarithmic growth in the mesh size of the effective time to transition. We thus studied global blowup of EG2/viscosity solutions under decreasing mesh size/viscosity including wellposedness, and not as in [6, 10, 11, 12] local blowup of exact Euler solutions without wellposedness.

5 Blowup Detection for Burgers Solutions

As an instructive model of the Euler equations, we consider *Burgers' equation*: Find the scalar function $u(x,t)$ defined on $\mathbb{R} \times [0,T]$ such that

$$\begin{aligned} \dot{u} + uu' = 0 \quad \text{in } \mathbb{R} \times (0,T], \\ u(0,x) = u^0(x), \end{aligned} \tag{3}$$

where u^0 is a given initial value (with compact support) and $\dot{u} = \frac{\partial u}{\partial t}$ and $u' = \frac{\partial u}{\partial x}$. It is well known that all initially smooth solutions in finite time develop into piecewise smooth discontinuous *shock solutions* with (a finite number of) jump discontinuities (with the left hand limit larger than the right hand limit). All initially smooth solution thus exhibit blowup into non-smooth shock solutions.

Wellposedness of a Burgers' solution u is governed by stability properties of the *linearized Burgers' equation*:

$$\begin{aligned} \dot{v} + uv' + u'v = 0 \quad \text{in } \mathbb{R} \times (0,T], \\ v(0,x) = v^0(x), \end{aligned} \tag{4}$$

where v represents a perturbation. This is a linear convection-reaction problem with convection coefficient u and reaction coefficient u'. The solution u is said to be *wellposed* in the L_2-norm $\|\cdot\|_0$ if for $t > 0$

$$\|v(\cdot,t)\|_0 \leq K\|v^0\|_0, \tag{5}$$

where K is of moderate size. If K is exponentially large ($K \geq \exp(C)$ with C of moderate size, e.g. $K = 10^{100}$ with $C = 100$), then the solution u is *illposed* (in L_2).

To study the well/illposedness of a Burgers solution u, we multiply the linearised equation (4) by v, and integrate in space to obtain

$$\frac{d}{dt}\|v(\cdot,t)\|_0^2 + \int_{\mathbb{R}} u'(x,t)v^2(x,t)\,dx = 0, \tag{6}$$

which shows that the growth of $\|v(\cdot,t)\|^2$ directly connects to the reaction coefficient $u'(x,t)$. If $u' \geq 0$ everywhere, there is only decay, while if somewhere $u'(x,t) < 0$, then exponential growth is possible. By a Gronwall estimate, we have

$$\|v(T)\|_0^2 \leq \exp(C(u'))\|v(0)\|_0^2, \tag{7}$$

where

$$C(u') = \int_0^T \|u'_-(\cdot,t)\|_\infty dt, \tag{8}$$

with u'_- the negative part of u'. We thus may expect for the corresponding perturbation growth

$$\frac{\|v(T)\|_0}{\|v(0)\|_0} \approx \exp(C(u')/2) \tag{9}$$

if the support of v suitably overlaps with the region where $-u'_-$ attains its maximum. In particular we expect strong perturbation growth in L_2 at shocks with $-u' >> 1$.

The linearized Burgers equation also governs the smoothness of Burgers solutions: For example, differentiating Burgers equation with respect to x leads to the linearized equation (4) with $v = u'$. For a shock solution (9) then expresses the strong growth of u' as a shock is forming and initial smoothness is lost. Other derivatives are handled similarly [1, 17].

This analysis shows that there is a direct coupling between shock development, blowup and illposedness: As a shock is forming from smooth initial data, $-u'$ blows up to infinity along with perturbations expressing illposedness. This identification makes it possible to detect blowup of a solution u in two different ways: (i) by shock development and (ii) by illposedness of the linearized equations. In both cases, the key question is if the detection can be made computationally on a mesh of finite mesh size.

We recall the discussion of (i) in [16]: Consider the stationary non-smooth shock solution $u(x) = 1$ for $x < 0$ and $u(x) = -1$ for $x > 0$, which can develop from smooth initial data. Let $h > 0$ and define $U_h(x) = 1$ for $x < -h$, $U_h(x) = -\frac{x}{h}$ for $-h \leq x \leq h$, $U_h(x) = -1$ for $x > h$, represent a corresponding computational (continuous piecewise linear finite element) solution on a mesh with mesh size h. We have

$$\int_{-h}^{h} (U_h U_h')^2 dx = \frac{2}{3h}$$

and thus the L_2-norm of the Burgers residual $U_h U_h'$ scales like $h^{-1/2}$. Detecting $h^{-1/2}$ increase of L_2-residuals under decreasing mesh size would then be identified with blowup into a shock, since in smooth parts the residual would decrease like h. Apparently, the shock would be correctly detected with a finite mesh size.

We may compare with detecting shocks by local blowup, resolving the flow pointwise as the shock is forming. This requires mesh refinement without limit, and like Achilles will never reach the goal.

We compare with a *rarefaction wave* $u(x,t) = \frac{x}{t}$ for $|x| \leq t$, $u(x,t) = -1$ for $x < -t$, $u(x,t) = 1$ for $x > t$, with corresponding initial data $u^0(x) = -1$ for $x < 0$ and $u^0(x) = 1$ for $x > 0$. In this case $u' \geq 0$, showing that $\|v(t)\| \leq \|v(0)\|$ indicating wellposedness and non-blowup. In this case we can assume $U_h = u$, because $u(\cdot, t)$ is continuous piecewise linear, and thus the Euler residual will be small and also indicate non-blowup.

More generally, to accurately detect a shock in a Burgers solution u, the smooth part of u has to be resolved in order to correctly single out a shock from a smooth part. Now, the general structure of Burgers solutions as being piecewise smooth with jumps allows detection with a finite mesh size. The structure of Burgers solutions with a resolvable smooth part and sharp shocks with no smallest scale, thus makes blowup detection possible without requiring the mesh size to be infinitely small. Thus beyond the resolvable scale of the smooth part there are can be no surprises to be found by decreasing the mesh size, since all there is are shocks separated by smooth parts.

Computational blowup detection by (ii) is based on solving the linearized equation with a suitable perturbation designed to capture strong perturbation growth. Since the local perturbation growth is governed by $-u'_-(x,t)$, we expect to find increasing local perturbation growth with mesh refinement since $-u'_-(x,t)$ is increasing locally with decreasing mesh size.

We can now combine (i) and (ii) into a double criterion of blowup based on increasing L_2-residuals and perturbation growth with decreasing mesh size, for which finite mesh size can be sufficient.

In [14] we show by a refined stability analysis of the linearized problem, that in EG2 computation a shock is in fact wellposed, because the Galerkin orthogonality of EG2 give EG2 perturbations a special quality escaping from the illposedness of general L_2-perturbations.

6 The Incompressible Euler Equations

We now turn to detection of blowup of solutions of the incompressible Euler equations expressing conservation of momentum and mass of an incompressible inviscid fluid enclosed in a volume Ω in $\mathbb{R}^3$ with boundary Γ : Find the velocity $u = (u_1, u_2, u_3)$ and pressure p depending on $(x, t) \in \bar{\Omega} \times I$ with $\bar{\Omega} = \Omega \cup \Gamma$, such that

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u + \nabla p &= f && \text{in } \Omega \times I, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times I, \\ u \cdot n &= g && \text{on } \Gamma \times I, \\ u(\cdot, 0) &= u^0 && \text{in } \Omega, \end{aligned} \tag{10}$$

where n denotes the outward unit normal to Γ, f is a given volume force, g is a given inflow/outflow velocity, u^0 is a given initial condition, $\dot{u} = \frac{\partial u}{\partial t}$ and $I = (0, T]$ a given time interval. We notice the *slip boundary condition* expressing inflow/outflow with zero friction.

7 Exponential Instability

Subtracting the Euler equations for two solutions (u, p) and $(\bar{u}, \bar{p})$ with corresponding (slightly) different data, we obtain the following linearized equation for the difference $(v, q) \equiv (u - \bar{u}, p - \bar{p})$:

$$\begin{aligned} \dot{v} + (u \cdot \nabla)v + (v \cdot \nabla)\bar{u} + \nabla q &= f - \bar{f} && \text{in } \Omega \times I, \\ \nabla \cdot v &= 0 && \text{in } \Omega \times I, \\ v \cdot n &= g - \bar{g} && \text{on } \Gamma \times I, \\ v(\cdot, 0) &= u^0 - \bar{u}^0 && \text{in } \Omega. \end{aligned} \tag{11}$$

With u and $\bar{u}$ given, this is a linear convection-reaction problem for (v, q) with the reaction term given by the 3×3 matrix $\nabla \bar{u}$, similar to the linearized Burgers equation. By the incompressiblity, the trace of $\nabla \bar{u}$ is zero, which shows that in general $\nabla \bar{u}$ has eigenvalues with real value of both signs, of the size of $|\nabla \bar{u}|$ (with $|\cdot|$ som matrix norm), thus with at least one exponentially unstable eigenvalue. Thus we expect to generically find exponential perturbation growth with exponent $|\nabla u|$ and thus illposedness.

Birkhoff questions in [2] if there is any stable Euler solution, but gets a devastating review in [18]. Fefferman states in [8]: "Many numerical computations appear to exhibit blowup for solutions of the Euler equations, but the extreme numerical instability of the equations makes it very hard to draw reliable conclusions". It is natural to ask why Fefferman views "the extreme numerical instability of the equations", not as a sign of instability or illposedness and blowup, but only as an obstacle to conclusion.

8 Gronwall Stability Estimates

Multiplying the momentum equation of (11) by v, assuming $\bar{f} = f$ and $\bar{g} = g$, and integrating in space, we obtain an estimate of the form

$$\frac{dw}{dt} \leq \|\nabla \bar{u}\|_\infty w \quad \text{on } I,$$

where $w(t) = \|v(t)\|^2_{L_2(\Omega)}$ and $\|\cdot\|_\infty$ the $L_\infty(\Omega)$-norm. By integration in time we obtain the following analogue of (8):

$$w(T) \leq \exp(C(\nabla\bar{u}))w(0),$$

where

$$C(\nabla\bar{u}) = \exp(\int_0^T \|\nabla\bar{u}(t)\|_\infty \, dt), \tag{12}$$

replaces the constant C_ω appearing in (2), c.f. [17].

9 The Vorticity Equation

Formally applying the curl operator $\nabla\times$ to the momentum equation we obtain the vorticity equation

$$\dot{\omega} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nabla\times f \quad \text{in } \Omega, \tag{13}$$

which is a convection-reaction equation in the vorticity $\omega = \nabla\times u$ with coefficients depending on u, of the same form as the linearized equation (11), with similar properties of exponential instability referred to as *vortex stretching*. The vorticity is thus locally subject to exponential growth with exponent $|\nabla u|$.

In classical analysis it is often argued that from the vorticity equation (13), it follows that vorticity cannot be generated starting from potential flow with zero vorticity and $f = 0$, which is *Kelvin's theorem.* But this is an incorrect conclusion, since perturbations of $\bar{f}$ of f with $\nabla\times\bar{f} \neq 0$ must be taken into account. What you effectively see in computations is local exponential growth of vorticity in vortex stretching, even if $\nabla\times f = 0$, which is a main route of instability to turbulence.

10 Viscous Regularization

We define the *Euler residual* by

$$R(u,p) \equiv \dot{u} + (u\cdot\nabla)u + \nabla p - f, \tag{14}$$

which is the residual of the momentum equation, assuming for simplicity that the incompressiblity equation $\nabla\cdot u = 0$ is not subject to perturbations. *The regularized Euler equations* take the form: Find (u_ν, p_ν) such that

$$\begin{array}{rcll} R(u_\nu,p_\nu) & = & -\nabla\cdot(\nu\nabla u_\nu) & \text{in } \Omega\times I, \\ \nabla\cdot u_\nu & = & 0 & \text{in } \Omega\times I, \\ u_\nu\cdot n & = & g & \text{on } \Gamma\times I, \\ u_\nu(\cdot,0) & = & u^0 & \text{in } \Omega, \end{array} \tag{15}$$

where $\nu > 0$ is a small *viscosity*, together with a homogeneous Neumann boundary condition for the tangential velocity. Notice that we keep the slip boundary condition $u_\nu\cdot n = g$, which eliminates viscous Dirichlet no-slip boundary layers. The turbulence we discover thus does not emanate from viscous no-slip boundary layers (which is a common misconception). We consider here for simplicity a standard ad hoc regularization and return to computational regularization below. Existence of a pointwise solution (u_ν, p_ν) of (15) (allowing ν to have a certain dependence on $|\nabla u|$), follows by standard techniques, see e.g. [6]. Notice that the Euler residual $R(u_\nu, p_\nu)$ equals the viscous term $-\nabla \cdot (\nu \nabla u_\nu)$, which suggests an interpretation of the viscous term in the form of the Euler residual.

The standard *energy estimate* for (15) is obtained by multiplying the momentum equation with u_ν and integrating in space and time, to get in the case $f = 0$ and $g = 0$,

$$\int_0^t \int_\Omega R(u_\nu, p_\nu) \cdot u_\nu \, dxdt = D(u_\nu; t) \equiv \int_0^t \int_\Omega \nu |\nabla u_\nu(s, x)|^2 dxds, \tag{16}$$

from which follows by standard manipulations of the left hand side,

$$K(u_\nu(t)) + D(u_\nu; t) = K(u^0), \quad t > 0, \tag{17}$$

where

$$K(u_\nu(t)) = \frac{1}{2} \int_\Omega |u_\nu(t, x)|^2 dx.$$

This estimate shows a balance of the *kinetic energy* $K(u_\nu(t))$ and the *viscous dissipation* $D(u_\nu; t)$, with any loss in kinetic energy appearing as viscous dissipation, and vice versa. In particular $D(u_\nu; t) \leq K(0)$ and thus the viscous dissipation is bounded (if $f = 0$ and $g = 0$).

Turbulent solutions of (15) are characterized by *substantial turbulent dissipation*, that is (for t bounded away from zero),

$$D(t) \equiv \lim_{\nu \to 0} D(u_\nu; t) >> 0. \tag{18}$$

That a positive limit (~ 1) exists is *Kolmogorov's conjecture*, which is consistent with

$$\|\nabla u_\nu\|_0 \sim \frac{1}{\sqrt{\nu}}, \quad \|R(u_\nu, p_\nu)\|_0 \sim \frac{1}{\sqrt{\nu}}, \tag{19}$$

where $\|\cdot\|_0$ denotes the $L_2(Q)$-norm with $Q = \Omega \times I$. On the other hand, it follows by standard arguments from (17) that

$$\|R(u_\nu, p_\nu)\|_{-1} \leq \sqrt{\nu}, \tag{20}$$

where $\|\cdot\|_{-1}$ is the norm in $L_2(I; H^{-1}(\Omega))$. Kolmogorov thus conjectures that the Euler residual $R(u_\nu, p_\nu)$ is strongly (in L_2) large, while being small weakly (in H^{-1}).

11 EG2 Regularization

An EG2 solution (U, P) on a mesh with local mesh size $h(x, t)$ according to [13], satisfies the following energy estimate (with $f = 0$ and $g = 0$):

$$K(U(t)) + D_h(U; t) = K(u^0), \tag{21}$$

where

$$D_h(U; t) = \int_0^t \int_\Omega hR(U, P)^2 \, dxdt, \tag{22}$$

is an analog of $D(u_\nu; t)$ with $h \sim \nu$. We see that the EG2 viscosity arises from penalization of a non-zero Euler residual $R(U, P)$ with the penalty directly connecting to the violation (according the theory of criminology). A turbulent solution is characterized by substantial dissipation $D_h(U; t)$ with $\|R(U, P)\|_0 \sim h^{-1/2}$, and

$$\|R(U, P)\|_{-1} \leq \sqrt{h} \tag{23}$$

in accordance with (19) and (20).

EG2 explains the occurence of viscous effects in Euler solutions in a new way, not simply assuming ad hoc that "there is always some small constant shear viscosity", but from the *impossibility of pointwise exact conservation of momentum.* EG2 viscosity is not a simple constant shear viscosity but rather a solution dependent bulk (or streamline) viscosity [13, 14].

12 Wellposedness of Mean-Value Outputs

Let $M(v) = \int_Q v\psi dxdt$ be a *mean-value output* of a velocity v defined by a smooth weight-function $\psi(x, t)$, and let (u, p) and (U, P) be two EG2-solutions on two meshes with maximal mesh size h. Let (φ, θ) be the solution to the *dual linearized problem*

$$\begin{array}{rcll} -\dot{\varphi} - (u \cdot \nabla)\varphi + \nabla U^\top \varphi + \nabla \theta & = & \psi & \text{in } \Omega \times I, \\ \nabla \cdot \varphi & = & 0 & \text{in } \Omega \times I, \\ \varphi \cdot n & = & g & \text{on } \Gamma \times I, \\ \varphi(\cdot, T) & = & 0 & \text{in } \Omega, \end{array} \tag{24}$$

where $\top$ denotes transpose. Multiplying the first equation by $u - U$ and integrating by parts, we obtain the following output error representation [13, 14]

$$M(u) - M(U) = \int_Q (R(u, p) - R(U, P)) \cdot \varphi \, dxdt \tag{25}$$

from which follows the *a posteriori error estimate*

$$|M(u) - M(U)| \leq S(\|R(u, p)\|_{-1} + \|R(U, P)\|_{-1}), \tag{26}$$

where the stability factor

$$S = S(u, U, M) = S(u, U) = \|\varphi\|_{H^1(Q)}. \tag{27}$$

In [13] we presented a variety of evidence, obtained by computational solution of the dual problem, that for global mean-value outputs such as drag and lift, $S << 1/\sqrt{h}$, while $\|R\|_{-1} \sim \sqrt{h}$ in conformity with (20). This allows an EG2 solution (U, P) to pass a wellposedness test of the form

$$S(U, U)\|R(U, P)\|_{-1} \leq TOL \tag{28}$$

for tolerances $TOL > 0$ and mesh sizes h of interest, because $S(U, U)$ of moderate size.

A crude analytical stability analysis of the dual linearized problem (24) using Gronwall type estimates, indicates that the dual problem is pointwise exponentially unstable because the reaction coefficient ∇U is locally very large, in which case effectively $S(U, U) = \infty$. This is consistent with massive observation that point-values of turbulent flow are non-unique or illposed.

On the other hand we observe computationally that S is not large for mean-value outputs of turbulent solutions. We explain in [13] this remarkable fact as an effect of *cancellation* from the following two sources:

(i) rapidly oscillating reaction coefficients of turbulent solutions,

(ii) smooth data in the dual problem for mean-value outputs.

For a laminar solution there is no cancellation, and therefore not even mean-values are unique. This is d'Alembert's paradox: A potential laminar solution has zero drag, while an arbitrarily small perturbation will turn it into a turbulent solution with substantial drag. The drag of a laminar solution is thus non-unique in the sense that an arbitrarily small perturbation will change the drag substantially. The stability factor is infinite for a laminar solution because of lack of cancellation [13].

13 Blowup Detection for Euler Solutions

To detect blowup in general should be an easier task than to accurately compute a wellposed output, because blowup can be viewed as a very crude output. We now consider the problem of detecting blowup of an incompressible Euler solution according to (i) and (ii) starting with (i).

Massive evidence indicates that incompressible Euler solutions have a general structure simular to that of a Burgers solution, with a smooth (laminar) part resolvable with finite mesh size combined with a turbulent part with no smallest scale and thus unresolvable on all meshes. Again we seem to have a dichotomy into a resolvable and an unresolvable part, allowing accurate blowup detection with finite mesh size/viscosity. Massive evidence indicates that a flow becomes partly turbulent if the *Reynolds number* Re is large enough, where $Re = \frac{UL}{\nu}$ with U a representative velocity, L a length scale and ν the viscosity. In particular, there is massive evidence that if a flow is partly turbulent for a particular Reynolds number, it will remain so under decreasing viscosity and increasing Reynolds number. This can be motivated by rescaling decreasing the length scale with the viscosity (thus focussing on a part of the fluid domain) keeping the Reynolds number constant and thus expecting the flow to remain turbulent.

This seems to open the possibility of detecting blowup of Euler solutions from finite mesh size computation, by detecting $\|R(U, P)\|_0 \sim h^{-1/2}$. This is possible partly because EG2 solutions satisfy slip boundary conditions and thus lack the thin boundary layers of slightly viscous Navier-Stokes solutions satisfying no-slip boundary conditions, which are unresolvable on any foreseeable computer. This connects back to Eulers original idea of the Euler equations as a useful model of slightly viscous flow, as developed in detail in [13], in contrast to the legacy of Prandtl discarding Euler solutions because they do not satisfy no-slip boundary conditions, thus requiring the resolution of thin boundary layers of Navier-Stokes solutions, which however is impossible. The Euler equations thus are useful, because solutions can be computed and can provide information, while the Navier-Stokes equations seem less useful, because solutions cannot be computed.

Using the approach (ii), we compute stability factors (by solving the dual problem computationally) with different data corresponding to different outputs. As shown in [13] we then find stability factors increasing much more quickly for pointwise outputs than for mean-value outputs with decreasing mesh size, and we can use the quick increase for pointwise outputs as a sign of blowup.

Altogether, we can combine (i) and (ii) to a double criterion of blowup, which can be tested computationally on a sequence of meshes with decreasing finite mesh size.

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