BLOWUP OF INCOMPRESSIBLE EULER SOLUTIONS

Johan Hoffman and Claes Johnson

www.bodysoulmath.org, www.fenics.org, www.csc.kth.se/cgjoh

D'ALEMBERT: CLAY: FLYING

- D'ALEMBERT'S PARADOX 1752:
  - ZERO DRAG/LIFT OF INVISCID FLOW!?
- HOW TO FLY? LIFT: KUTTA-ZHU 1903
  - LIFT/DRAG: PRANDTL 1904
- CLAY 2000 MILLENNIUM PROBLEM:
  - EXIST SMOOTH SOL OR BLOWUP
  - NAVIER-STOKES/EULER?

COST OF SEPARATION

- FLOW AROUND A BODY: ATTACH–SEPARATE
- DIVORCE:
  - LEAVING IDEAS-IDEALS
  - LEAVING PROPERTY-POWER
- GETTING OLDER:
  - BEYOND 60....

HAPPY (NON-SEPARATED) MAN!

CLAY PROBLEM: BLOWUP?

- NAVIER-STOKES/EULER (INCOMP)

(I) EXISTENCE SMOOTH SOL ALL DATA?

- or

(II) BLOWUP FOR SPECIFIC DATA?

- ALL DATA: PROOF(I) ANALYTICAL!
- SPECIFIC DATA: PROOF(II) COMPUT???
- ONE MILLION DOLLAR!

BLOWUP OR NON-BLOWUP?

- BEALE-KATO-MAJDA: BLOWUP at T>0 IFF

$$\int_0^T \int_\Omega \|\omega(\cdot,t)\|_\infty \, dt = \infty \qquad (1984)$$

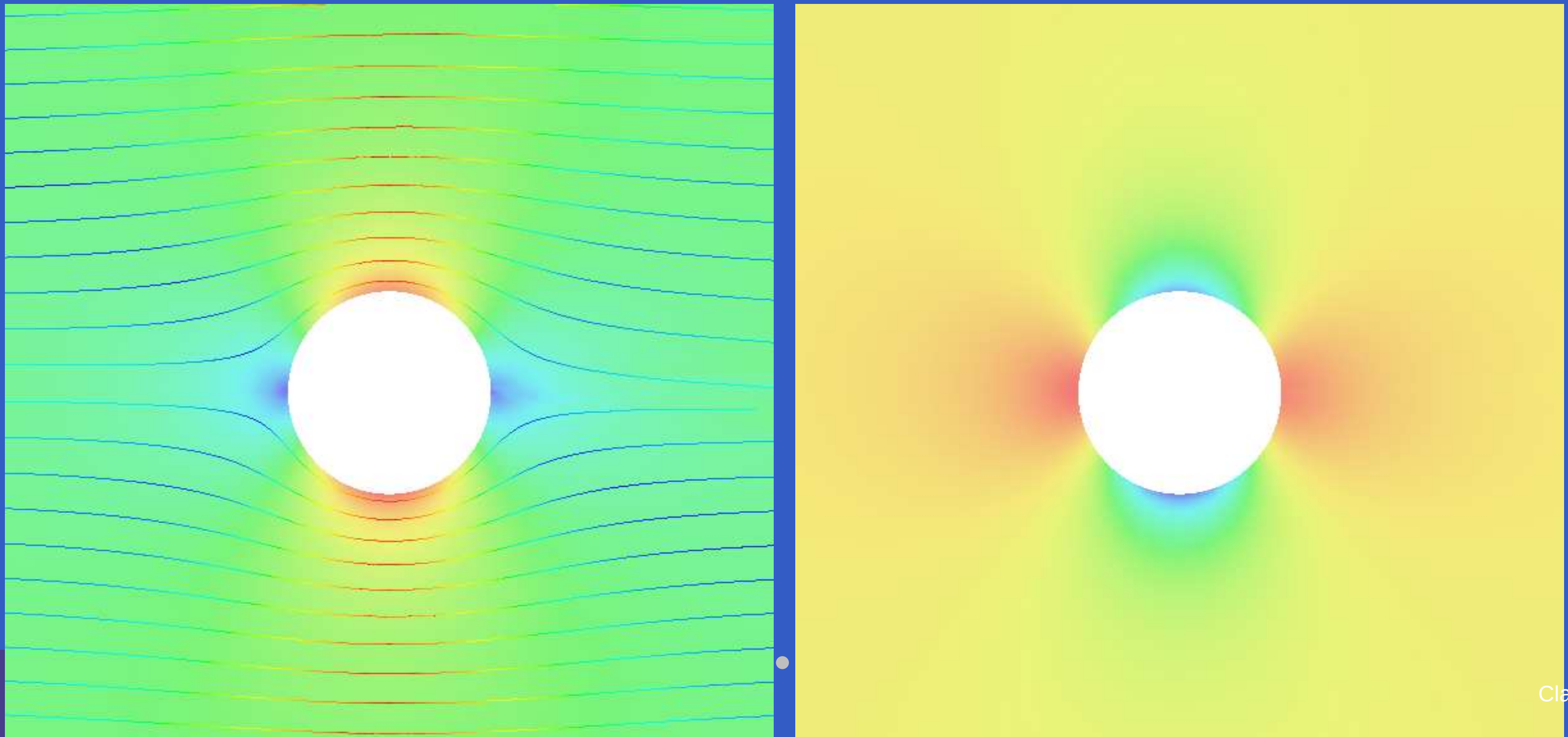
- KELVIN'S THEOREM (ω VORT)

$$\omega(\cdot,0) = 0 \quad \Rightarrow \quad \omega(\cdot,t) = 0 \quad t > 0.$$

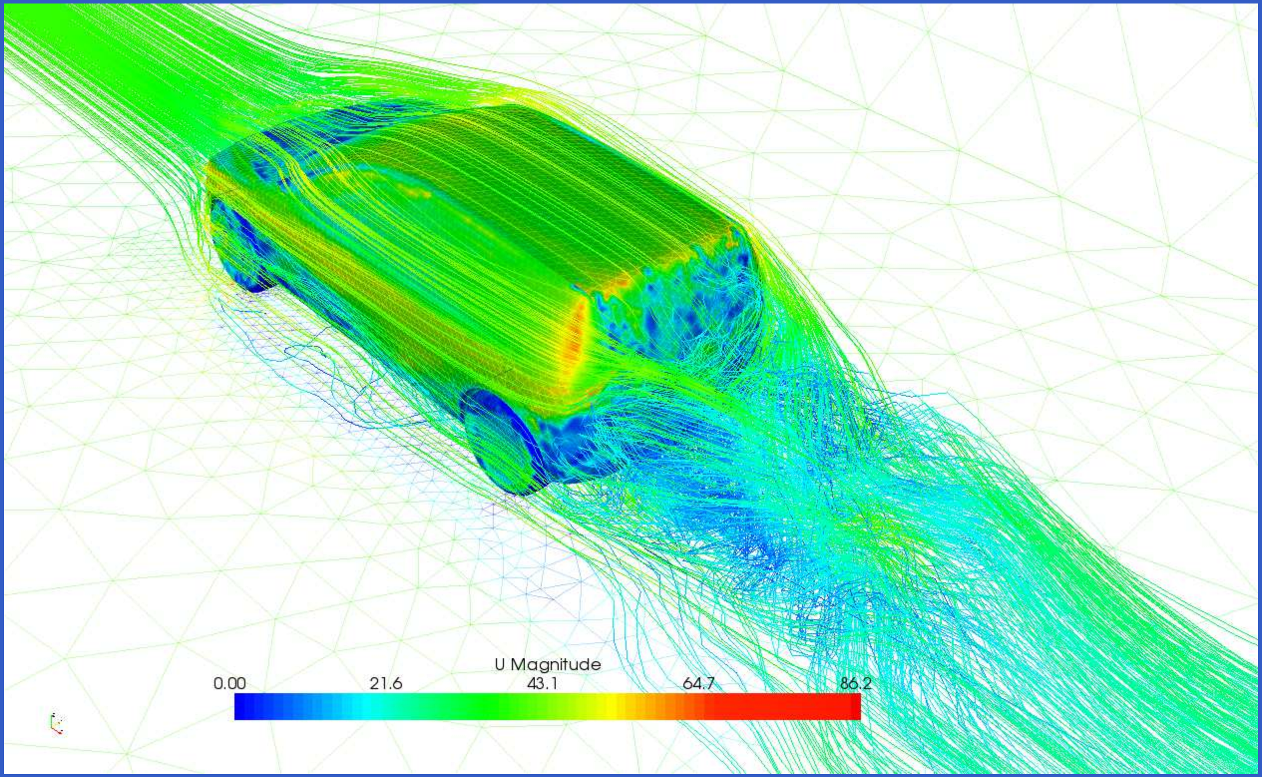
- KERR 1993-: COMPUT BLOWUP
- HOU 2006-: COMPUT NON-BLOWUP
- CONSTANTIN: MAIN PROBLEM ???

POT SOL CIRC CYL: DRAG $= 0$

- INCOMPR INVISC IRROT EULER SOL SLIP
- DRAG $=$ 0! D'ALEMBERT'S PARADOX
- NON-BLOWUP ACC BKM



VOLVO CAR: BLOWUP EULER



- DRAG ≈ 0.33, SLIP BC
- TURBULENT INCOMPRESSIBLE EULER

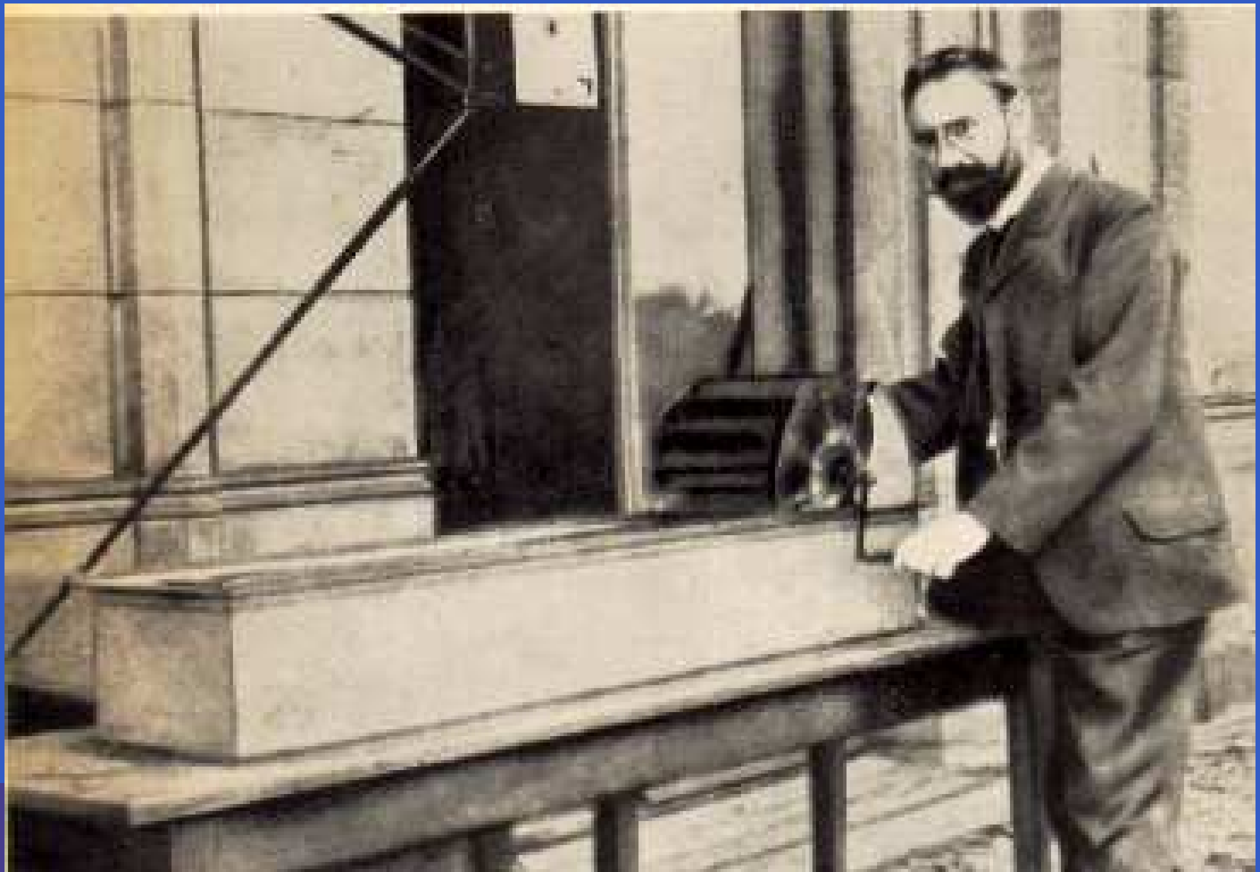
IMPOSSIBLE – POSSIBLE

- MOIN: IMPOSSIBLE: $> 10^{16}$ MESH POINTS
- HENNINGSON KTH: WILL TAKE 50 YEARS
- DAVIDSON CHALMERS: IMPOSSIBLE
- BUT POSSIBLE!! WITH 10^6 MESH POINTS

- TURBULENT INCOMPRESSIBLE EULER
- SLIP BC

WHAT’S WRONG WITH POT SOL?

- PRANDTL (FATHER FM) 1904: SLIP BC:
- DRAG/LIFT FROM NO-SLIP BOUND LAYER



WHAT'S WRONG WITH PRANDTL?

- NO-SLIP BC
- MAIN DRAG NOT FROM BOUND LAYER
- SKIN FRICTION SMALL
- ANOTHER SOURCE OF DRAG: WHAT?
- ONE-MILLION DOLLAR QUESTION!

BLOWUP=NON-SMOOTH=TURB

- SMOOTH:
  - SMALL EFFECT OF REGULARIZATION
- NON-SMOOTH:
  - LARGE EFFECT OF REG
  - = TURBULENT
  - = BLOWUP

FINITE MESH SIZE COMP

- BLOWUP $=$ TURB!
- DETECT TURB ON MESH SIZE H!
- PROVES BLOWUP FOR ALL $h < H$?
- TURB $RE_H = 1/H$
- $\Rightarrow$ TURB FOR ALL $RE > RE_H$?!
- TURB NOT DISAPPEAR INCREAS RE!
- BUCKLING: NOT DISAPP INCREAS LENGTH

COMPARE:

- LAMINAR FOR H or RE_H
- $\Rightarrow$ LAMINAR ALL $h < H$, $RE > RE_H$?
- FALSE!

MORAL:

- DETECT NON-SMOOTH ON FIN MESH
- CANNOT DETECT SMOOTH ON FIN MESH
- CLAY PROBLEM: DETECT NON-SMOOTH!!
- POSSIBLE ON FINITE MESH!
- DICHOTOMY: LAMINAR/SMOOTH – TURBULENT/NON-SMOOTH
- CF BURGERS: SMOOTH – SHOCKS

WELLPOSEDNESS

- HADAMARD 1902
- SMALL PERTURBATIONS
- $\Rightarrow$ SMALL EFFECT
- ON OUTPUT
- ONLY WELLPOSED MEANINGFUL
- NOT WELLPOSED NOT MEANINGFUL!!
- $\nu > 0$ SAME AS $\nu \geq 0$
- EULER INCLUDED IN CLAY NS PROBLEM!

EXACT SOL from APPROXIMATE

- IS THERE x: $D(x) = d$?
- $D(X) = d + R(X)$, X APPROX SOL
- $2S|R(X)| < TOL$
- $S = |D'(X)^{-1}|$
- $S|D'(y) - D'(X)| < \frac{1}{2}$ for $|x - X| < TOL$
- THEN $D(x) = d$ with $|x - X| < TOL$
- $|x - X| \leq 2S|R(X) - R(x)|$
- CONTRACTION MAP $x \to x - D'(X)^{-1}D(x)$

WELLPOSEDNESS

- $D(x) = d$ “EXACT” SOL x
- SUBJECT to PERTURBATION R
- $D(X) = d + R$
- X EXACT SOL of PERTURBED DATA
- X “AS GOOD” AS x IF
- $S = D'(X)^{-1}$ MODERATE SIZE!!
- TEST: WELLPOSEDNESS of COMPUTED X
- X REPRESENTATIVE SOLUTION

OUTPUT WELLPOSEDNESS

- $M(x)$ OUTPUT FUNCTIONAL WEIGHT ψ
- $M(X)$ WELLPOSED IF S MODERATE SIZE
- $|M(x) - M(X)| \leq S\|R(x) - R(X)\|_{-1}$
- $\|\cdot\|_{-1}$ WEAK NORM
- $S = \|\varphi\|_1$
- $D'(X)^\top \varphi = \psi$

EQ WITHOUT EXACT SOLUTIONS

- EULERS EQUATIONS IN FLUID MECH
- WELLPOS EXACT SOL DO NOT EXIST
- WELLPOS COMP SOLUTIONS DO EXIST!
- APPROX OF NON-EXISTING EXACT SOL!
- SIMULATIONS OF NON-EXISTING REALITY!

- HYPERREALITY

EQ WITHOUT EXACT SOLUTIONS

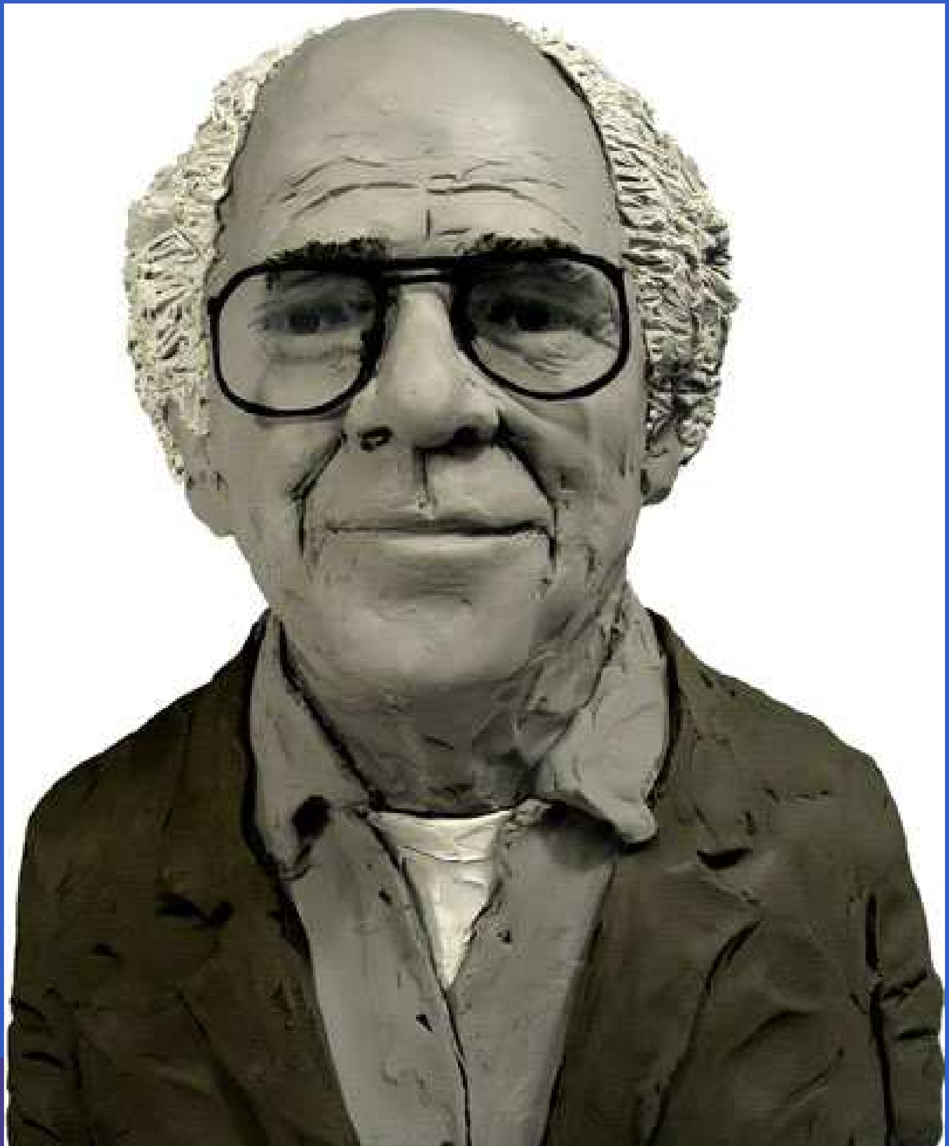
- SCHRÖDINGER'S
- KOHN NOBEL PRIZE 1998
- $\mathbb{R}^{3N}$ WAVE FUNCTION DOES NOT EXIST IF $N \geq 100$.
- SUPERPOSITION?? SCHRÖD CAT?
- HARTREE APPROX: N-SYSTEM in $\mathbb{R}^3$

- QM HYPERREALITY

BAUDRILLARD (1929-2007)

- REAL $=$ WHAT CAN BE SIMULATED
- HYPER-REAL $=$ WHAT IS SIMULATED
- SIMULATION of NON-EXIST REALITY
- MODELS of REAL without REAL ORIGIN
- MASKS NON-EXIST of REAL REALITY

SIMULATION of BAUDRILLARD



1ST-2ND ORDER SIMULATION

- BORGES
- EXACTITUDE in SCIENCE
- MAP COVERS TERRITORY

3RD ORDER SIM: HYPERREAL

- MAP REPLACES TERRITORY
- OUTSIDE REALM of GOOD and EVIL
- ONLY PERFORMATIVITY COUNTS
- CONTROL

BANK ROBBERY: GOOD-EVIL

- REAL: PUNISHED for BEING REAL
- SIMULATED: NOT PUNISHED for being SIMULATION
- SIMULATED: PUNISHED for UPSETTING JUDICIARY SYSTEM

DISNEYLAND

- IMAGE of
- AMERICAN SOC NEVER EXISTING
- MASKS NON-EXIST of REAL REALITY
- REPLACES REAL
- MODELS of WANTED REALITY

MAGRITTE

MODERN vs POSTMODERN

- MODERN: OBJ EXIST REAL WORLD
- POST-MODERN:
- HYPERREAL SIMULACRA of
- NON-EXIST REAL WORLD

DIJKSTRA

- Originallly I viewed it as the function of the ABSTRACT MACHINE to provide a truthful picture of the physical reality. Later, however, I learned to consider the abstract machine as the *TRUE* one, because that is the only one we can *THINK*; it is the PHYSICAL MACHINE's purpose to supply a *working model*, a (hopefully) sufficiently accurate physical SIMULATION OF THE TRUE ABSTRACT MACHINE.

HYPERREAL PHYSICS

- SPACE-TIME
- STATISTICAL MECHANICS
- QUANTUM MECHANICS

HYPERREAL PHYSICS

- SIMULATION of
- NON-EXISTING PHYSICS

- APPROXIMATIONS of
- NON-EXISTING EXACT SOLUTIONS

CLAY INST $\$1$ MILLION PRIZE

- WILL SHOW BLOWUP OF EULER SOLUTIONS
- NON-EXISTENCE OF EXACT SOLUTIONS
- HYPERREALITY

EULER EQUATIONS

- AIR/WATER: SMALL VISC ($= 0$)
- VELOCITY u PRESSURE p
- SLIP BC!

$$
\begin{aligned}
\dot{u} + u\cdot\nabla u + \nabla p &= f \quad \text{in } \Omega\times I \\
\nabla\cdot u &= 0 \quad \text{in } \Omega\times I \\
u\cdot n &= 0 \quad \text{on } \Gamma\times I \\
u(\cdot,0) &= u^0 \quad \text{in } \Omega
\end{aligned}
$$

LIN EQ: $v = u - \bar{u}$ WELLPOS?

$$
\begin{aligned}
\dot{v} + (u\cdot\nabla)v + (v\cdot\nabla)\bar{u} + \nabla q &= f - \bar{f} && \text{in } \Omega\times \\
\nabla\cdot v &= 0 && \text{in } \Omega\times \\
v\cdot n &= g - \bar{g} && \text{on } \Gamma\times \\
v(\cdot,0) &= u^0 - \bar{u}^0 && \text{in } \Omega.
\end{aligned}
\quad (1)
$$

- CONVECTION u REACTION $\nabla\bar{u}$
- TRACE $\nabla\bar{u} = \nabla\cdot\bar{u} = 0$
- UNSTABLE/STABLE EIGENVALUES
- EXP UNSTABLE IN RETARD

VORTICITY EQUATION

$$\dot{\omega} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nabla \times f \quad \text{in } \Omega, \qquad (2)$$

- EXPONENTIALLY UNSTABLE POINTW
- DIFFERENT SIGN OF REACT
- EXP UNSTABLE IN ACC

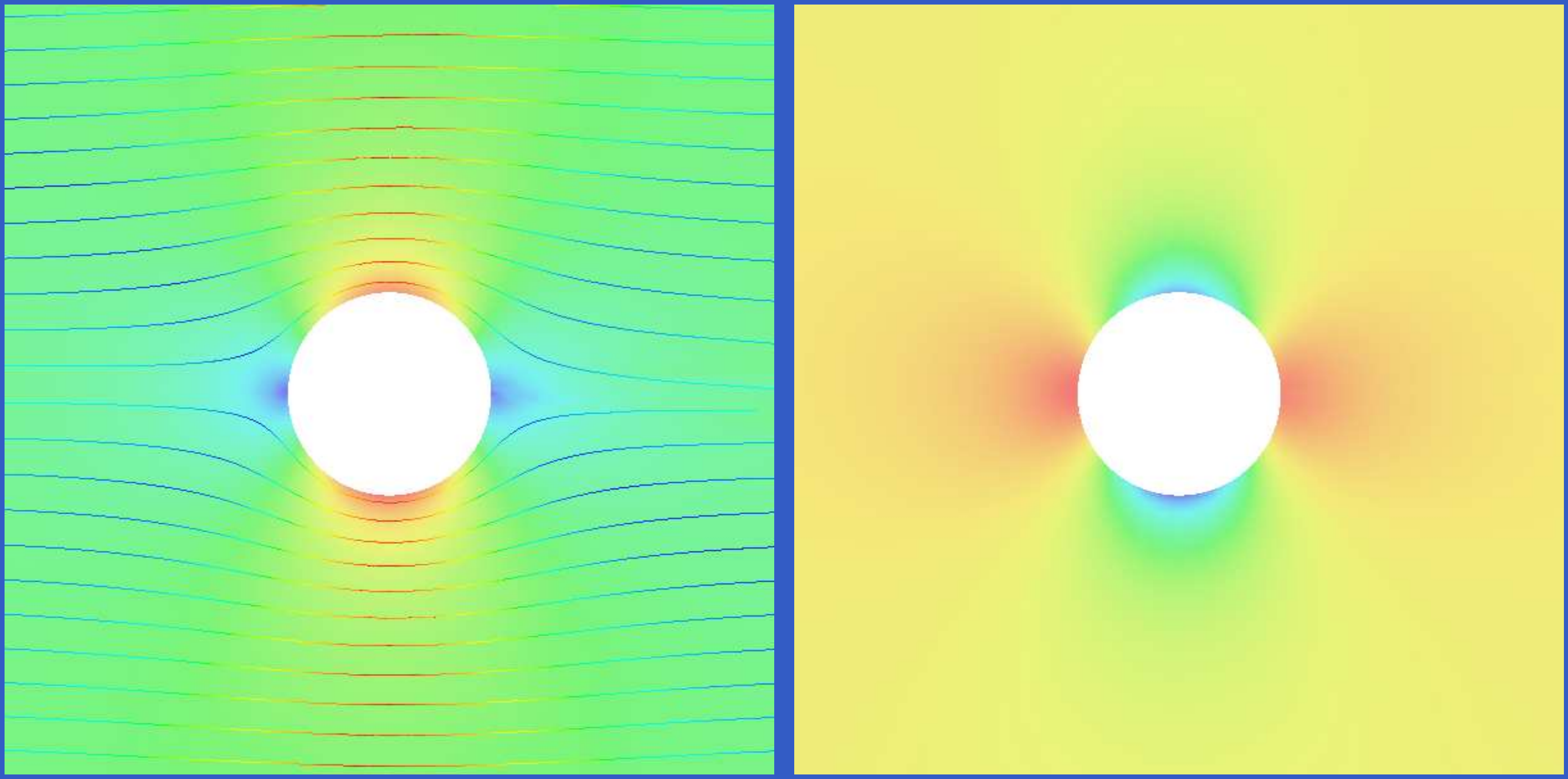
EG2 EULER GENERAL GALERKIN

- RESIDUAL LS STAB GALERKIN: hR^2
- MESH SIZE h
- SLIP BC
- NO PARAMETER (VISC $=$ 0)

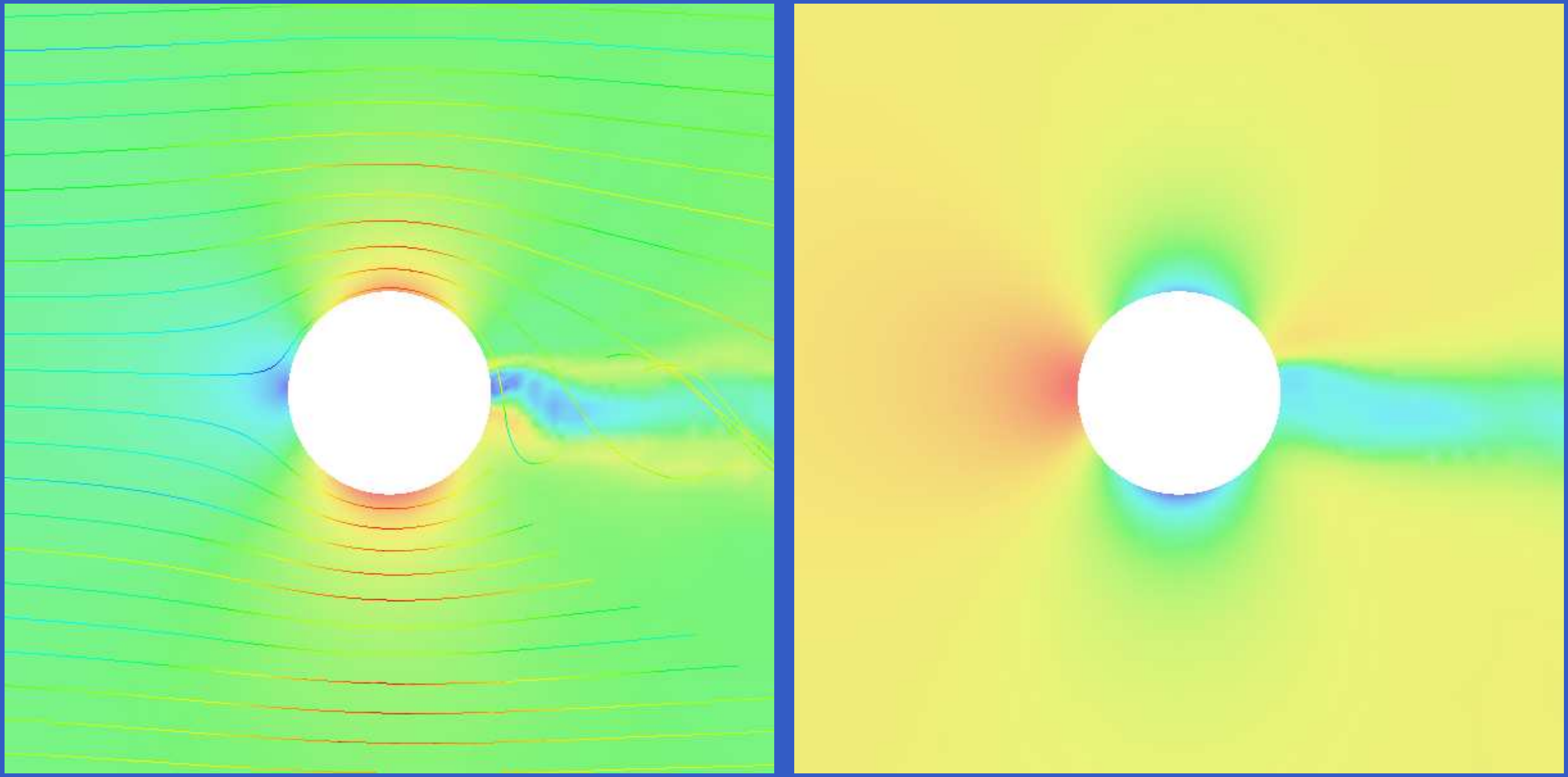
$$K(T) + D_h(T) = K(0), \quad K(T) \quad \text{KIN ENERGY}$$

$$D_h(t) = \int_0^T \int_\Omega hR^2 dxdt >> 0 \quad \text{TURB-NONSMOOTH}$$

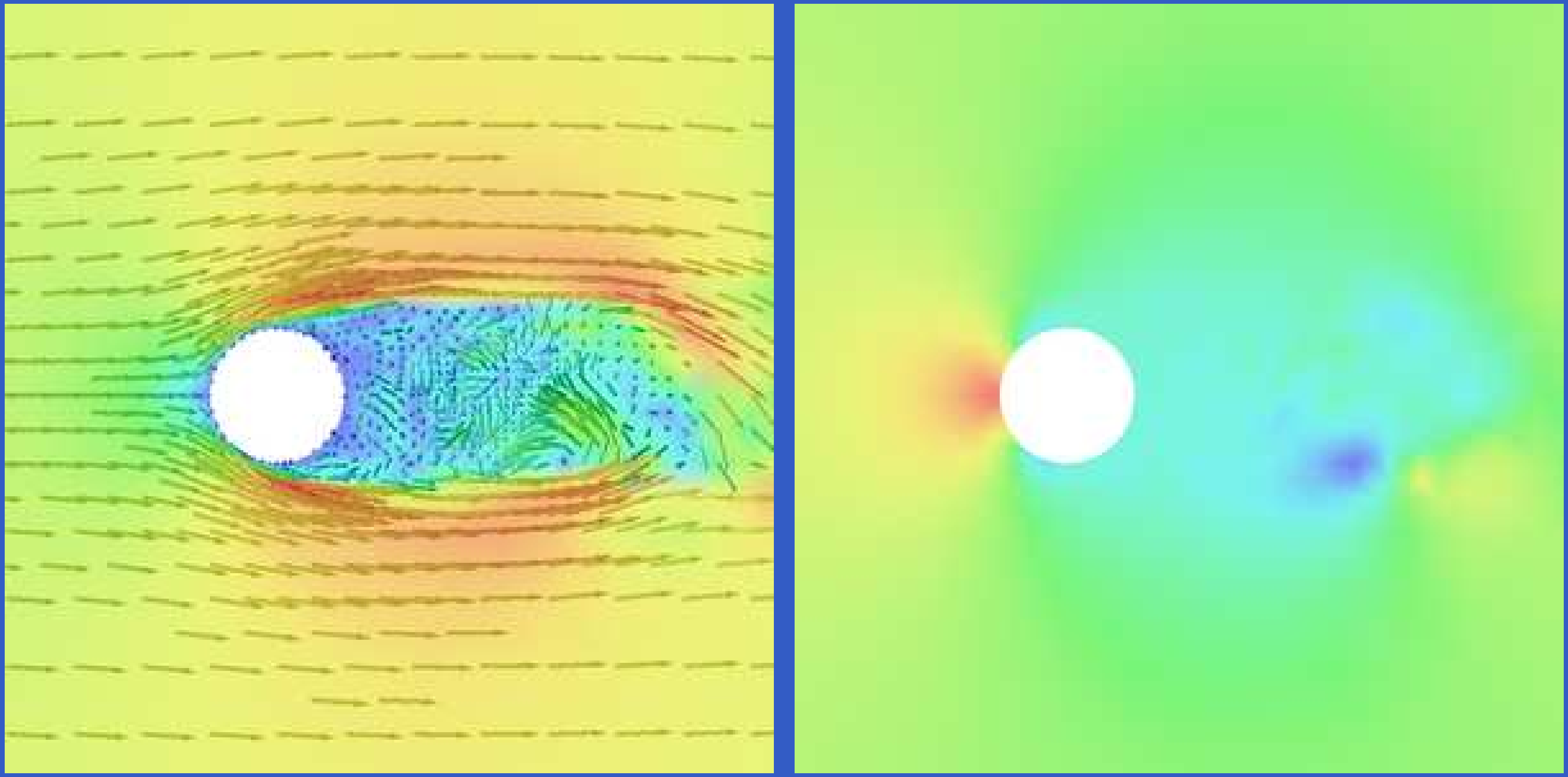
POT SOL CIRC CYL: DRAG $= 0$



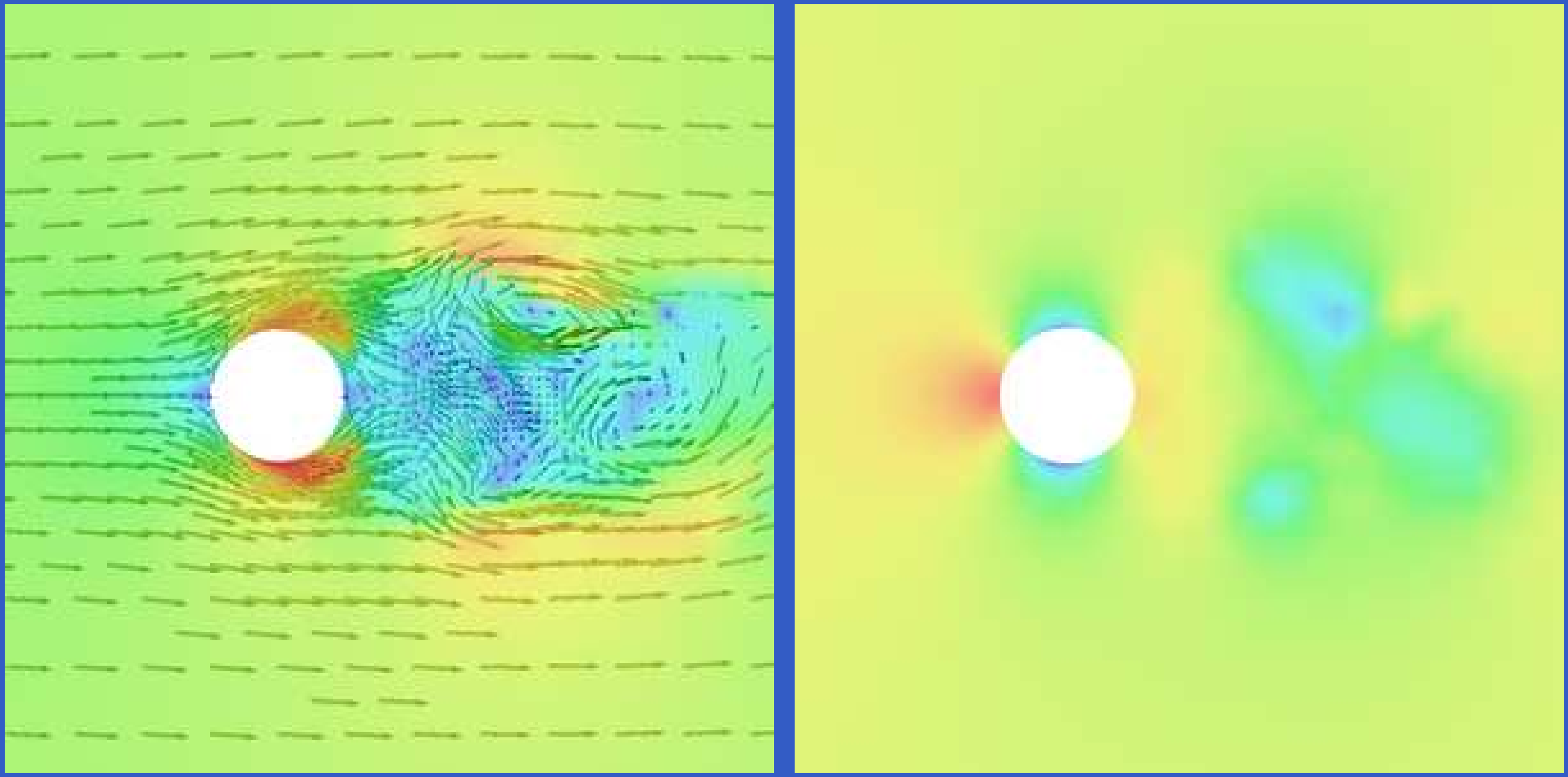
EG2 REAL SOL DRAG ≈ 1



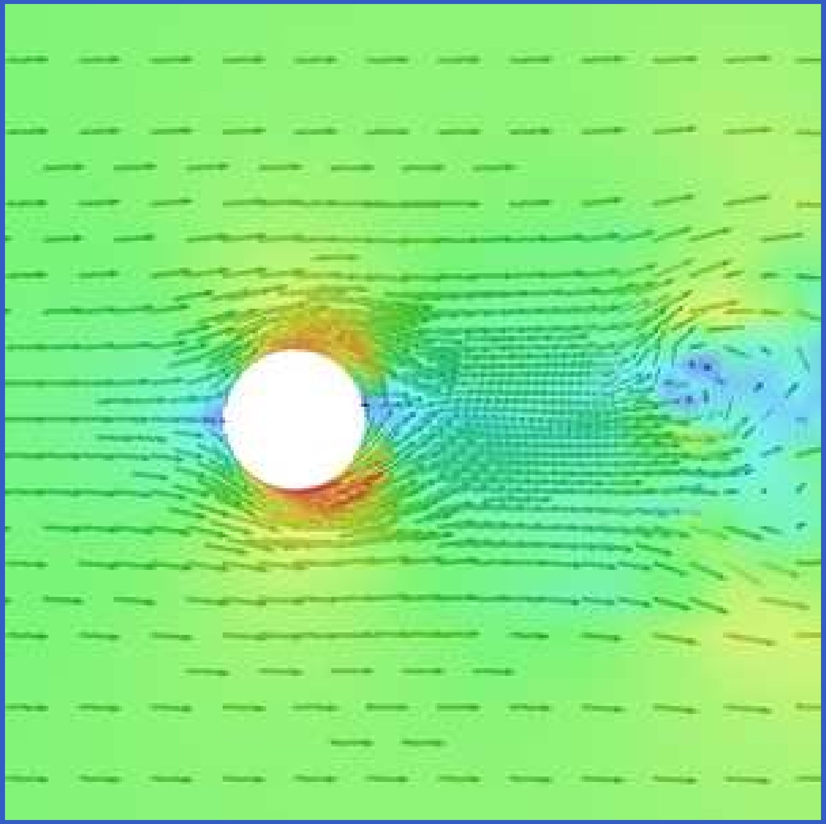
START NO-SLIP: $c_D = 1.03$

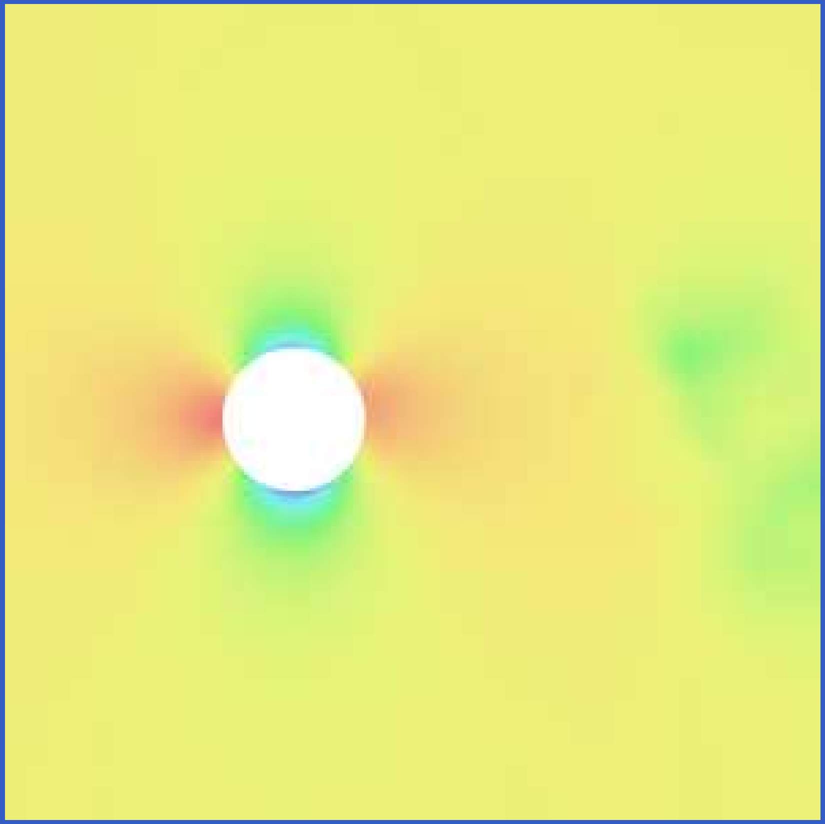


SLIP: t=0.25: $c_D = 0.06$

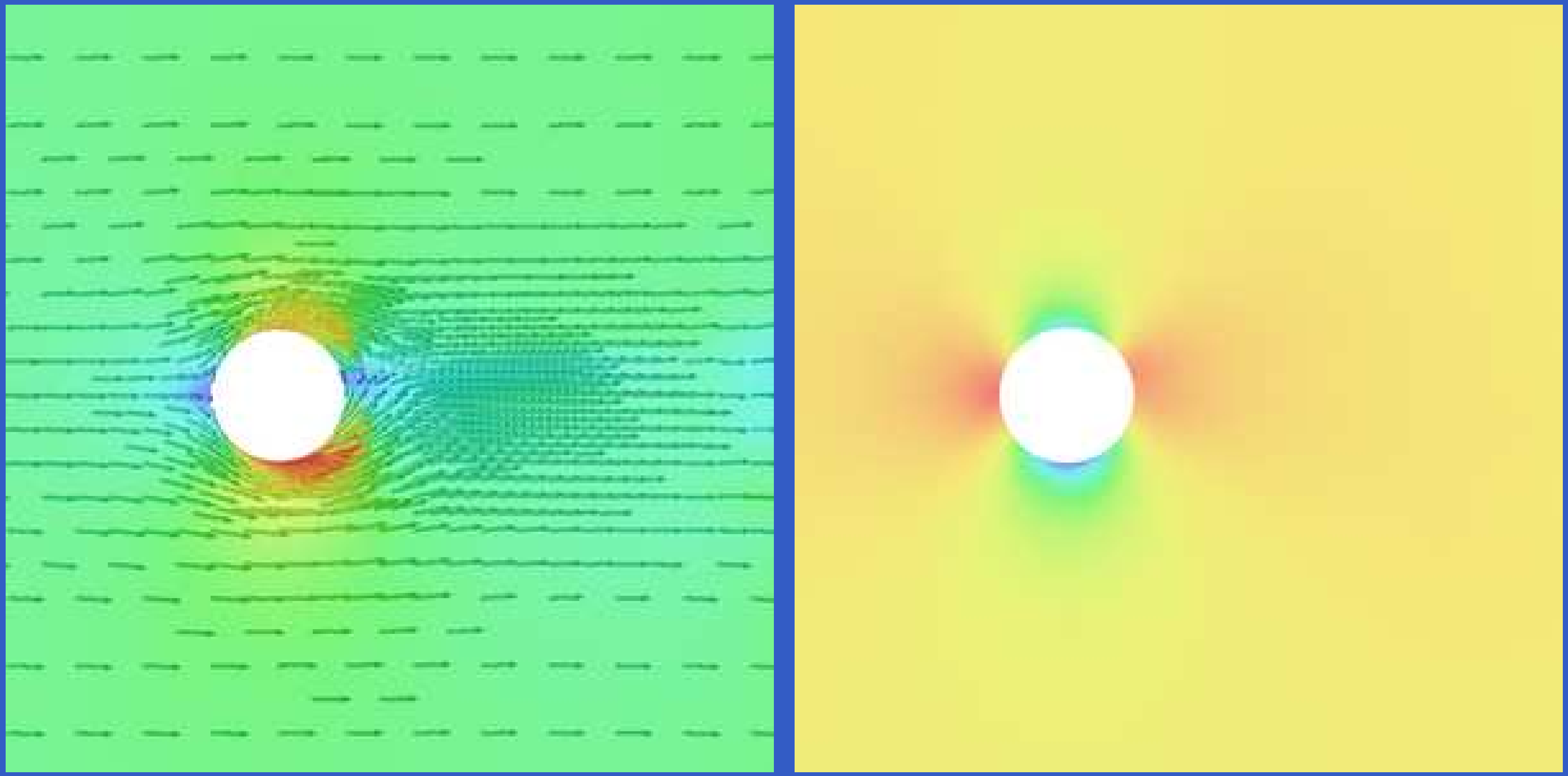


Velocity & pressure: t=0.5: $c_D = 0.10$

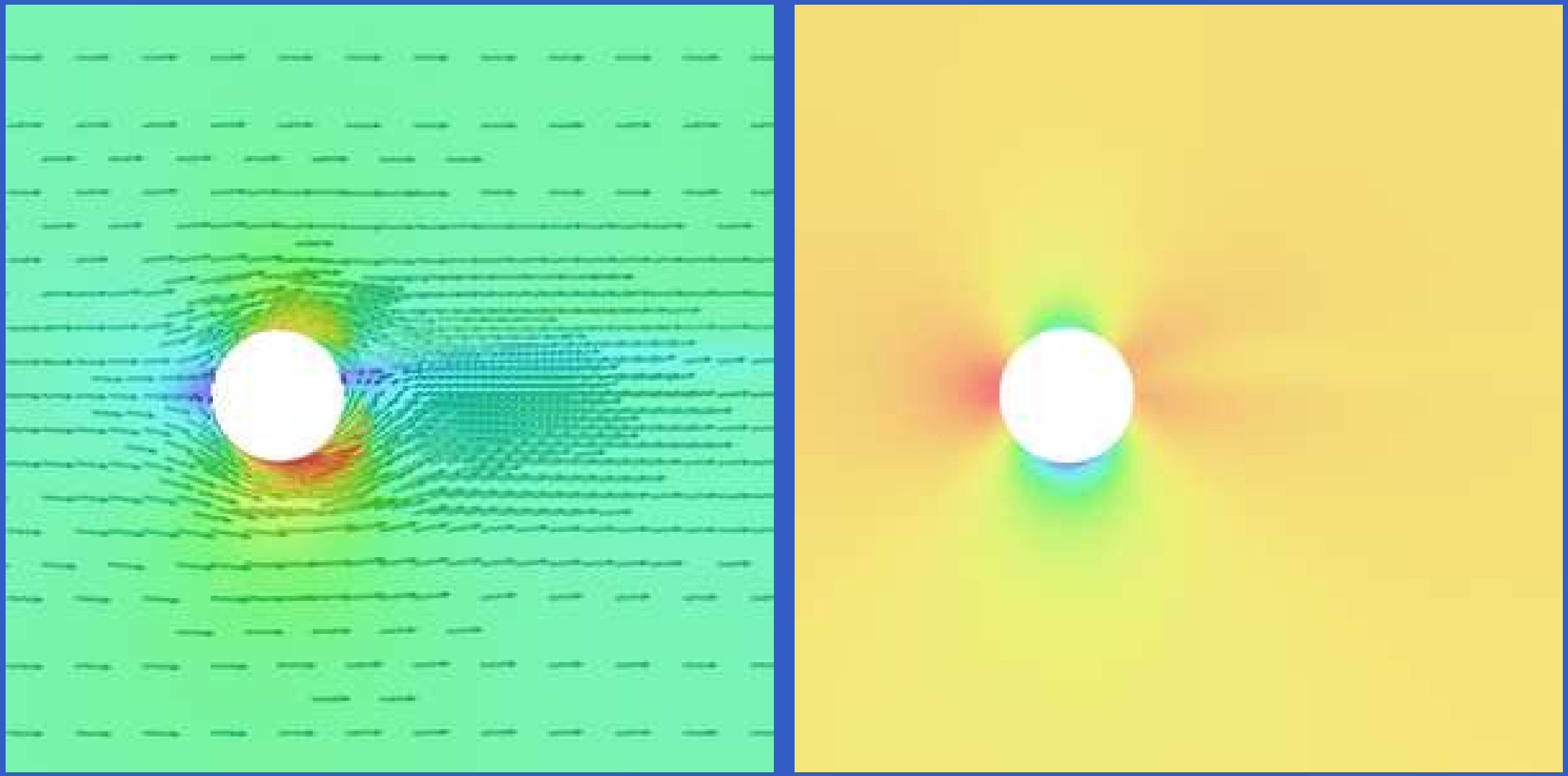




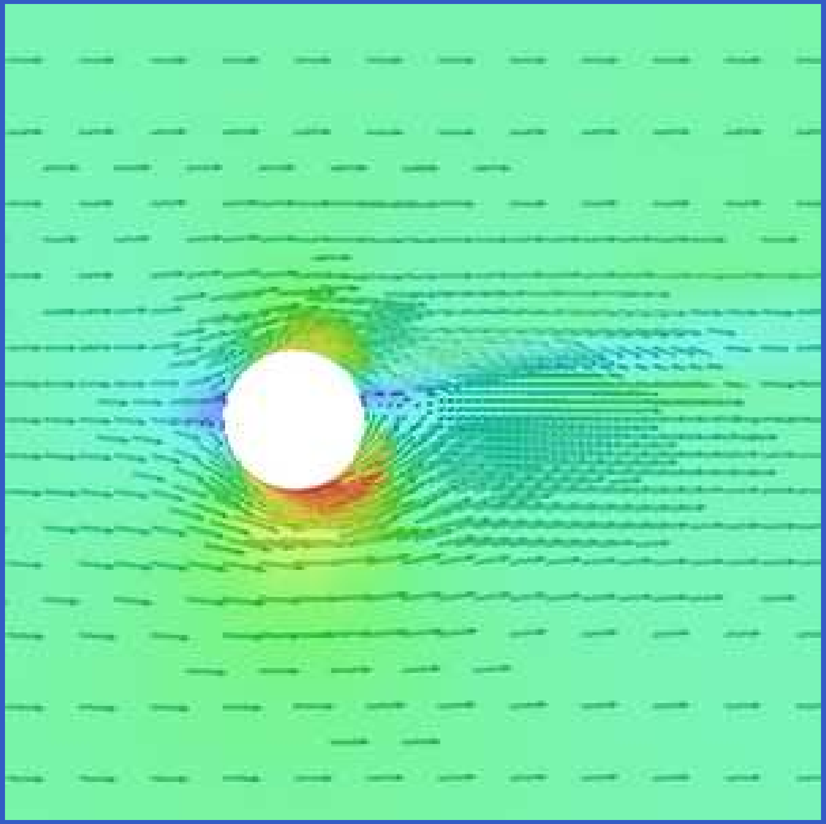
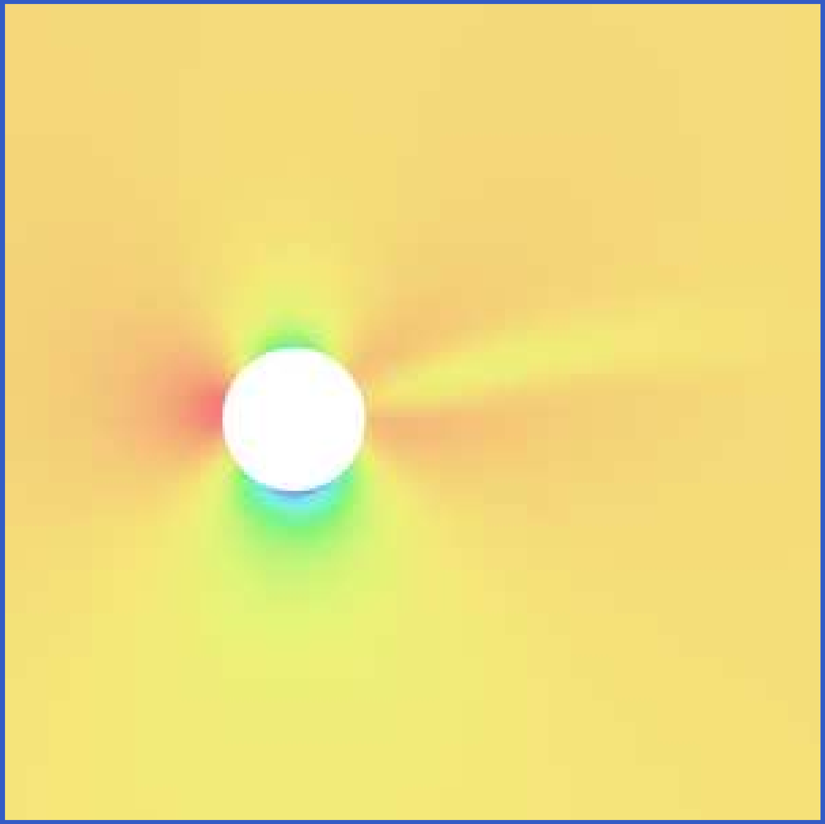
Velocity & pressure: t=0.75: $c_D = 0.15$



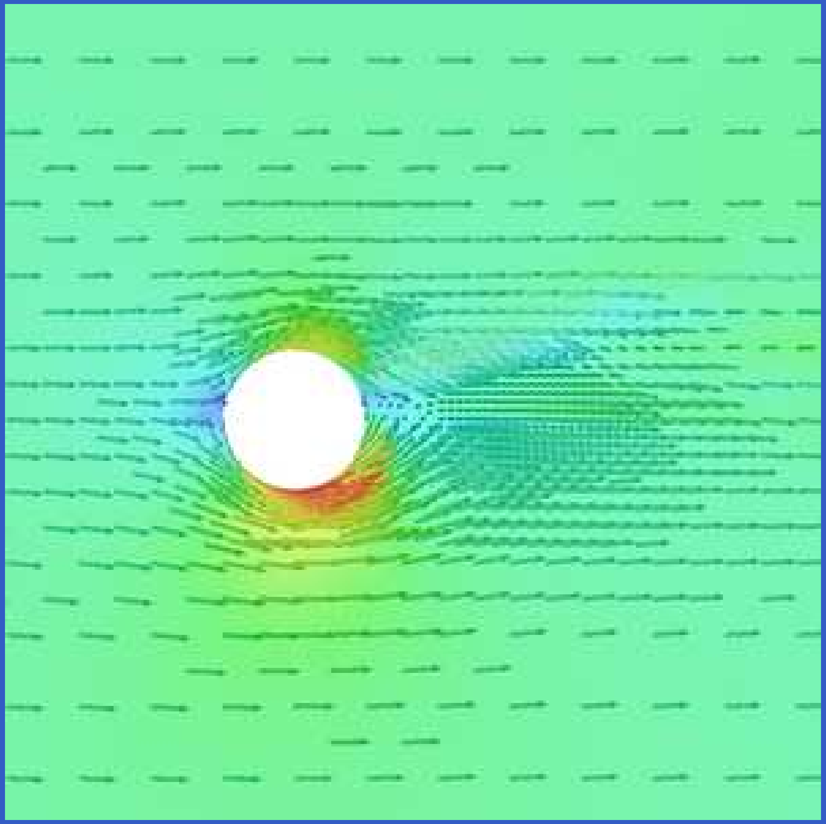
Velocity & pressure: t=1.0: $c_D = 0.22$

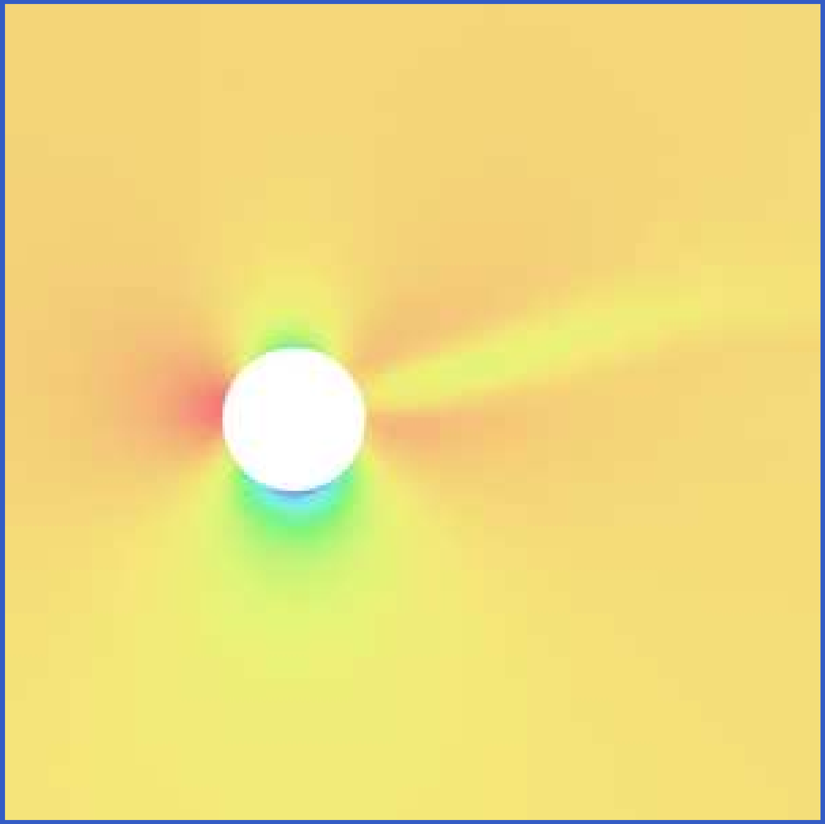


Velocity & pressure: t=1.25: $c_D = 0.25$

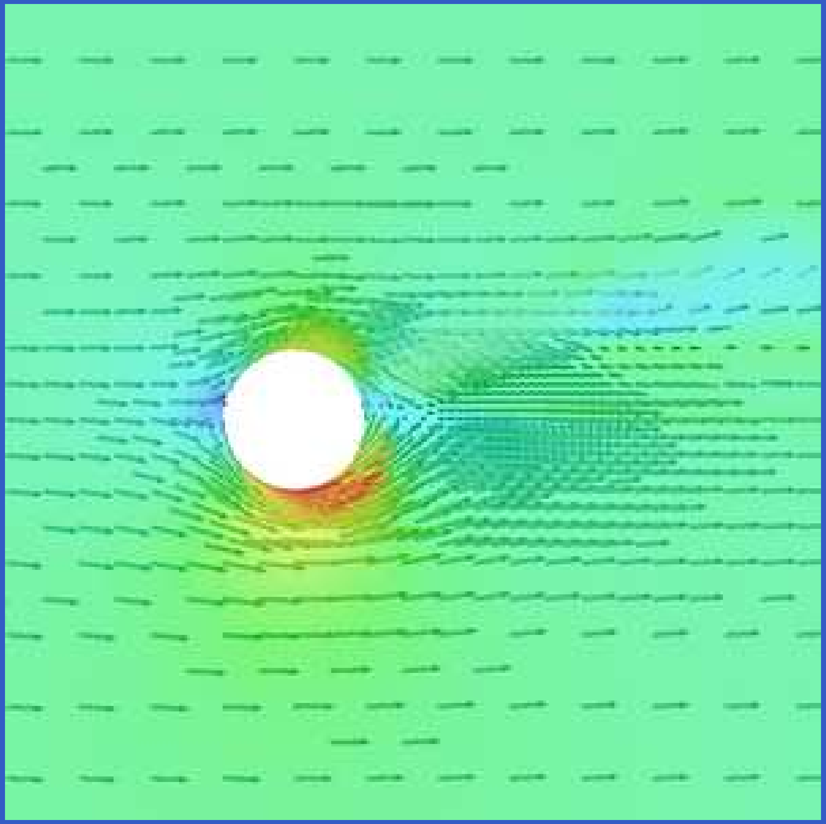
 

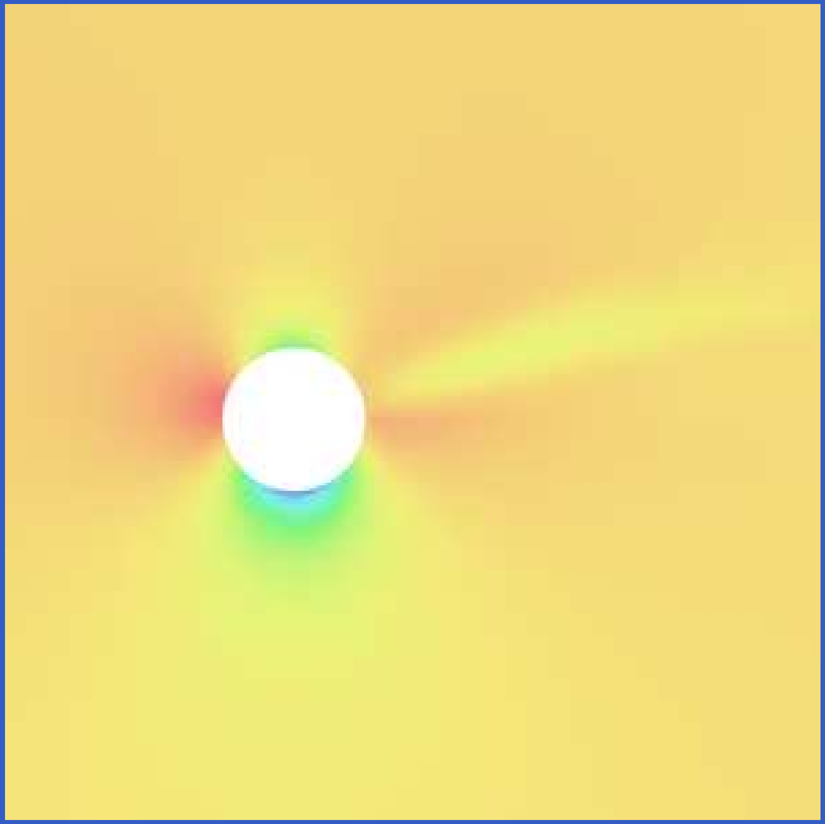
Velocity & pressure: t=1.5: $c_D = 0.28$



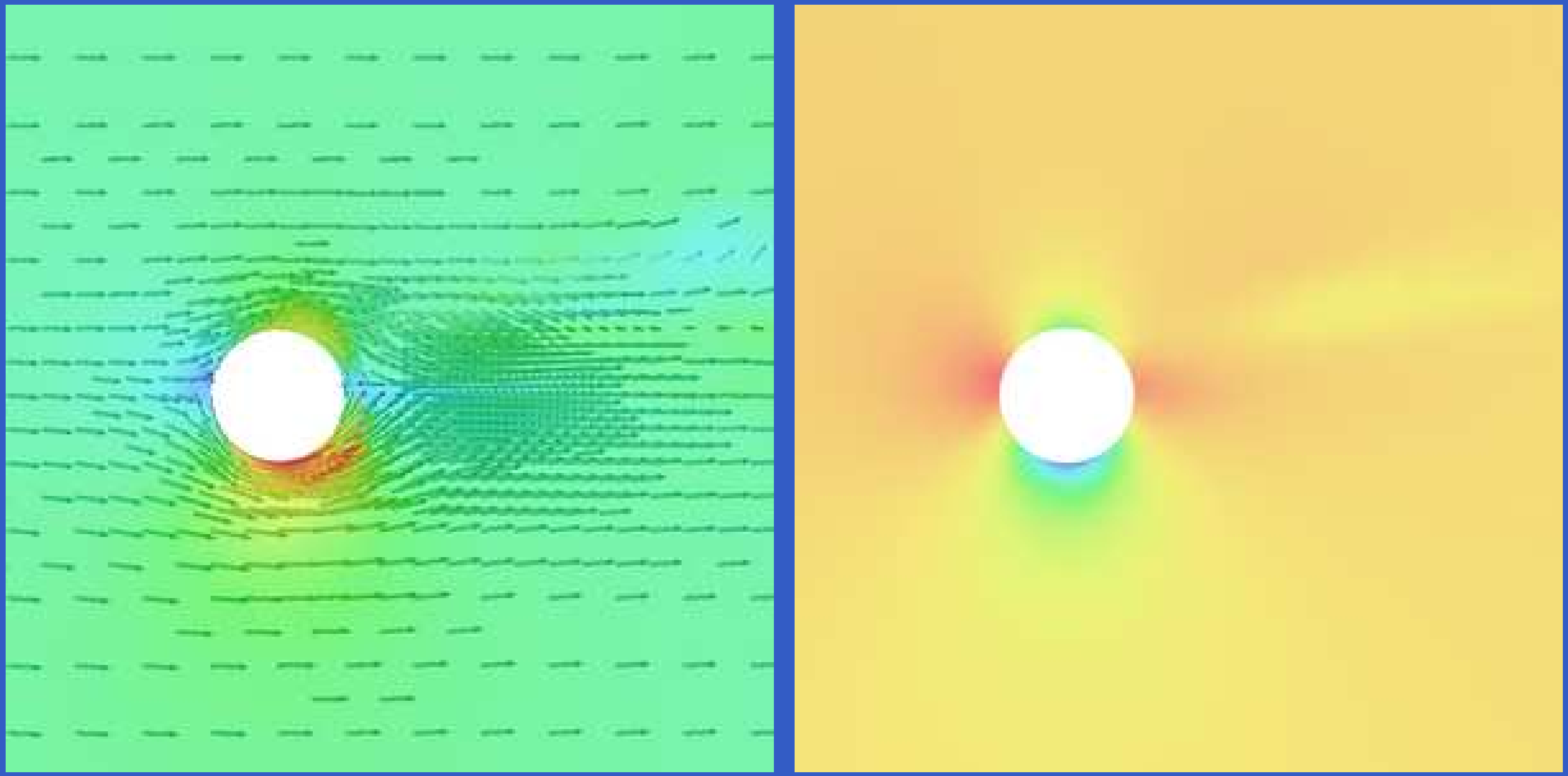


Velocity & pressure: t=1.75: $c_D = 0.36$

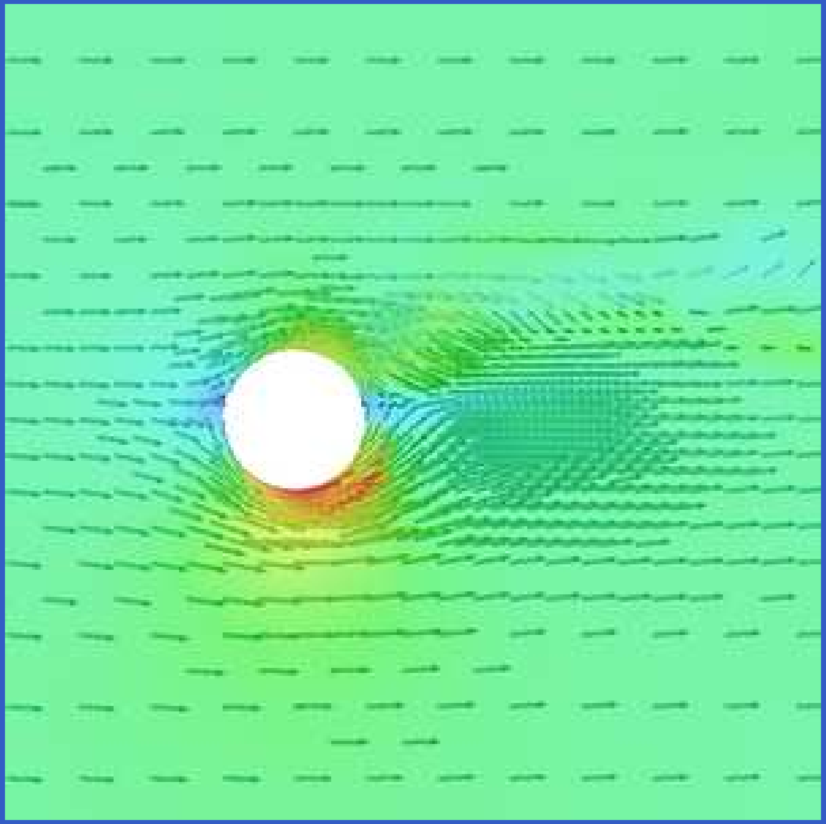


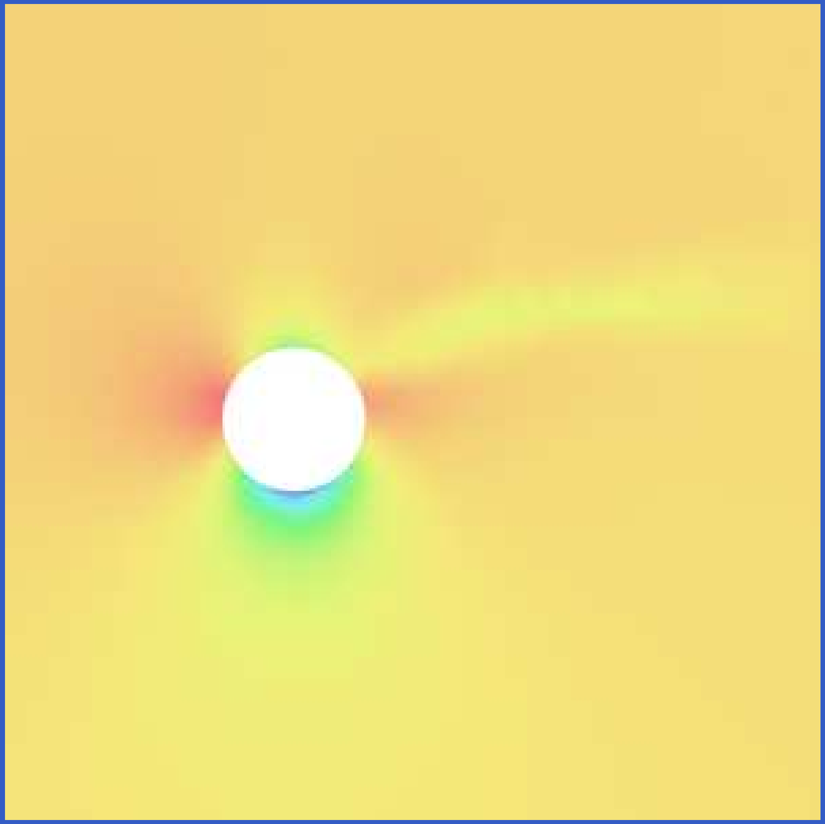


Velocity & pressure: t=2.0: $c_D = 0.51$

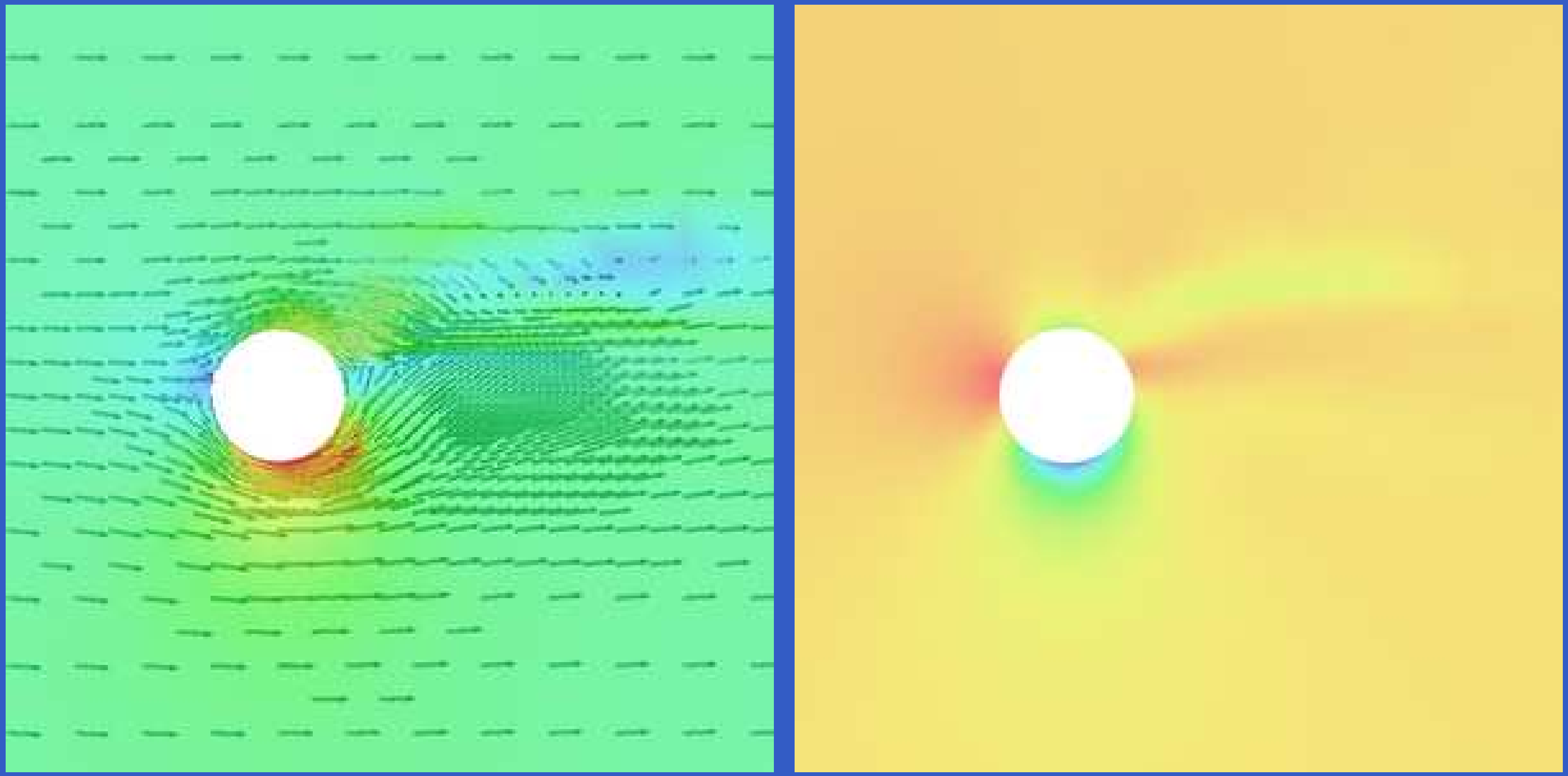


Velocity & pressure: t=2.25: $c_D = 0.78$

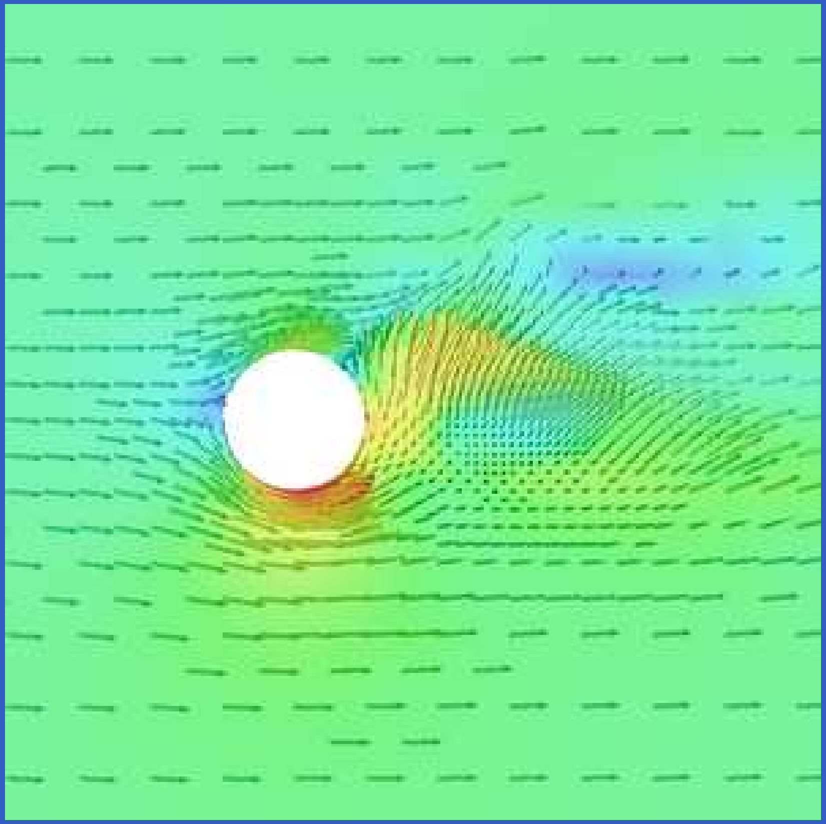


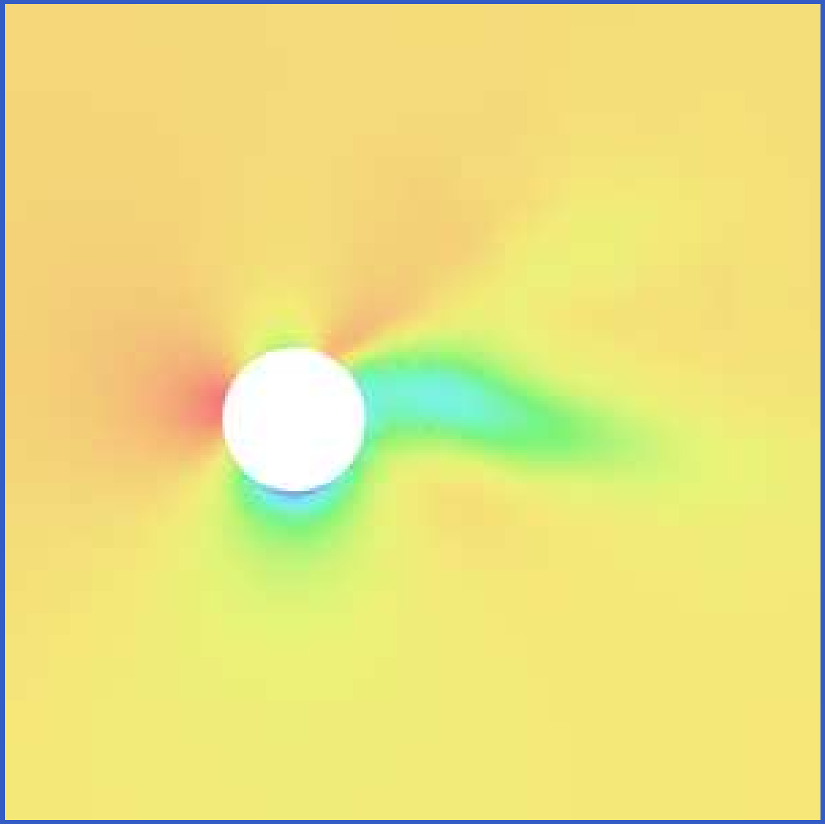


Velocity & pressure: t=2.5: $c_D = 1.14$

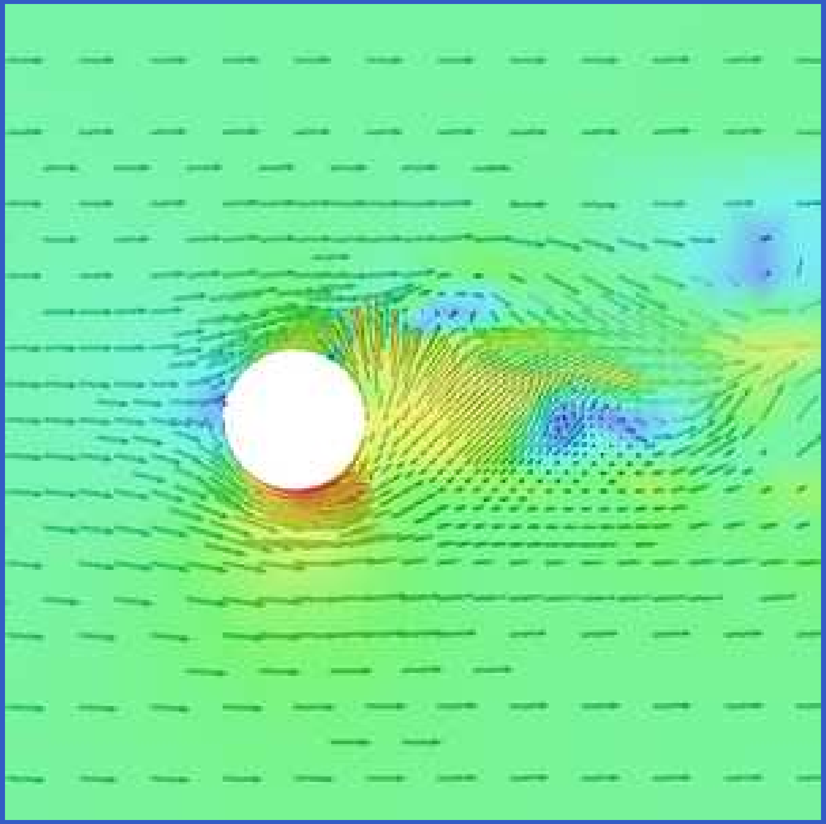


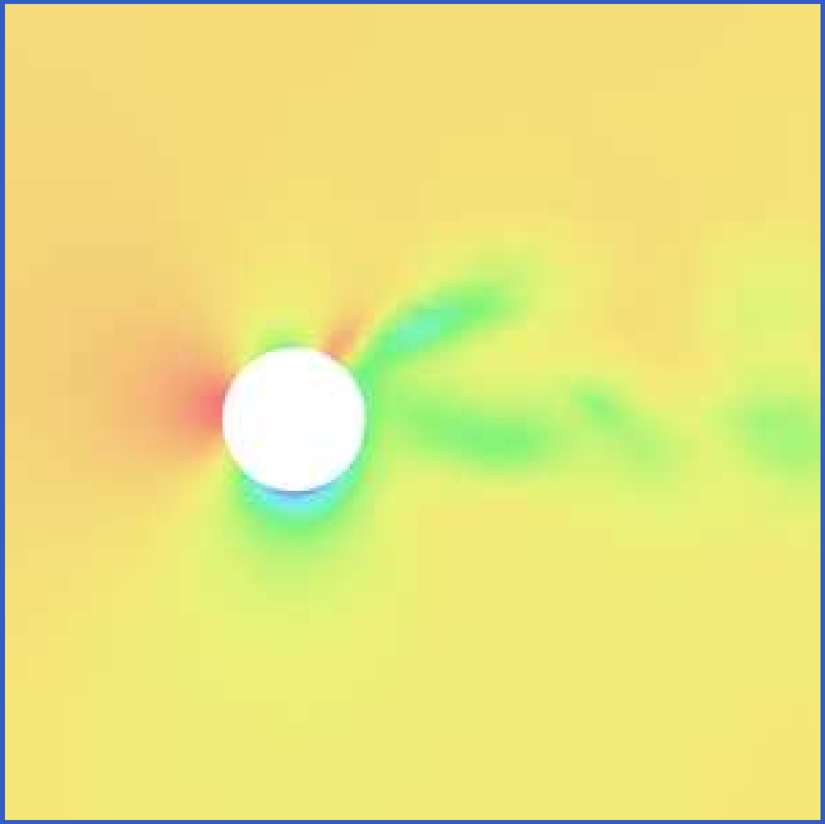
Velocity & pressure: t=2.75: $c_D = 1.04$



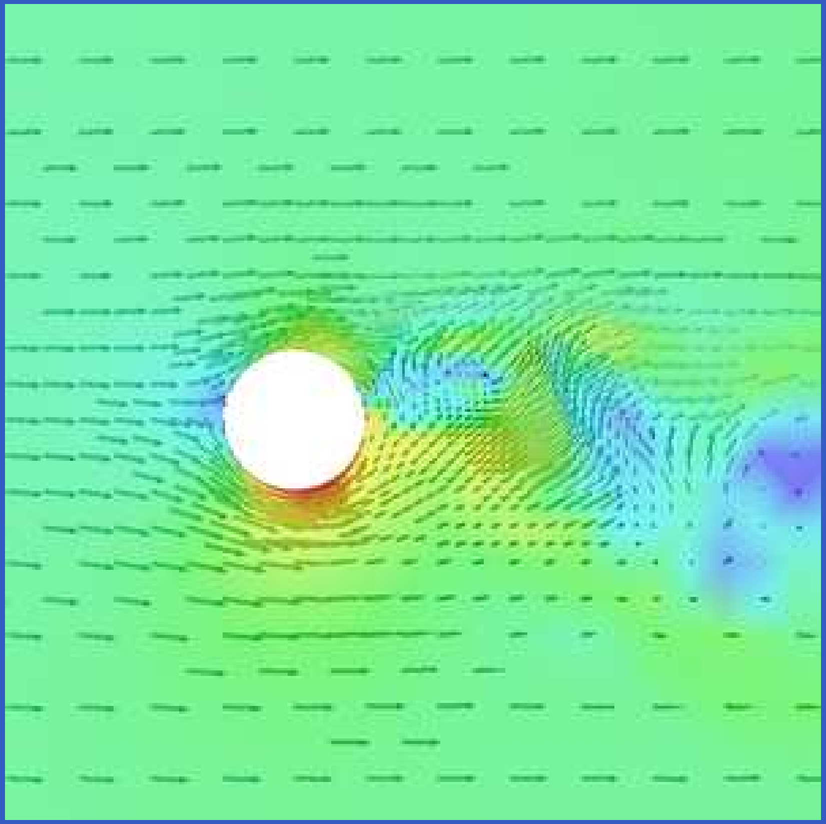


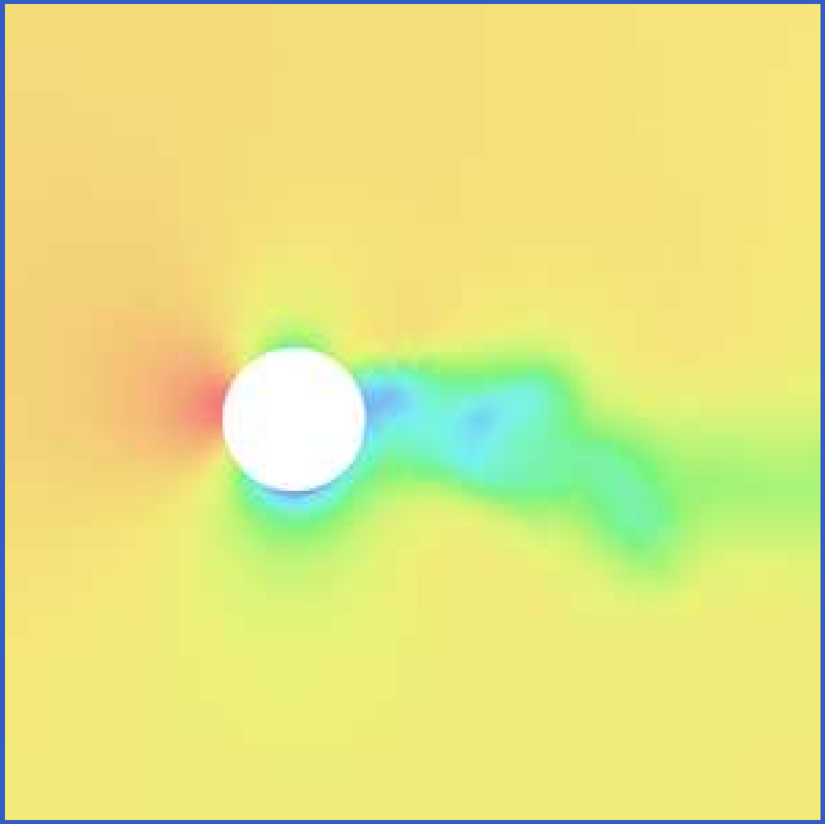
Velocity & pressure: t=3.0: $c_D = 1.05$



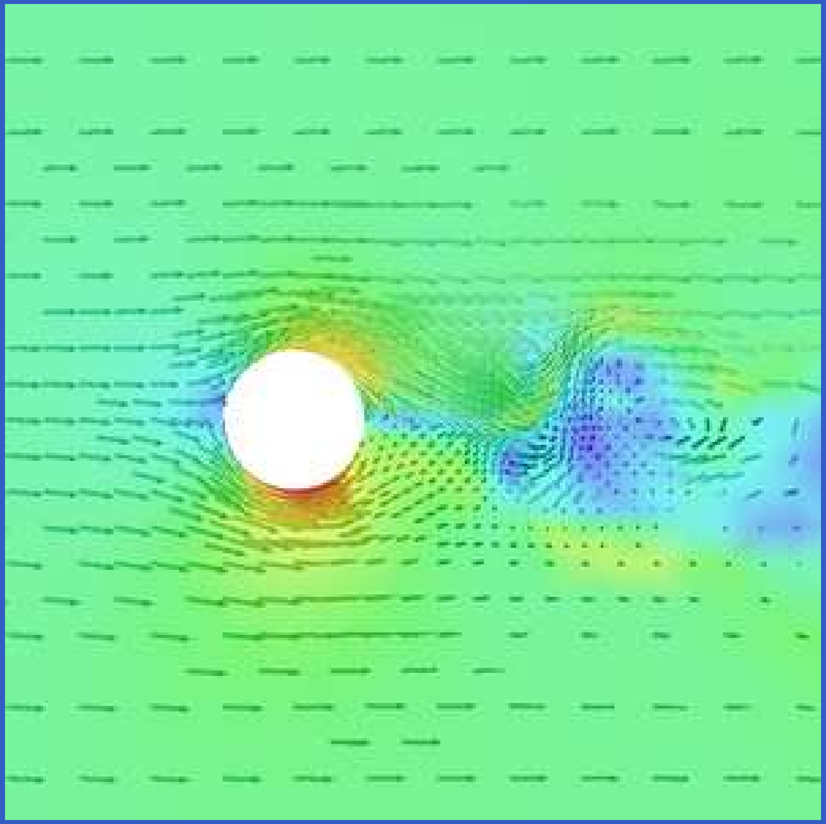


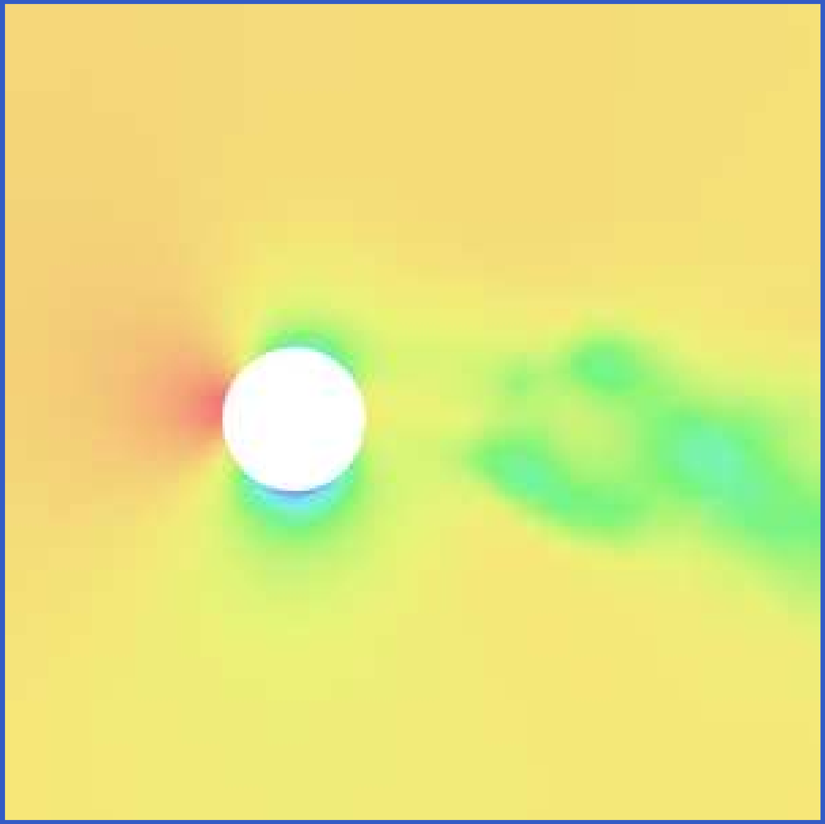
Velocity & pressure: t=4.5: $c_D = 1.63$



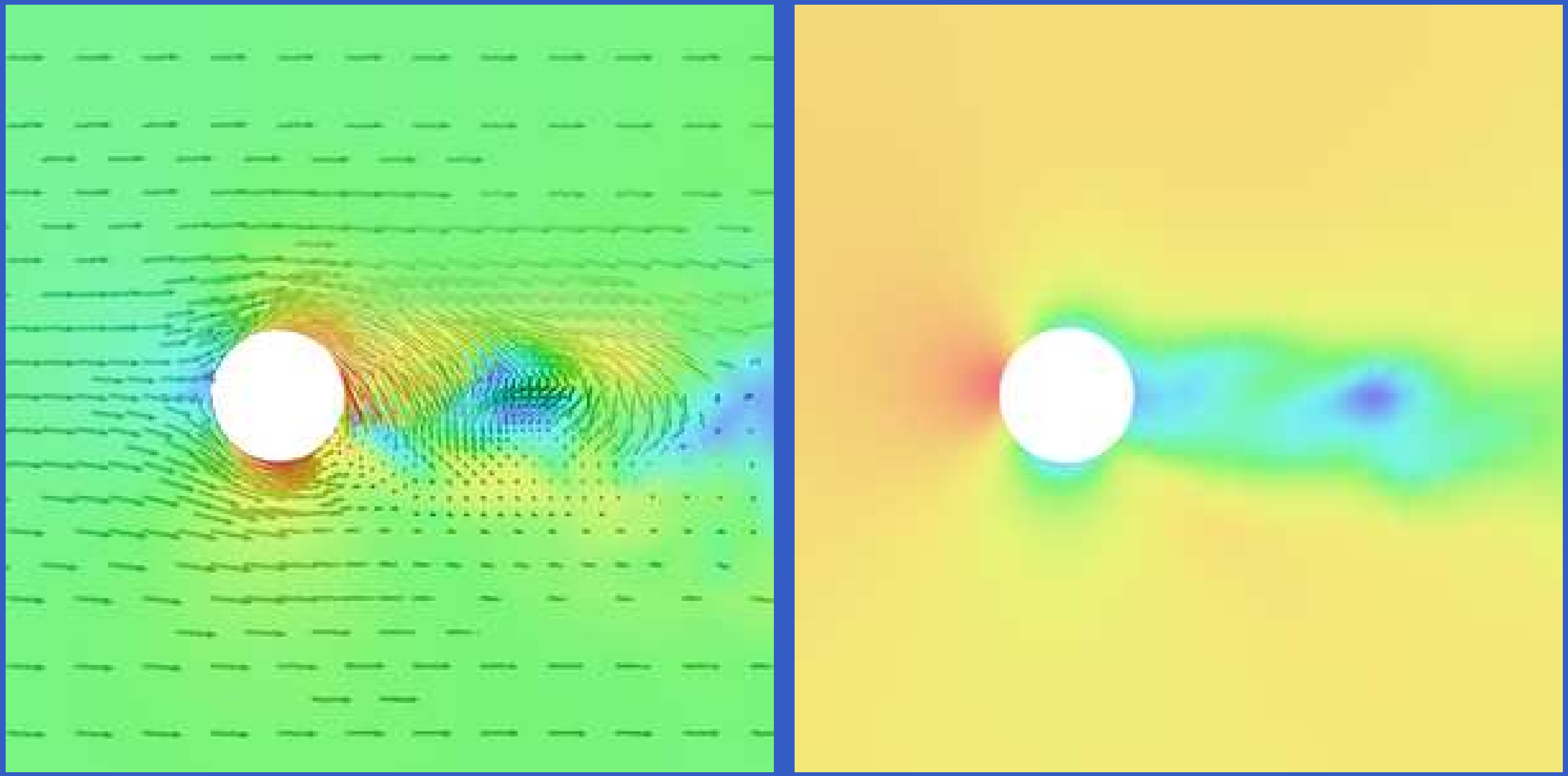


Velocity & pressure: t=5.0: $c_D = 1.79$

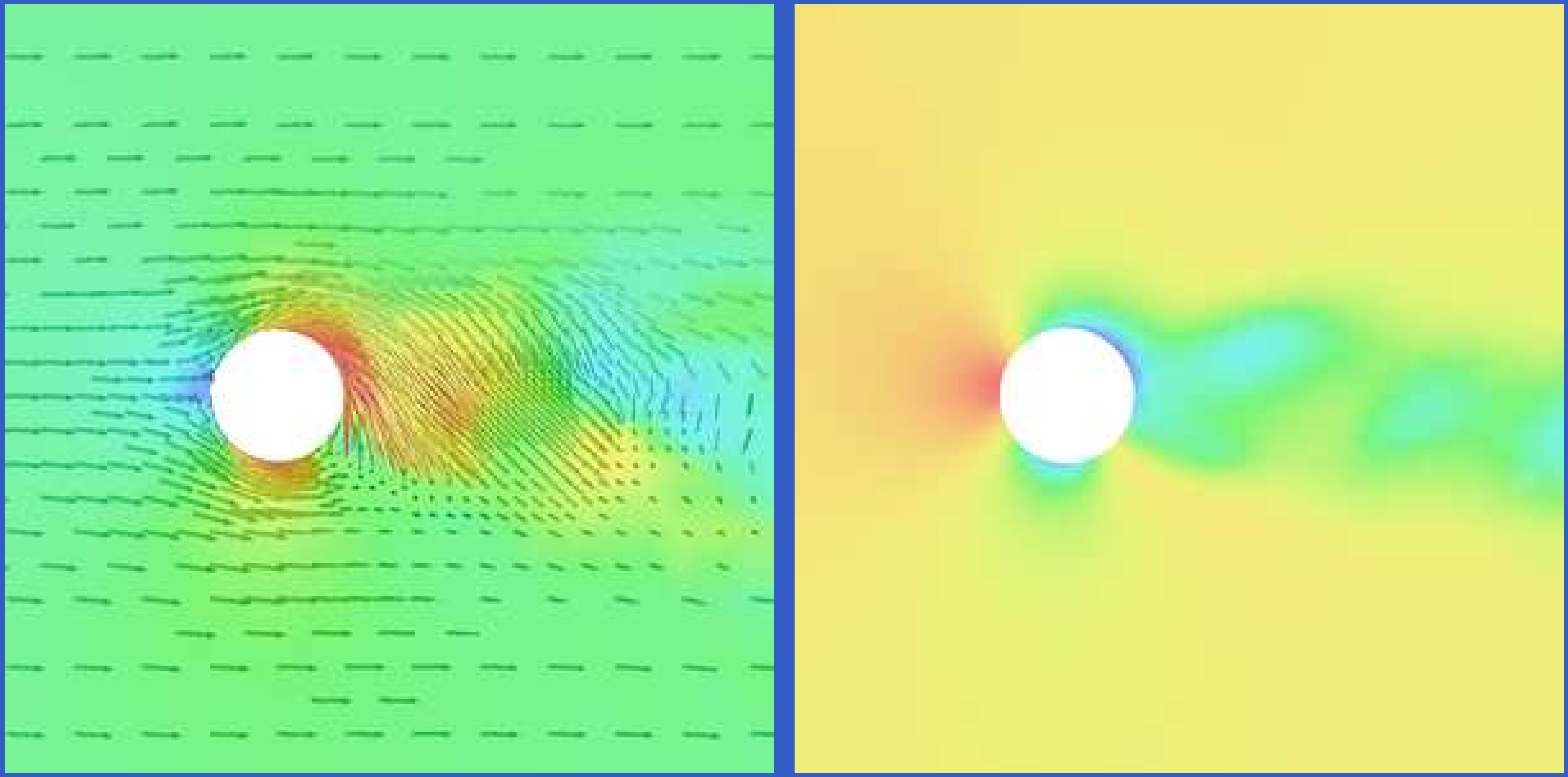




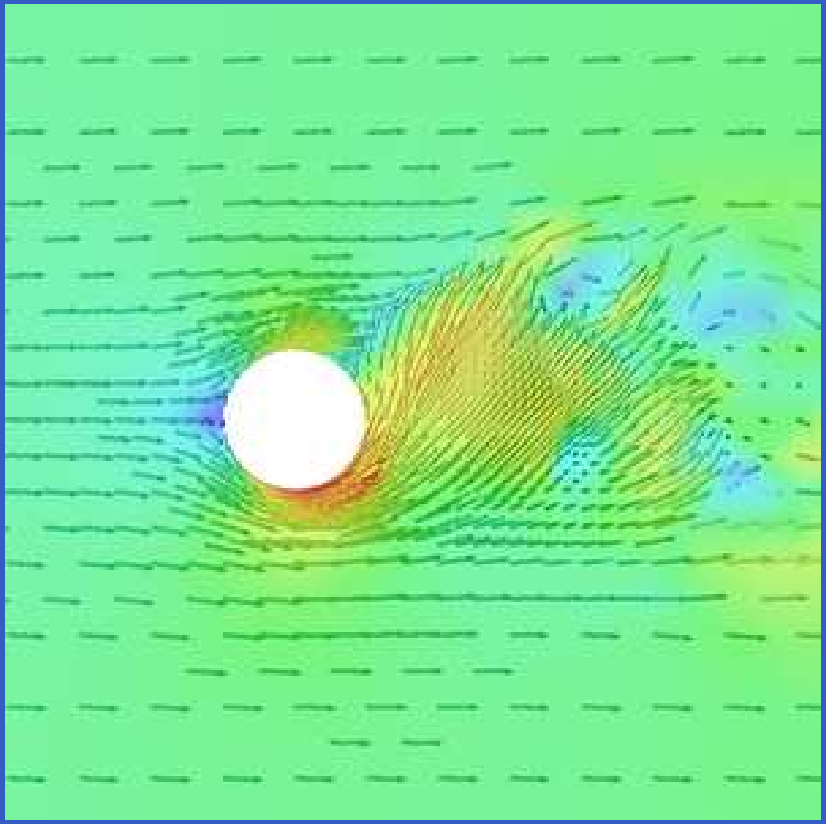
Velocity & pressure: t=5.5: $c_D = 1.96$

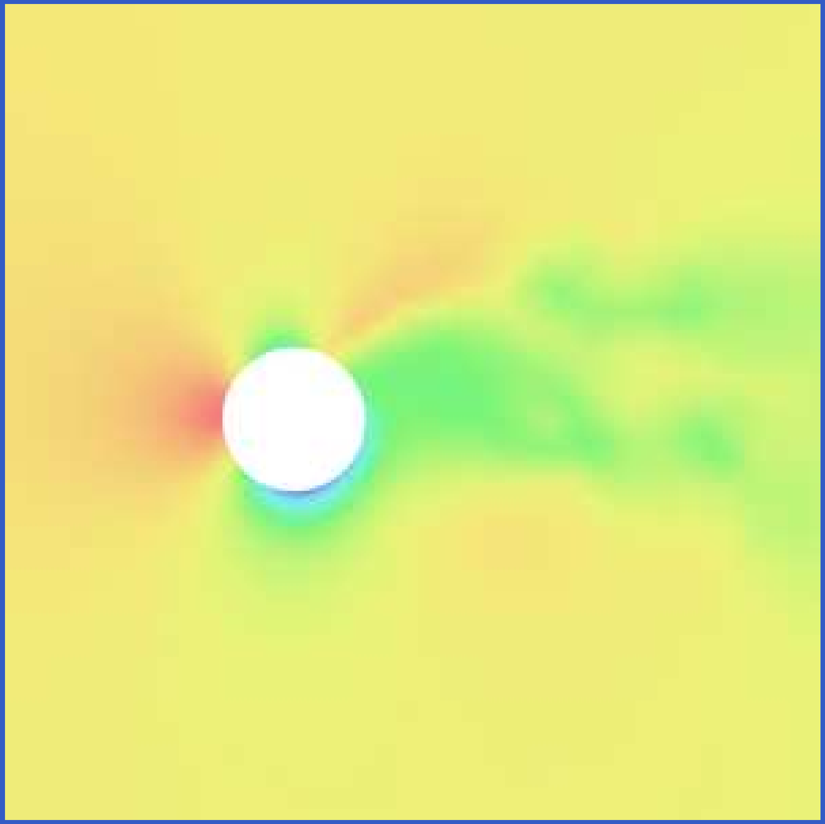


Velocity & pressure: t=5.75: $c_D = 1.90$

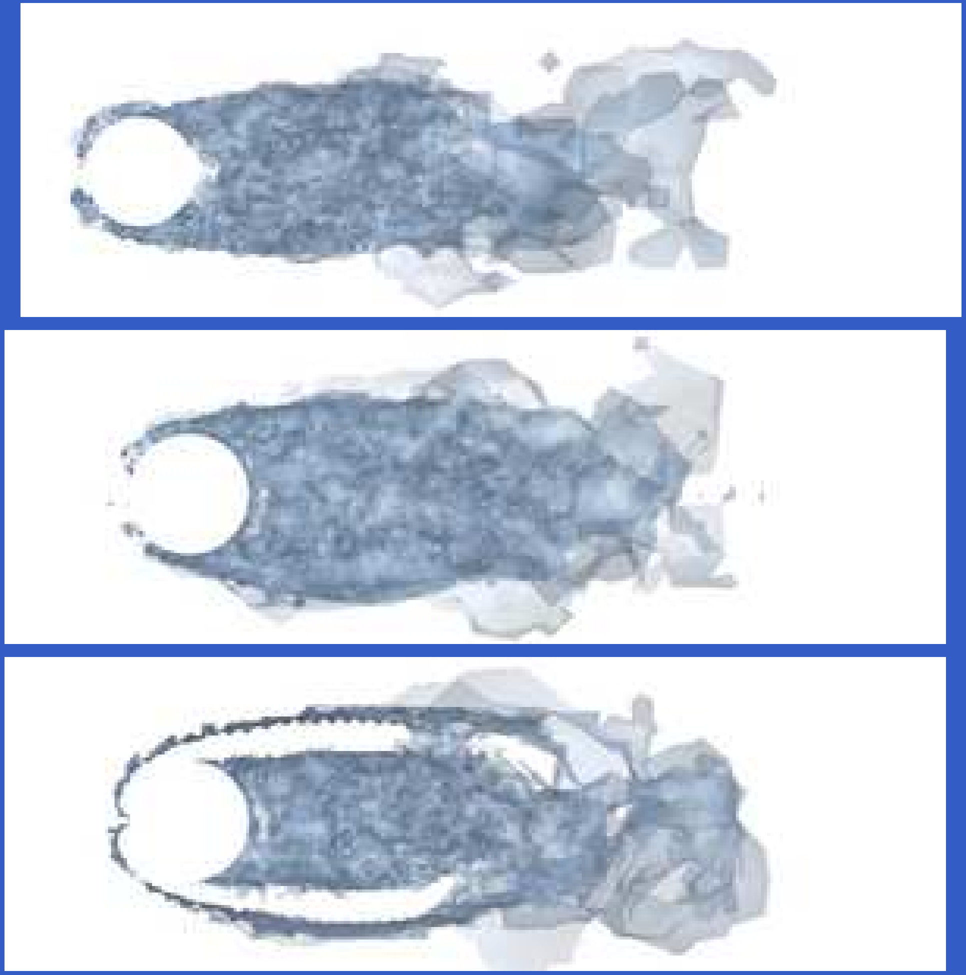


Velocity & pressure: t=11.0: $c_D = 1.82$

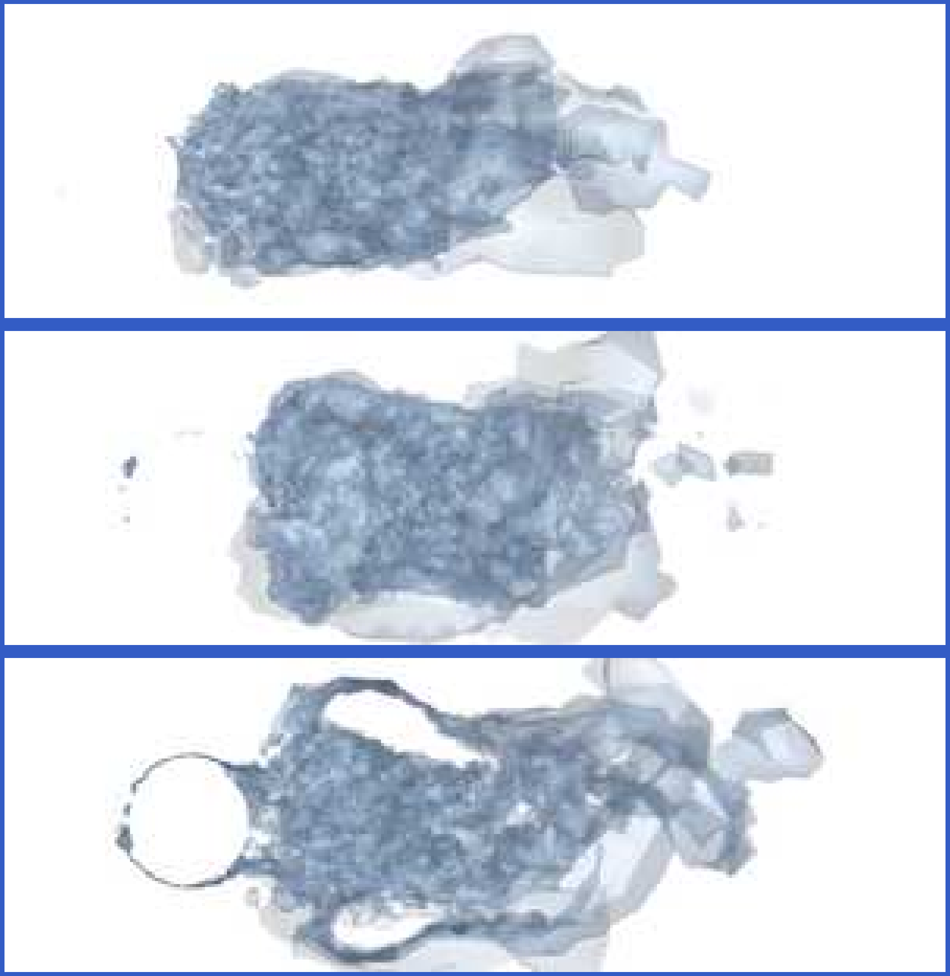




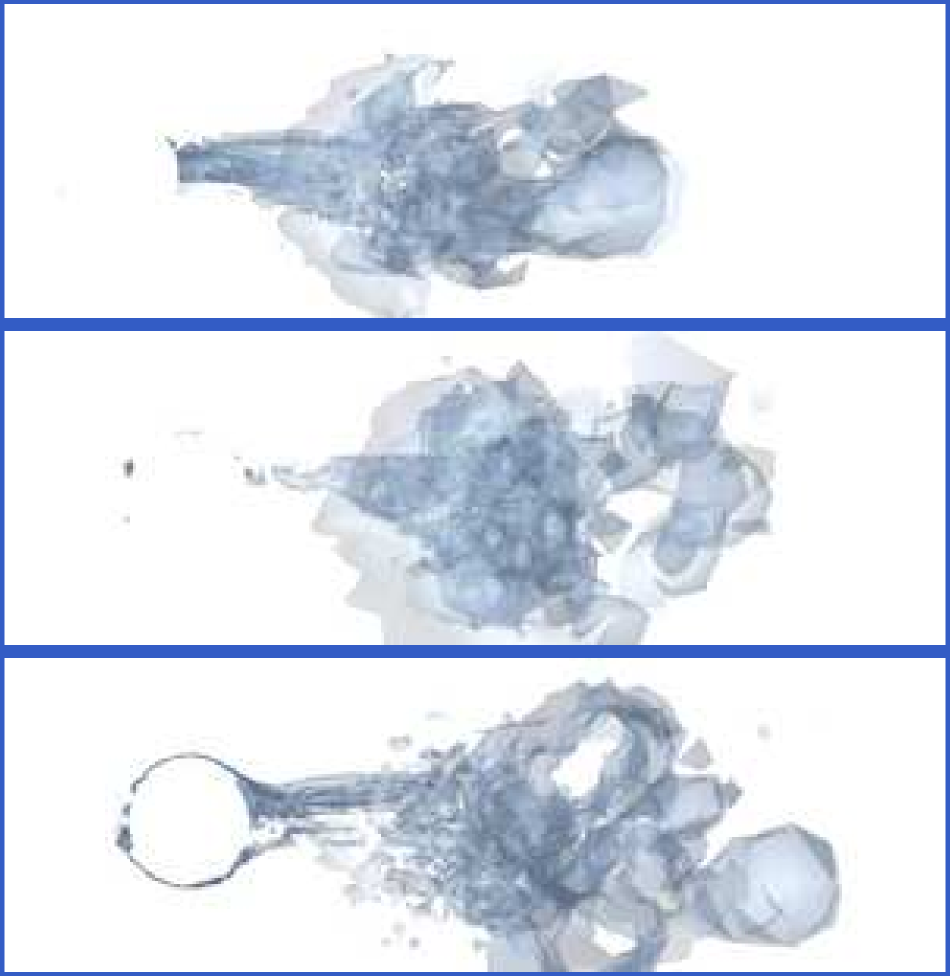
Vorticity: ω_1, ω_2, ω_3: t=0.0: $c_D = 1.03$



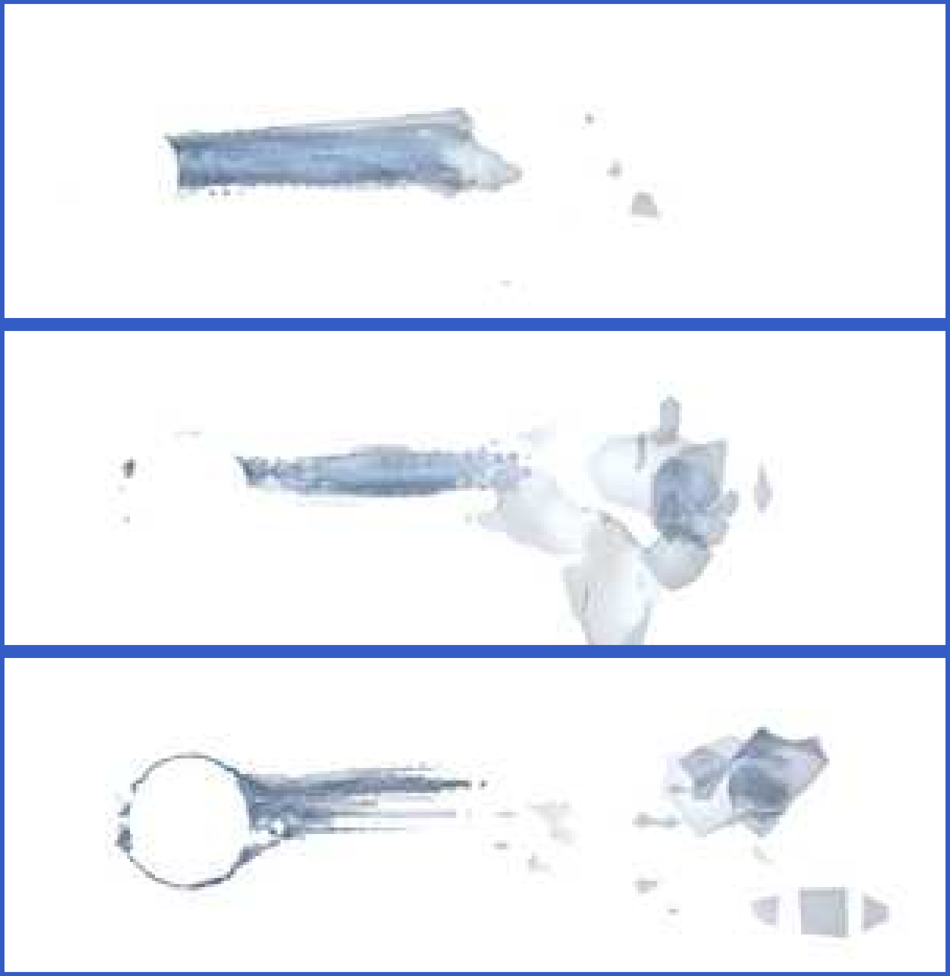
Vorticity: ω_1, ω_2, ω_3: t=0.25: $c_D = 0.06$



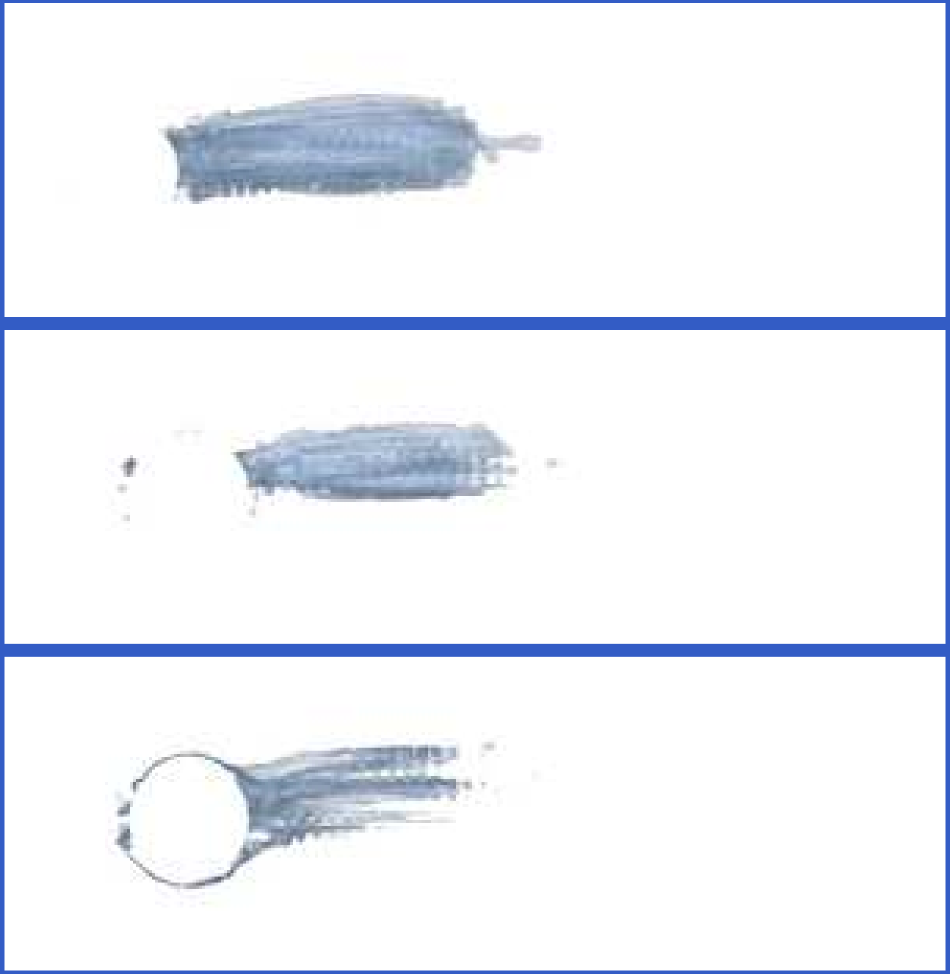
Vorticity: ω_1, ω_2, ω_3: t=0.5: $c_D = 0.10$



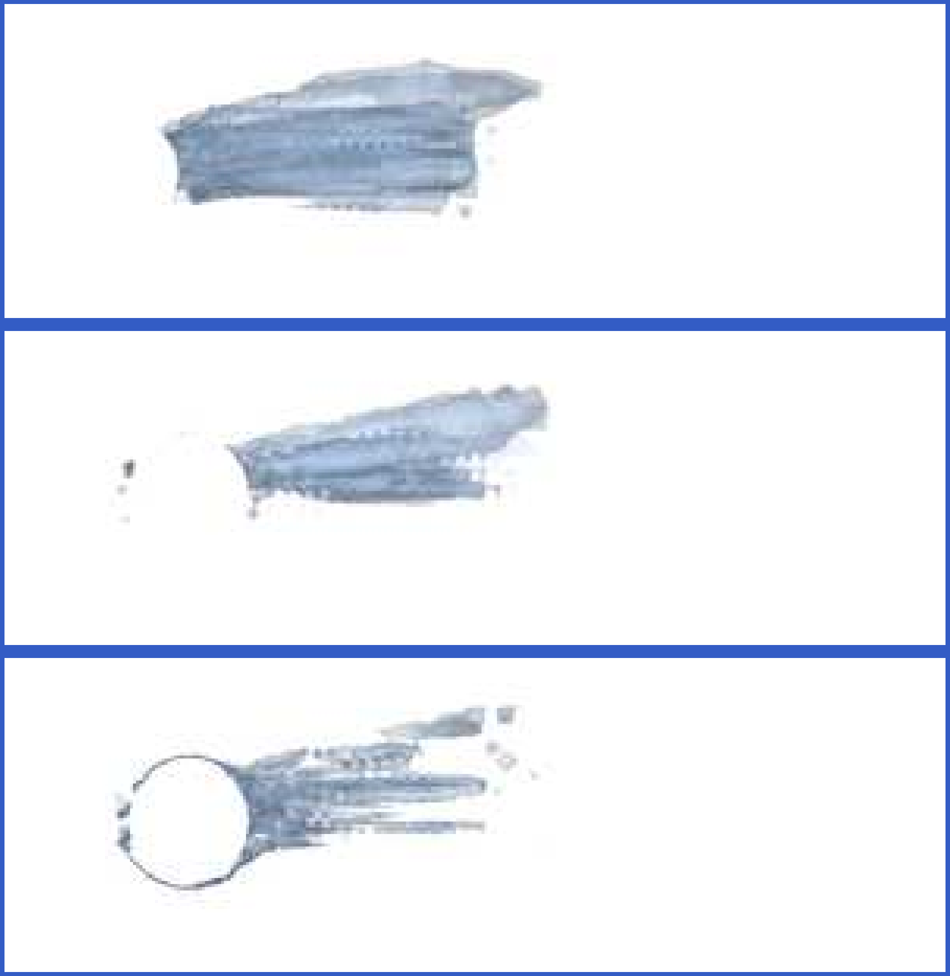
Vorticity: ω_1, ω_2, ω_3: t=0.75: $c_D = 0.15$



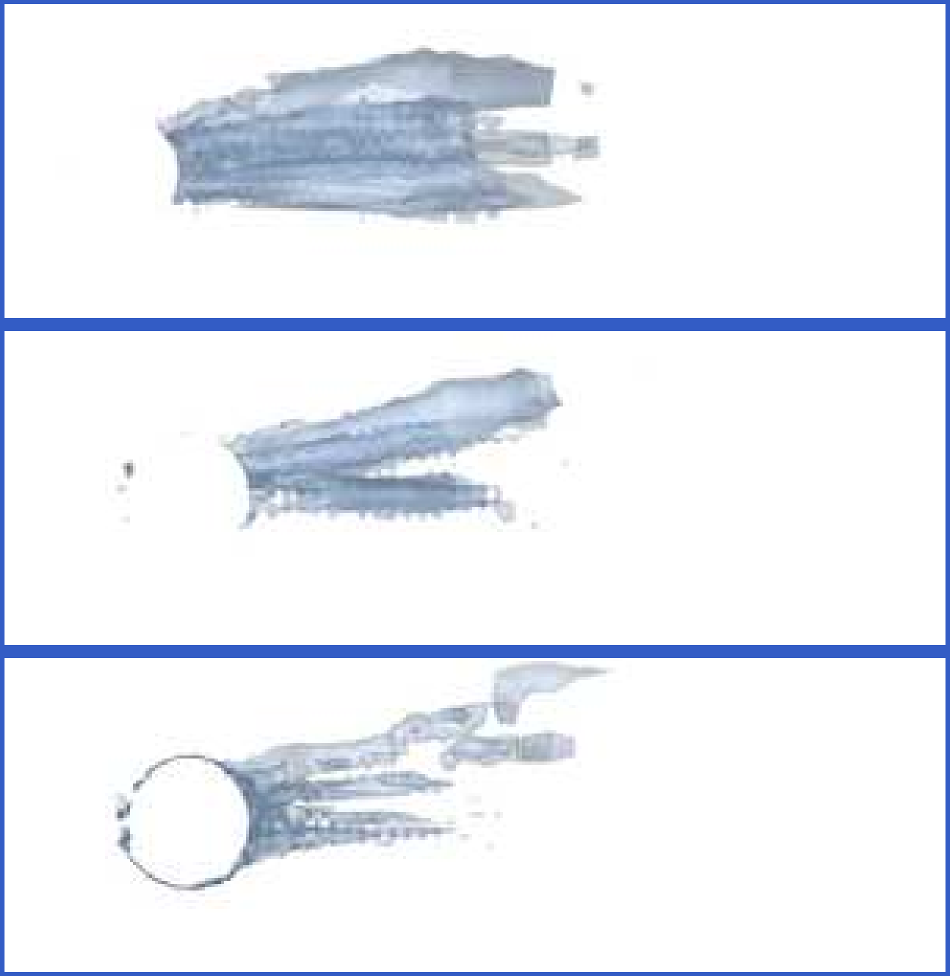
Vorticity: ω_1, ω_2, ω_3: t=1.0: $c_D = 0.22$



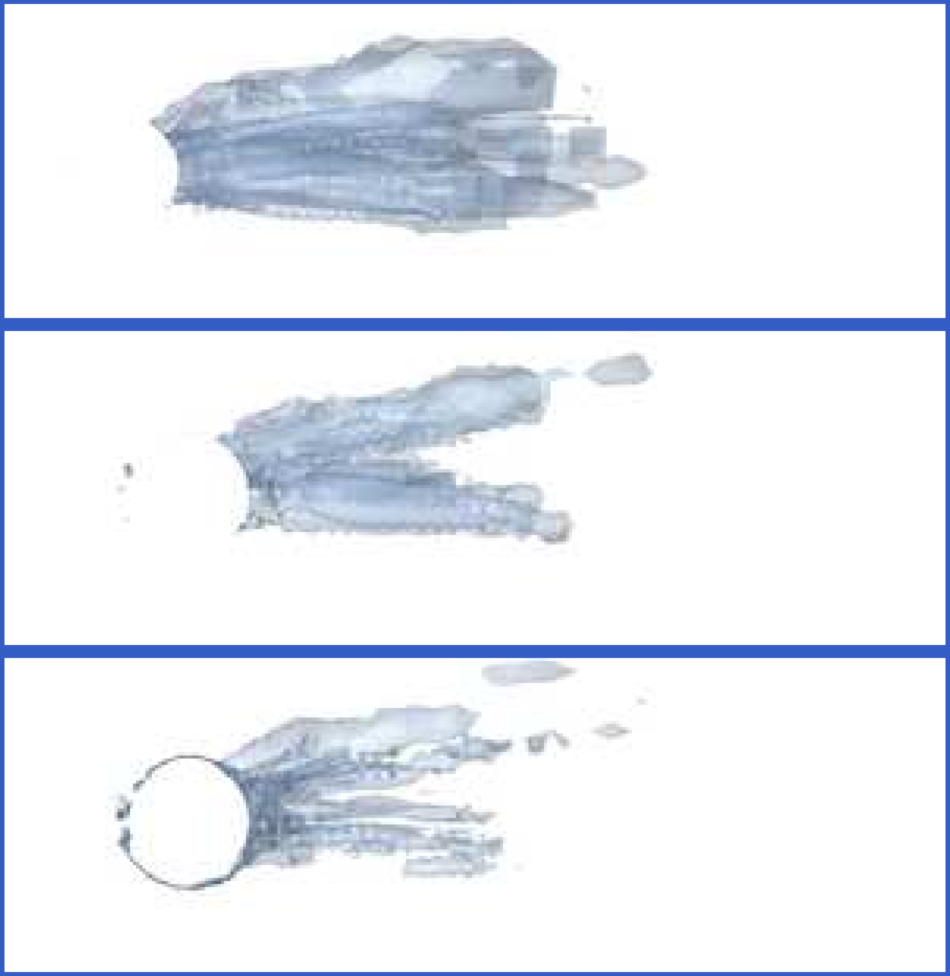
Vorticity: ω_1, ω_2, ω_3: t=1.25: $c_D = 0.25$



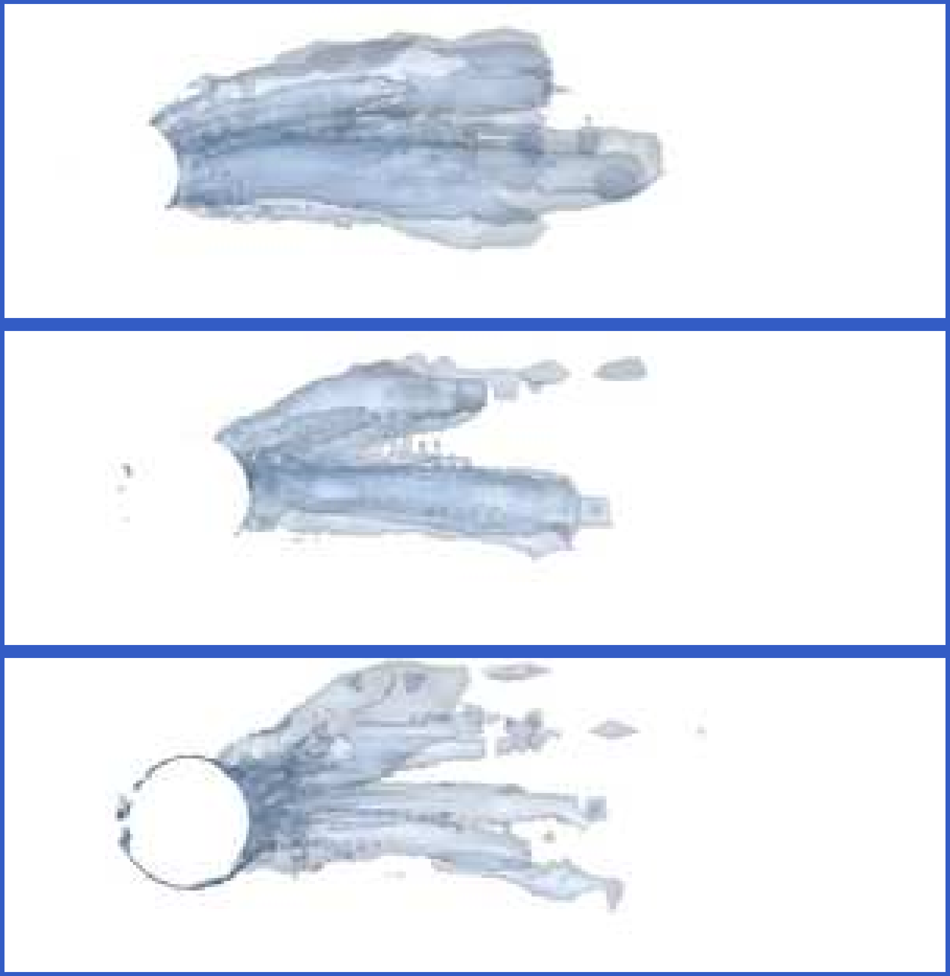
Vorticity: ω_1, ω_2, ω_3: t=1.5: $c_D = 0.28$



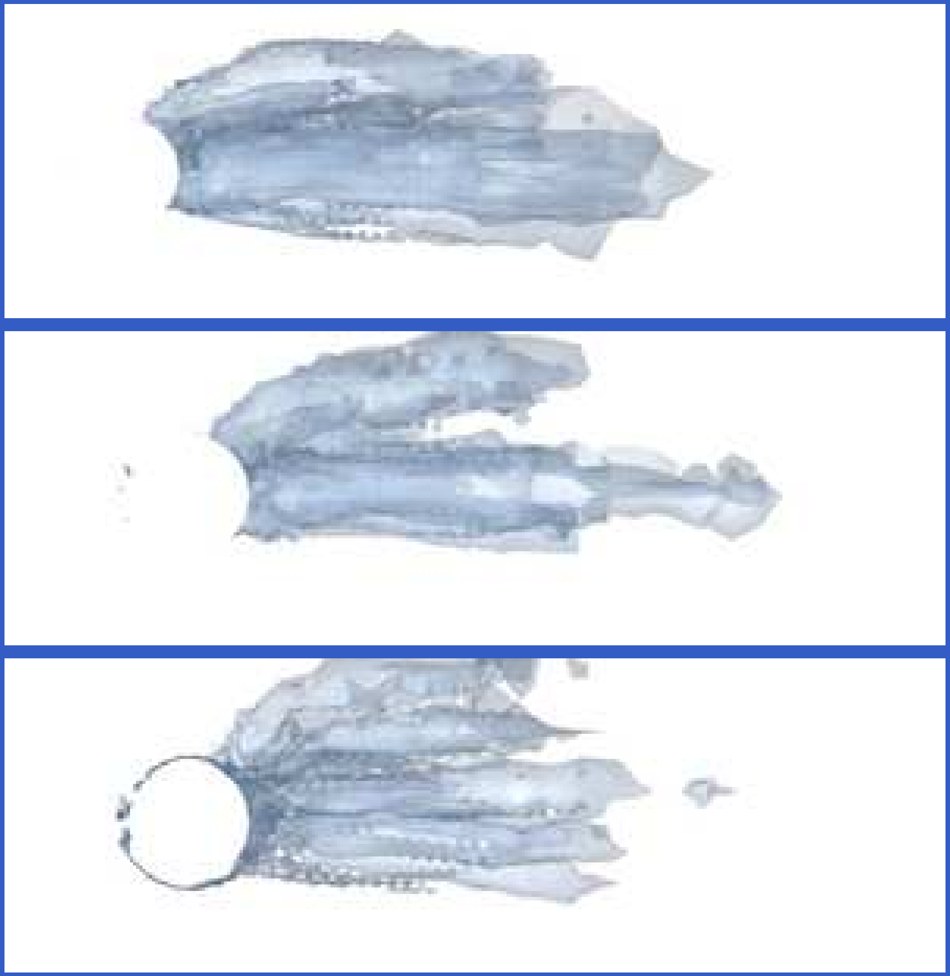
Vorticity: ω_1, ω_2, ω_3: t=1.75: $c_D = 0.36$



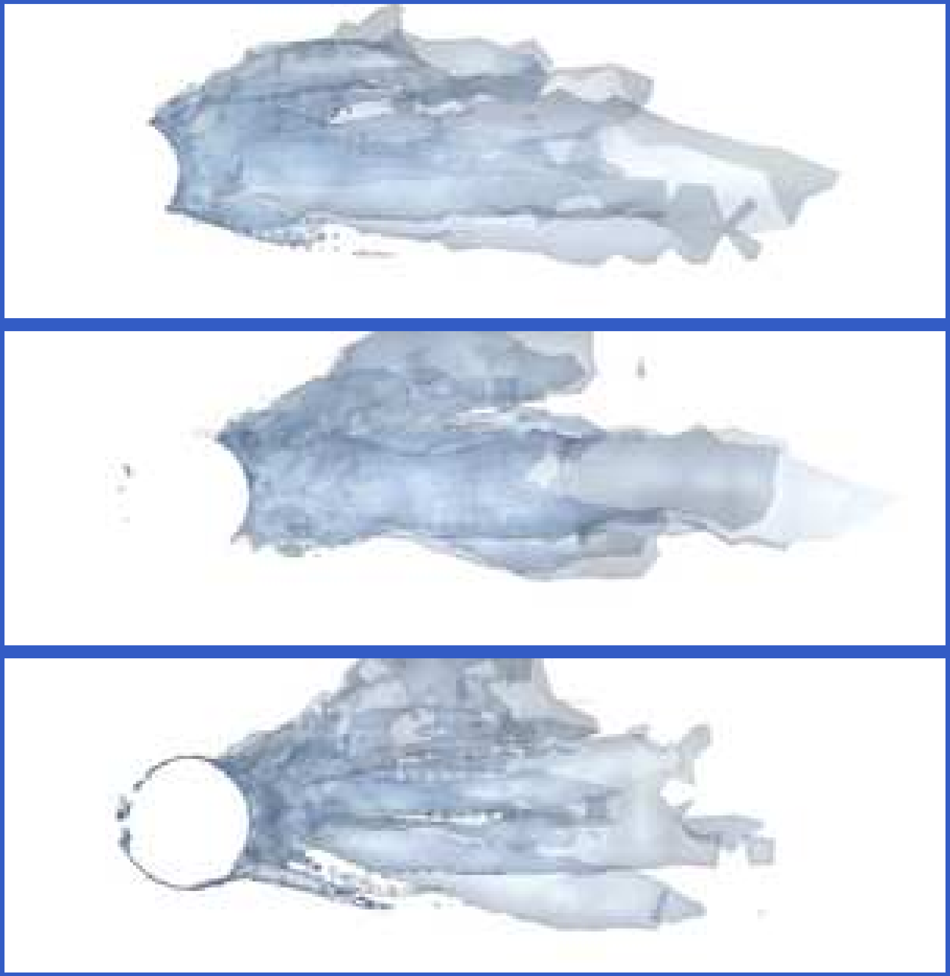
Vorticity: ω_1, ω_2, ω_3: t=2.0: $c_D = 0.51$



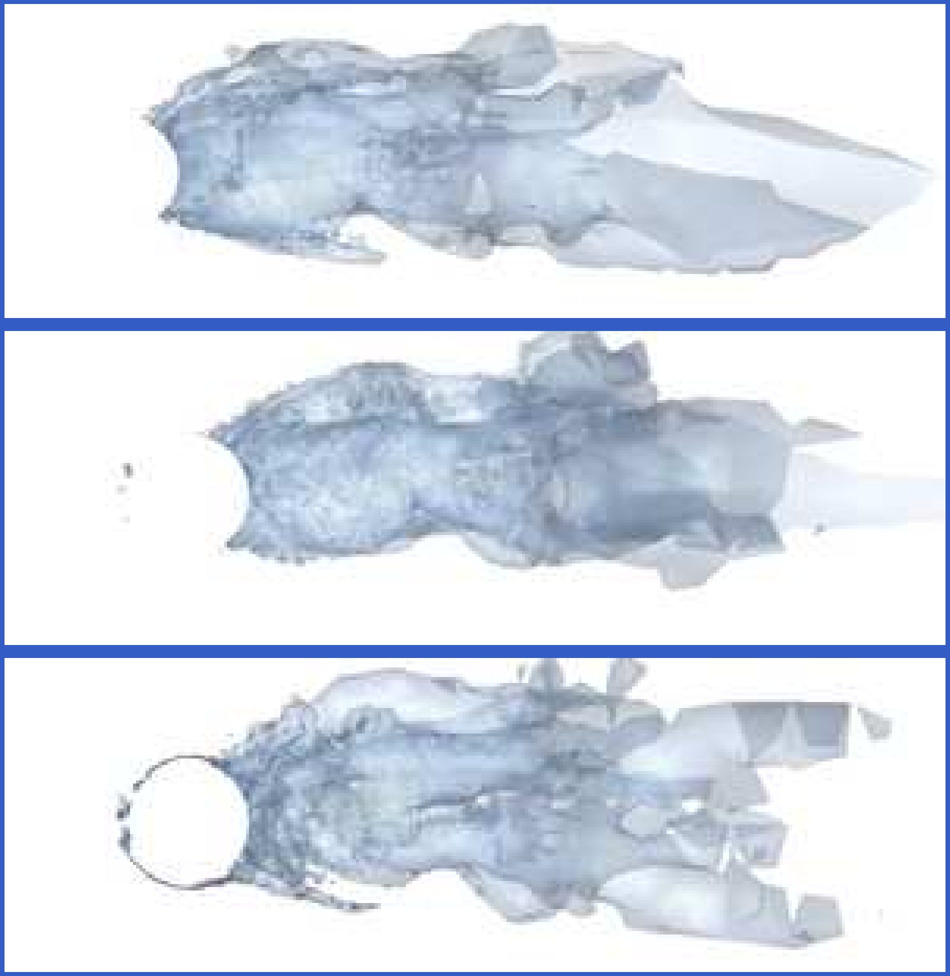
Vorticity: ω_1, ω_2, ω_3: t=2.25: $c_D = 0.78$



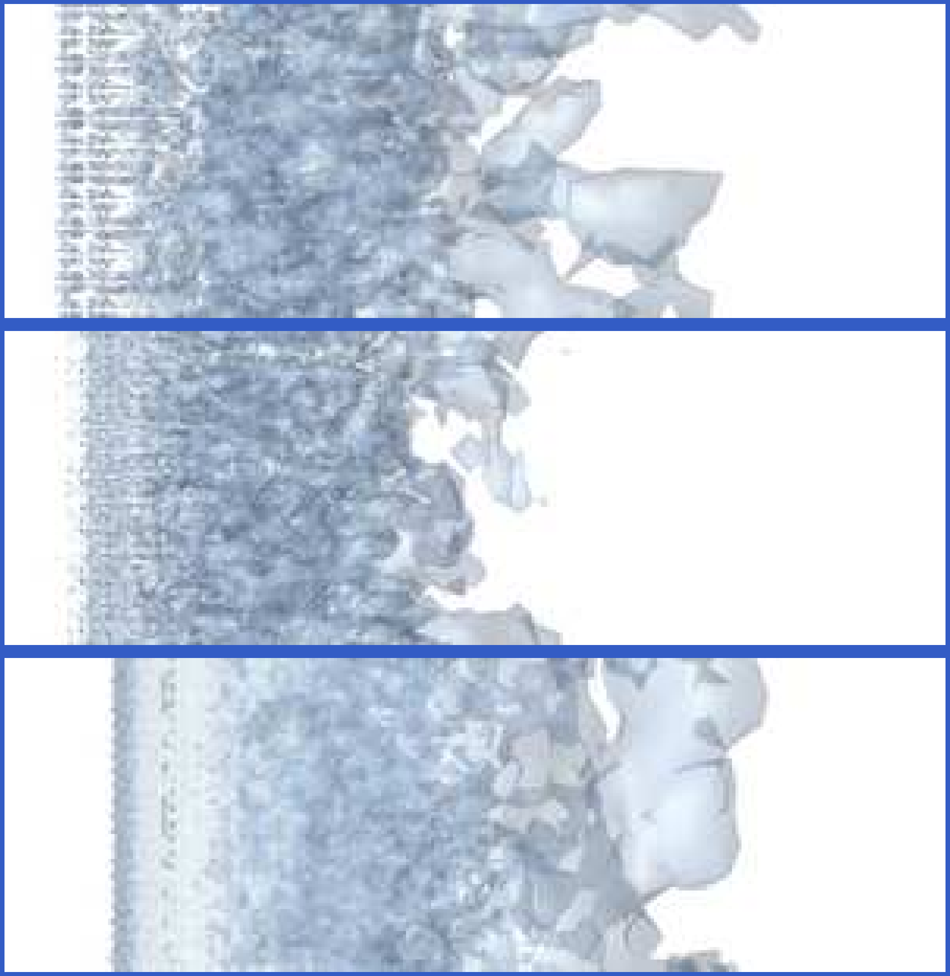
Vorticity: ω_1, ω_2, ω_3: t=2.5: $c_D = 1.14$



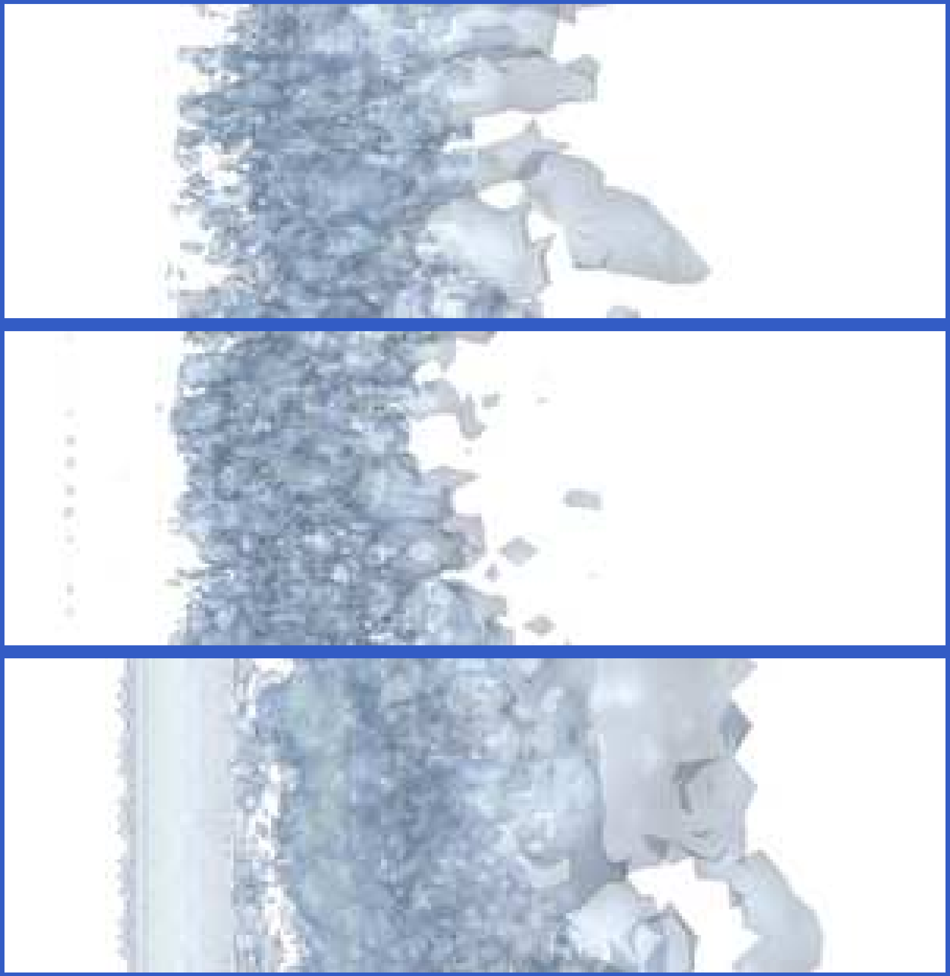
Vorticity: ω_1, ω_2, ω_3: **t=2.75:** $c_D = 1.04$



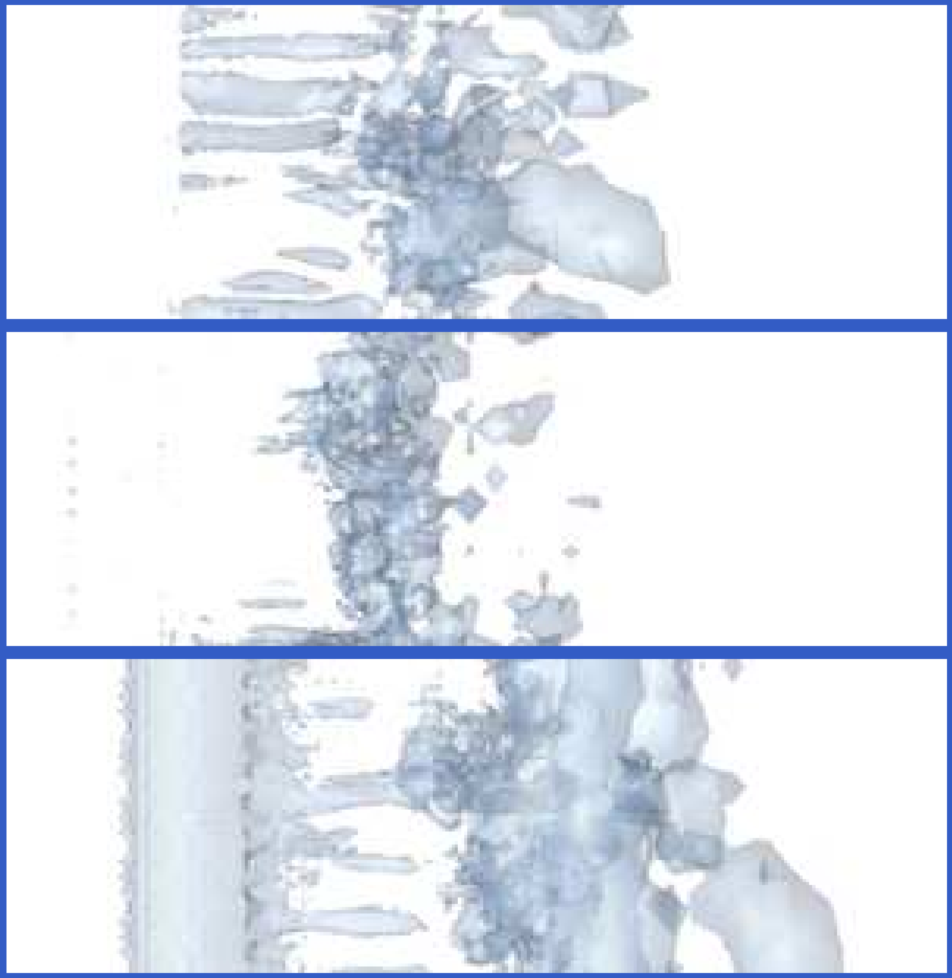
Vorticity: $\omega_1, \omega_2, \omega_3$**: t=0.0:** $c_D = 1.03$



Vorticity: ω_1, ω_2, ω_3: t=0.25: $c_D = 0.06$

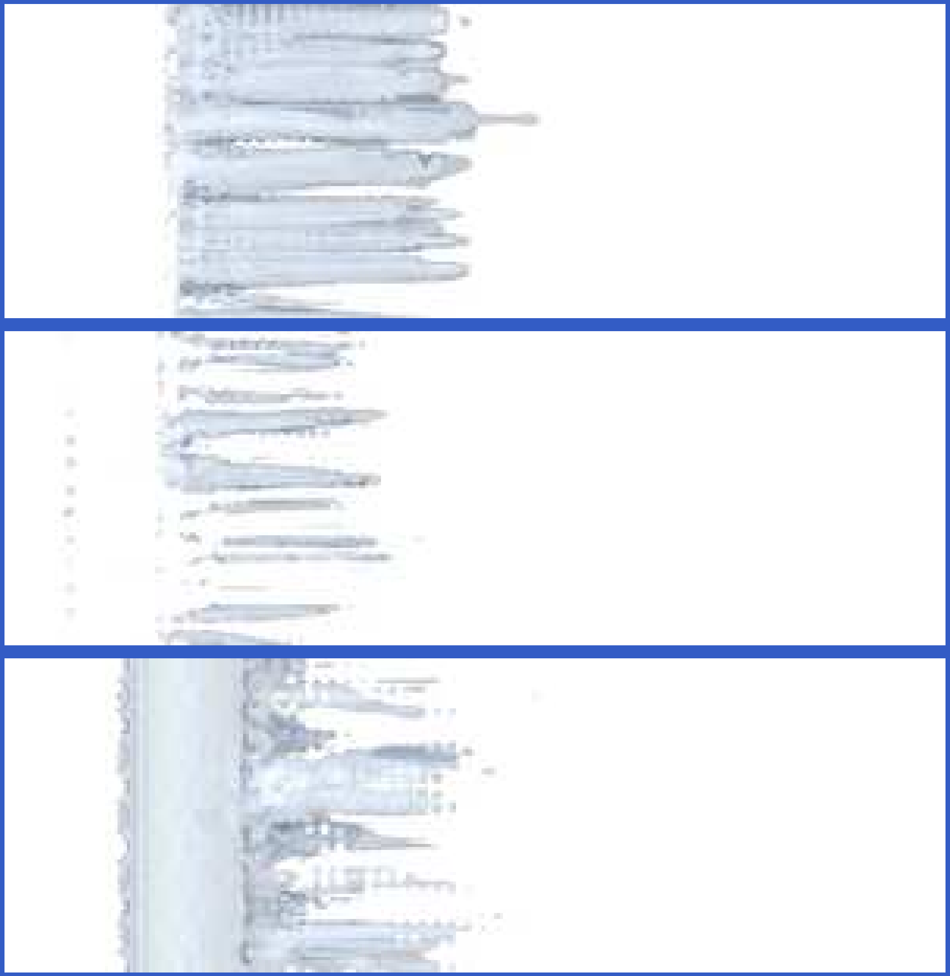


Vorticity: ω_1, ω_2, ω_3: **t=0.5:** $c_D = 0.10$

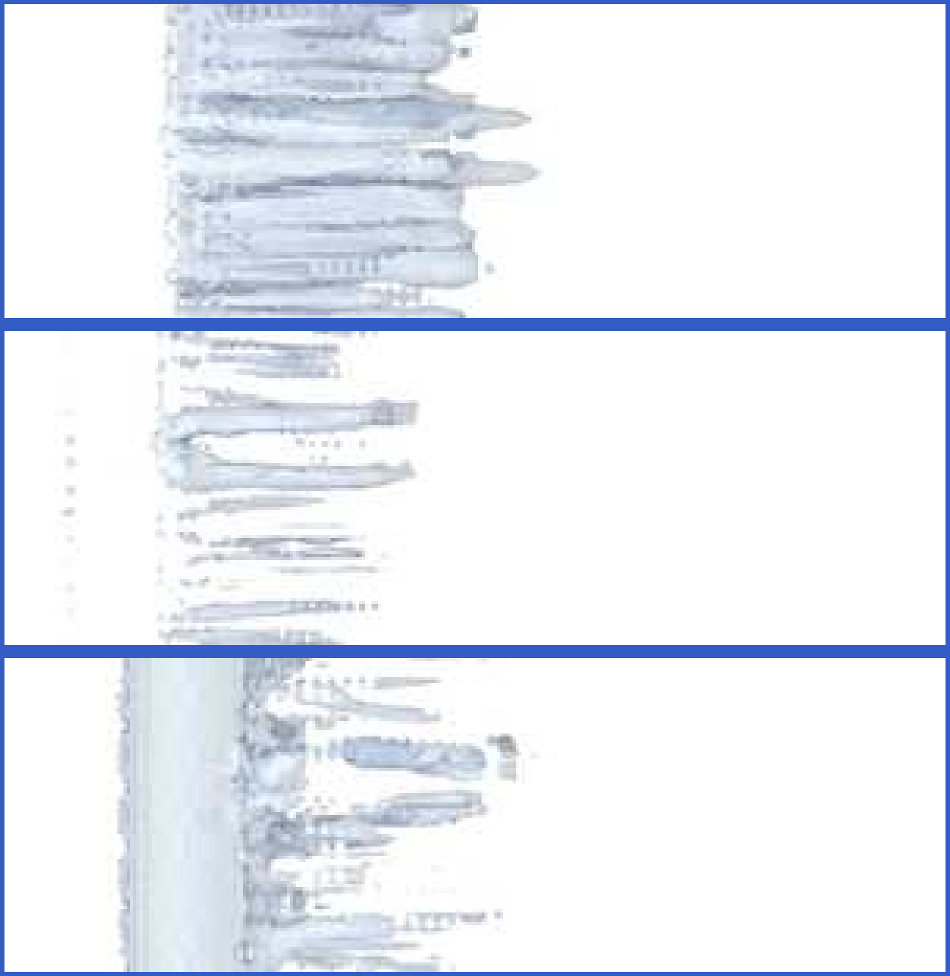


Vorticity: ω_1, ω_2, ω_3: **t=0.75:** $c_D = 0.15$

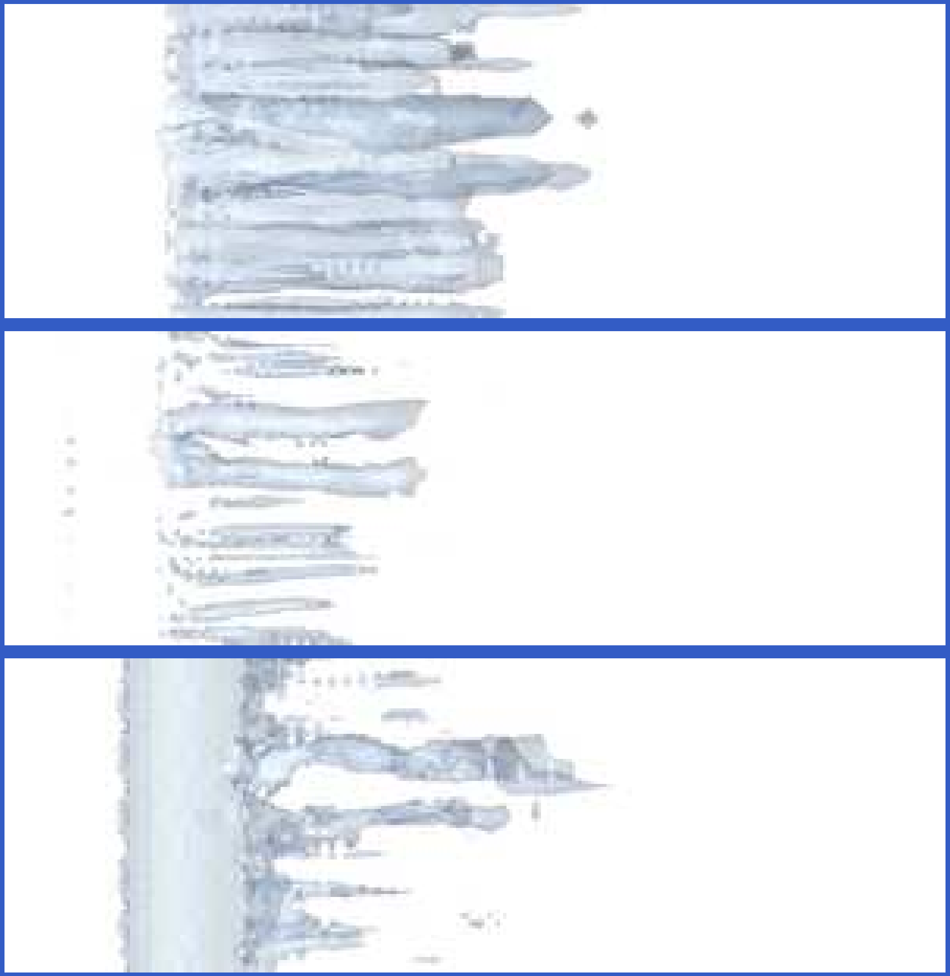
Vorticity: $\omega_1, \omega_2, \omega_3$: t=1.0: $c_D = 0.22$



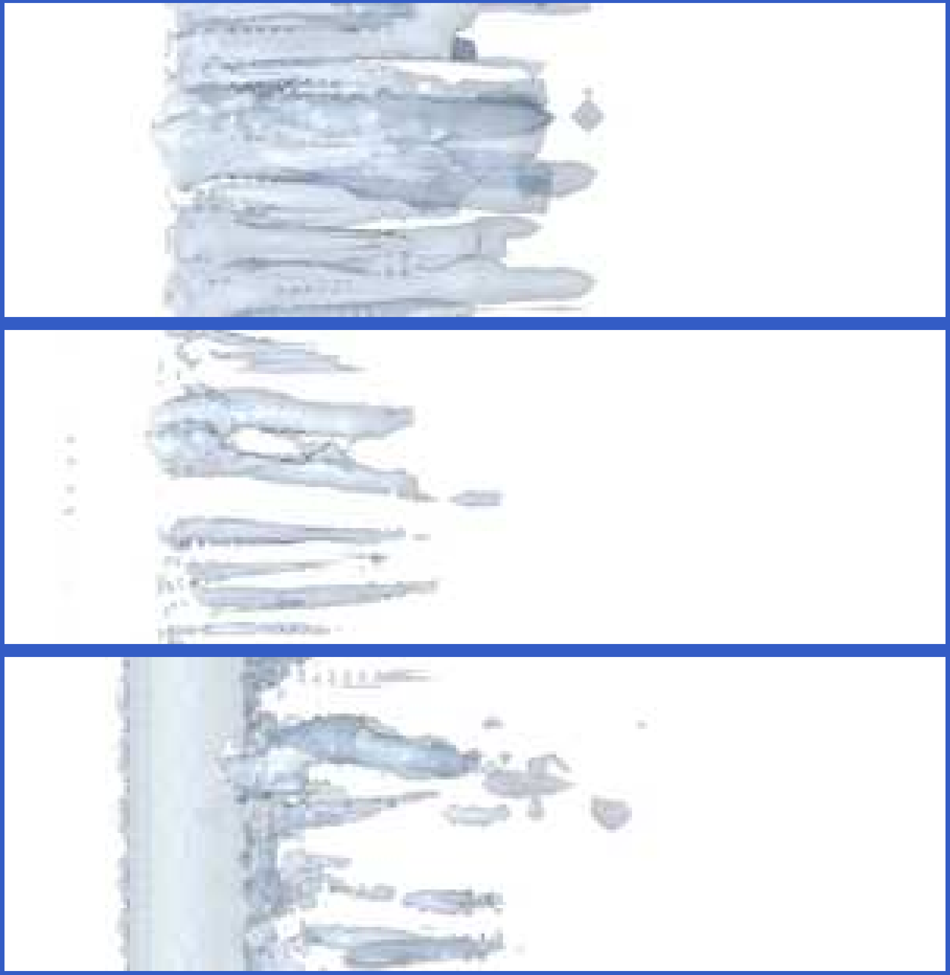
Vorticity: ω_1, ω_2, ω_3: t=1.25: $c_D = 0.25$



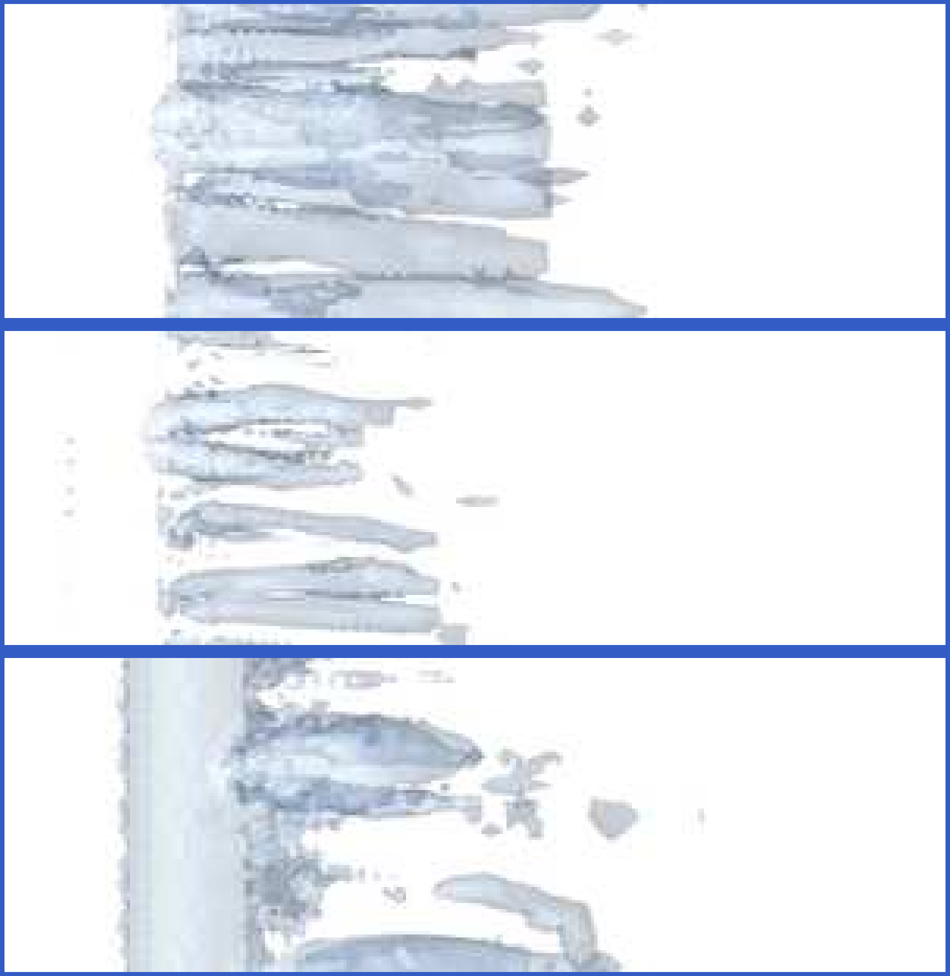
Vorticity: ω_1, ω_2, ω_3: **t=1.5:** $c_D = 0.28$



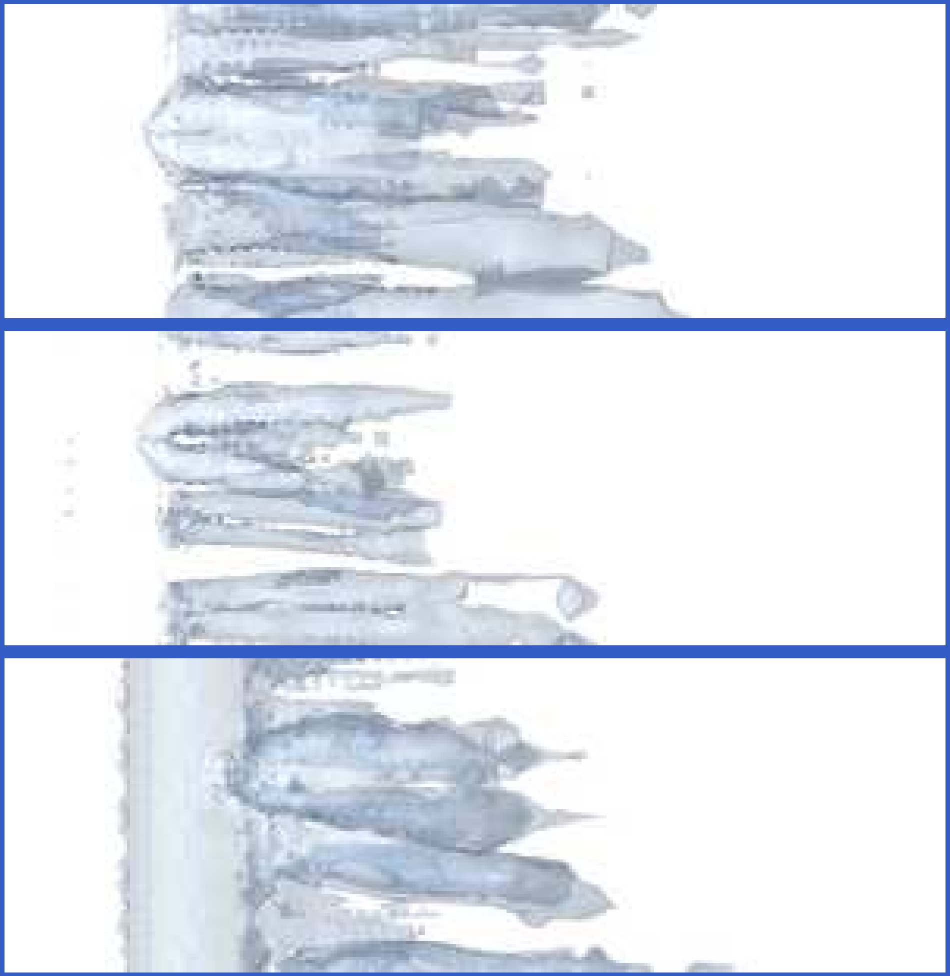
Vorticity: ω_1, ω_2, ω_3: t=1.75: $c_D = 0.36$



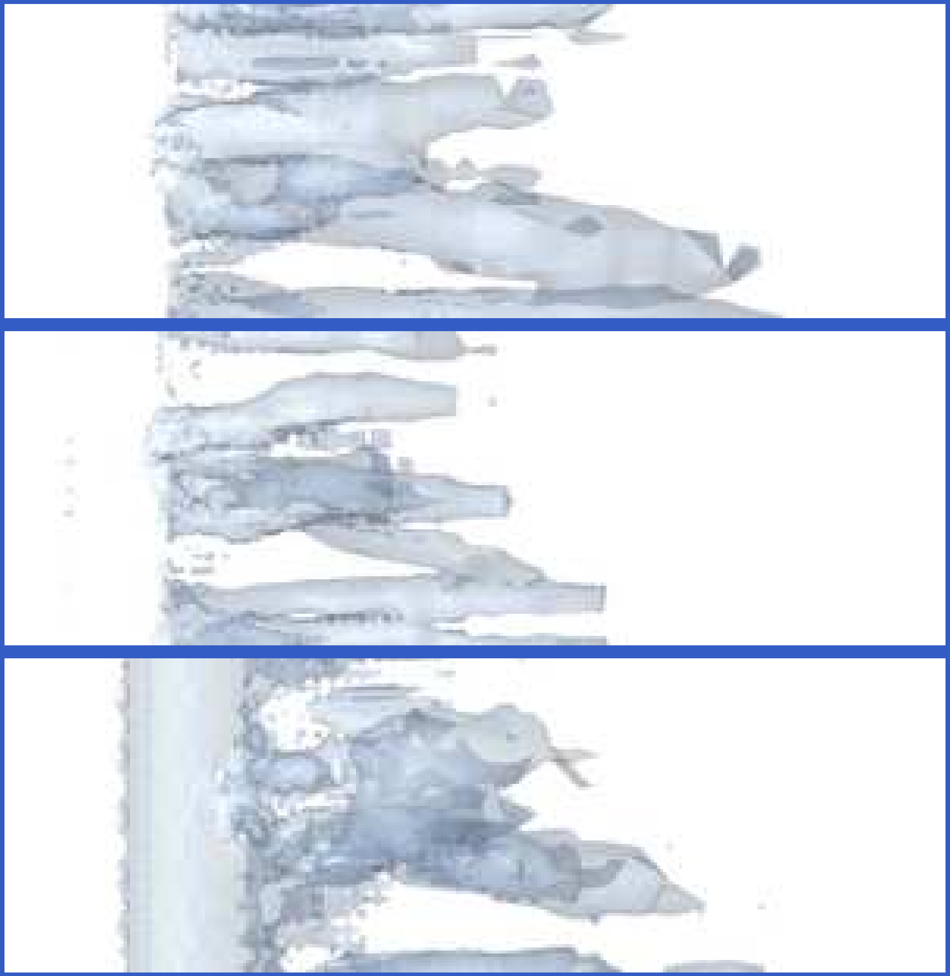
Vorticity: ω_1, ω_2, ω_3: t=2.0: $c_D = 0.51$



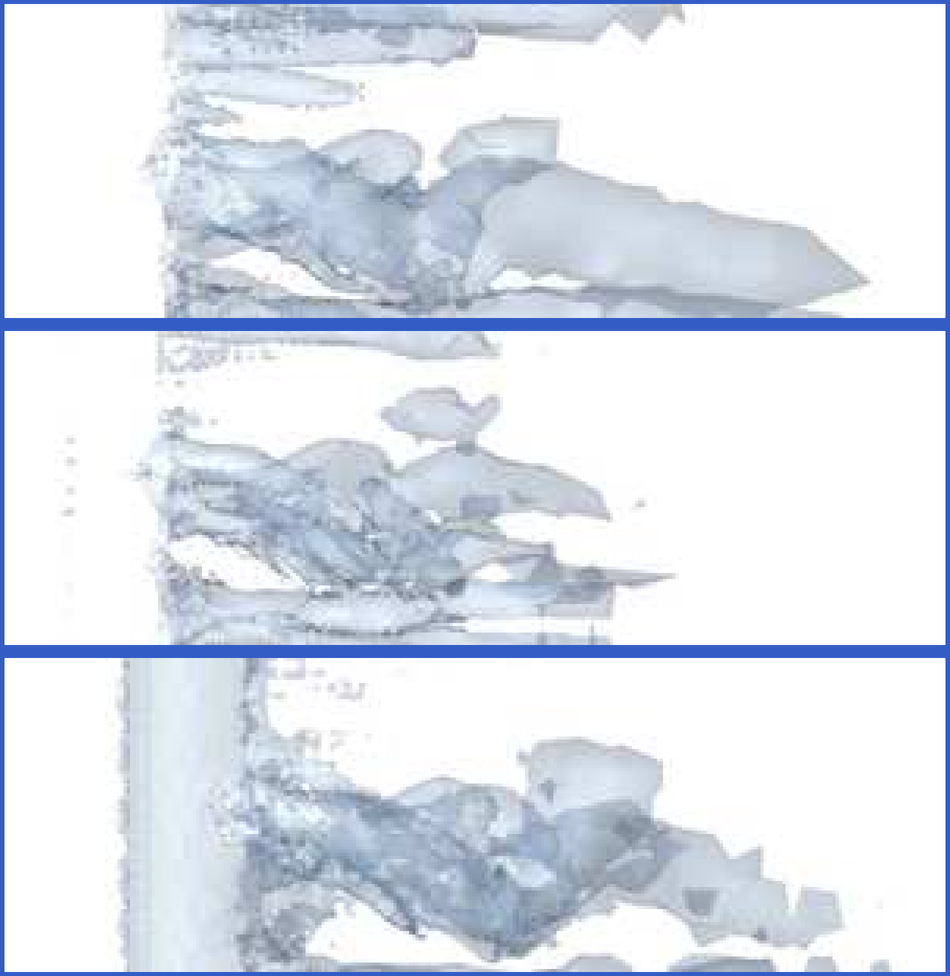
Vorticity: $\omega_1, \omega_2, \omega_3$: t=2.25: $c_D = 0.78$



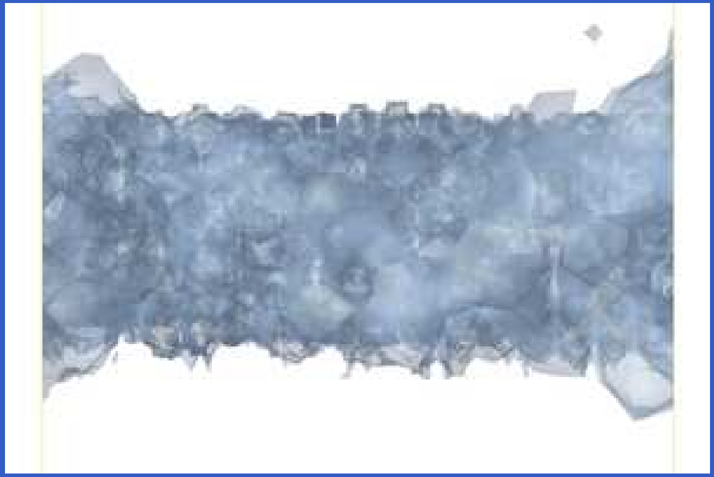
Vorticity: ω_1, ω_2, ω_3: t=2.5: $c_D = 1.14$

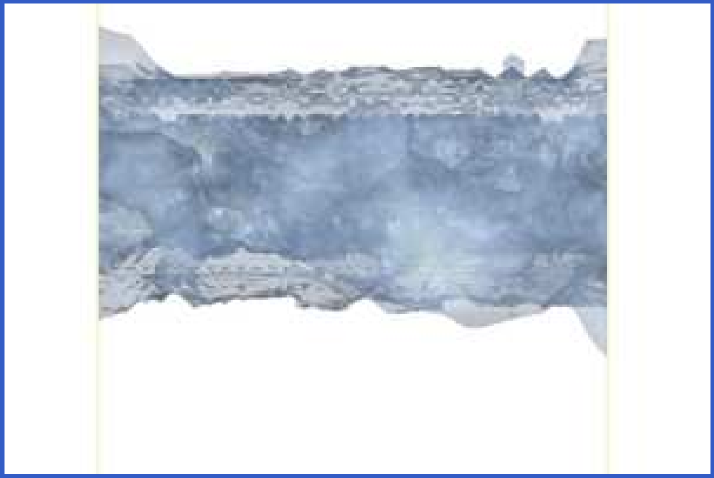


Vorticity: ω_1, ω_2, ω_3: **t=2.75:** $c_D = 1.04$

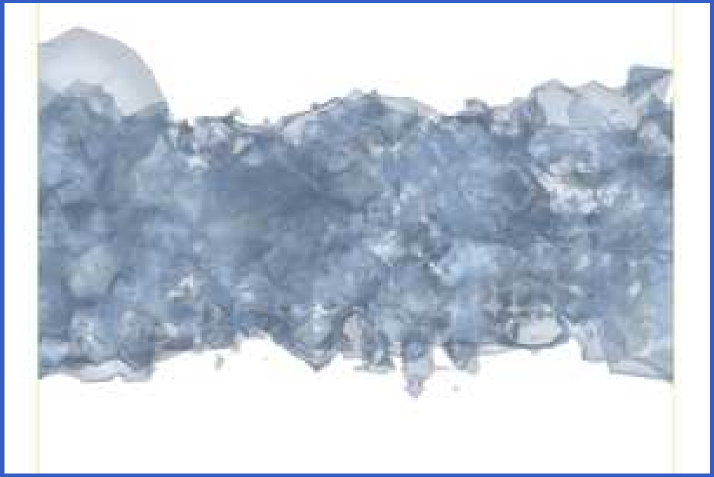


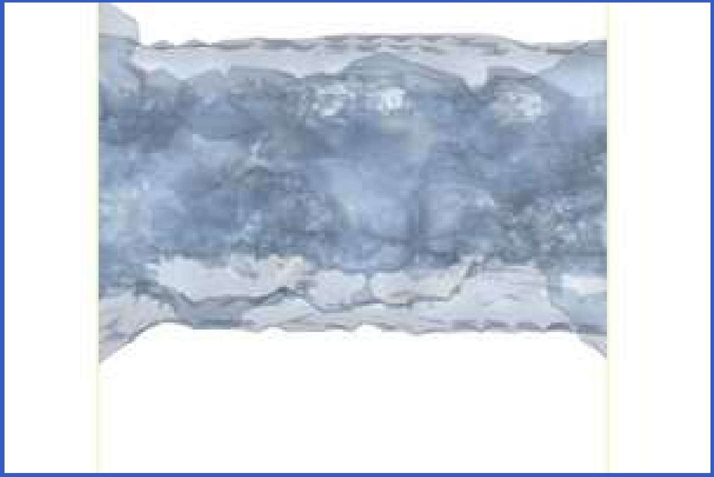
Vorticity: $\omega_1, \omega_2, \omega_3$: **t=0.0:** $c_D = 1.03$



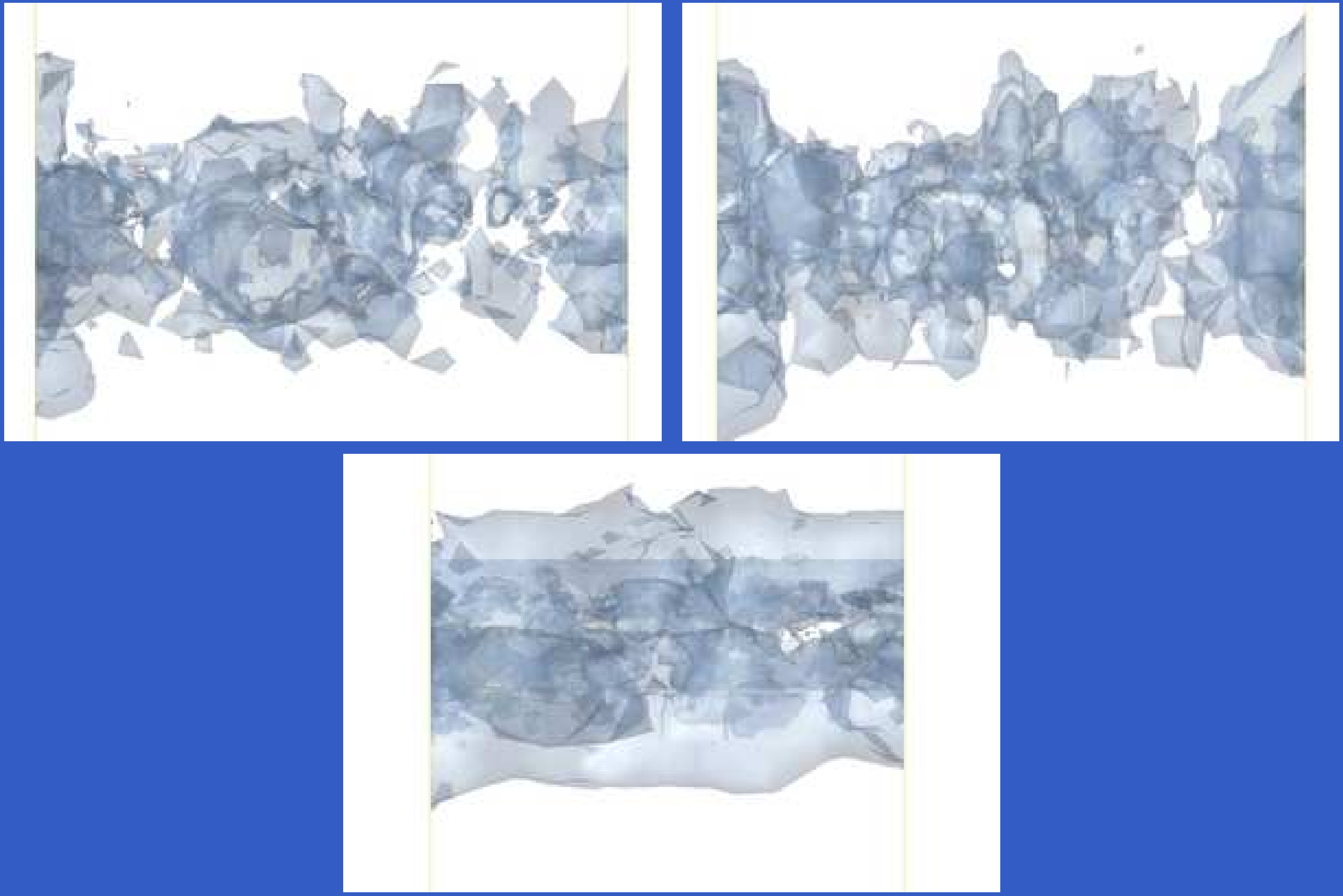


Vorticity: ω_1, ω_2, ω_3: t=0.25: $c_D = 0.06$

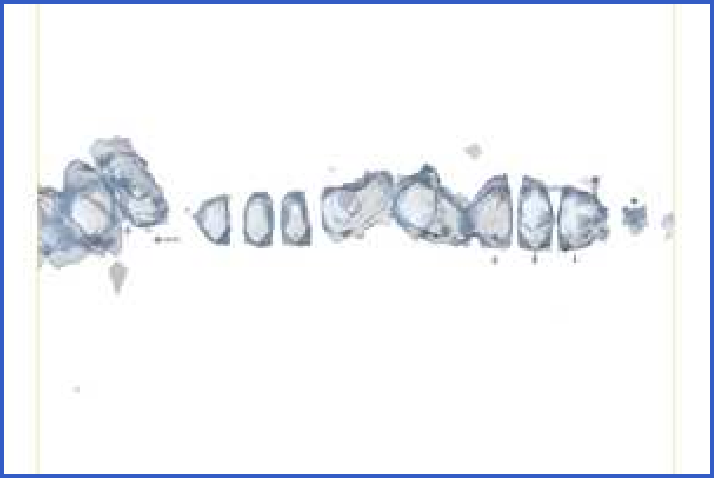


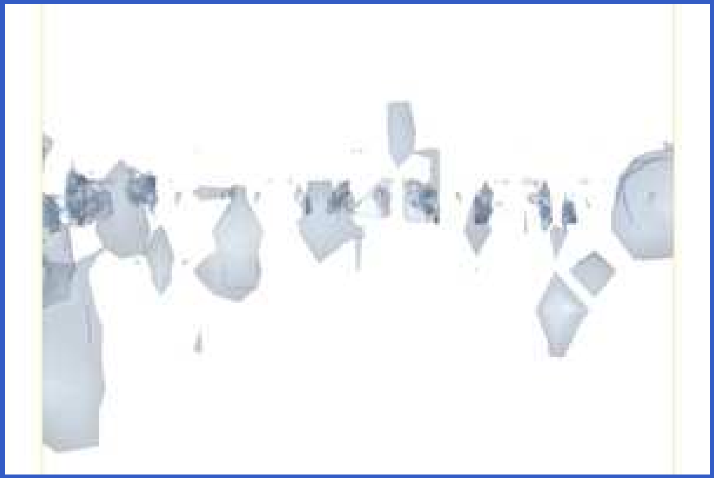


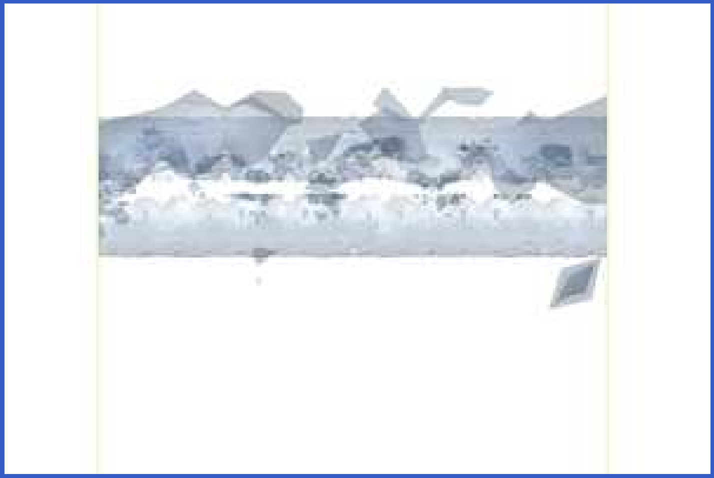
Vorticity: ω_1, ω_2, ω_3: t=0.5: $c_D = 0.10$



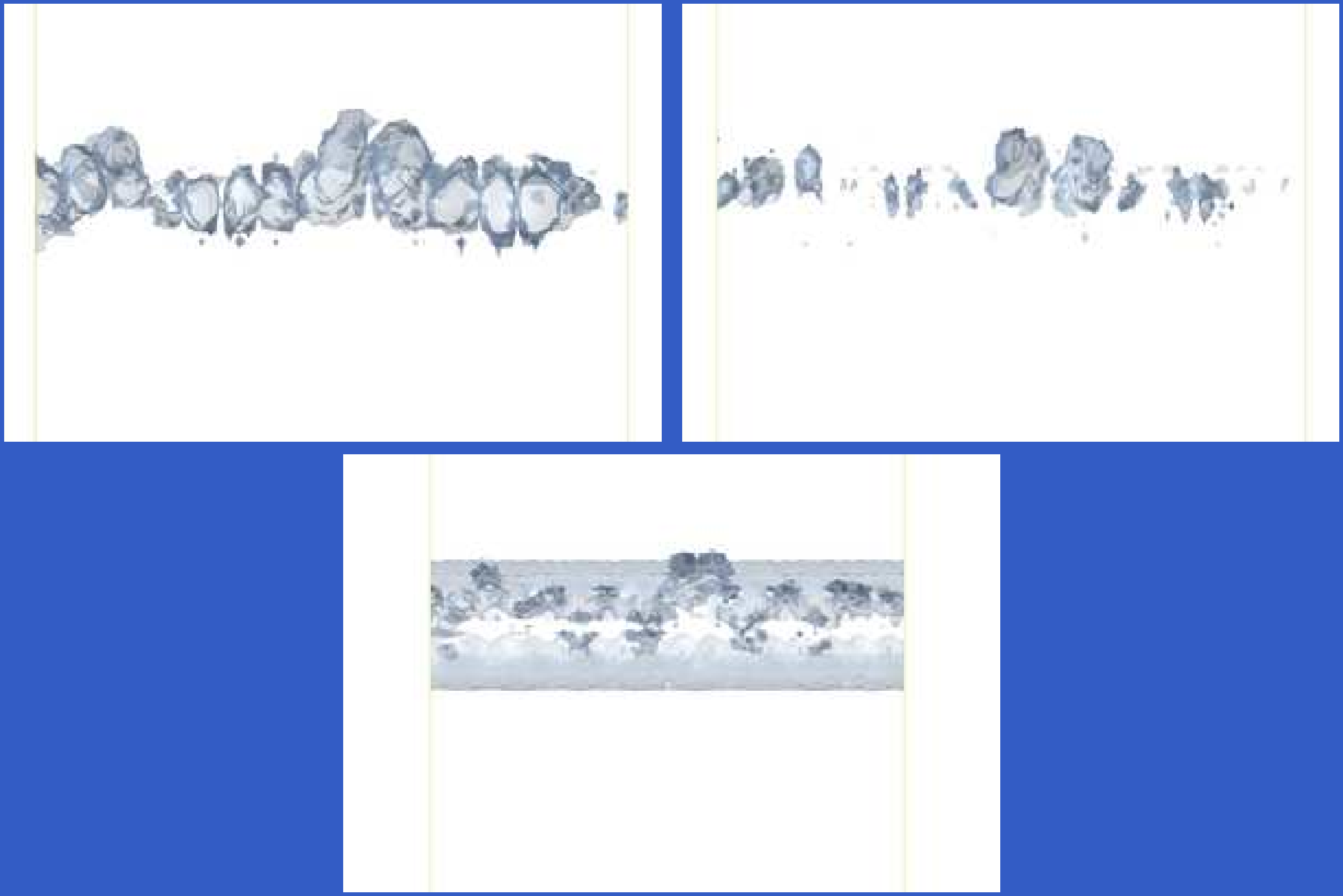
Vorticity: $\omega_1, \omega_2, \omega_3$: t=0.75: $c_D = 0.15$



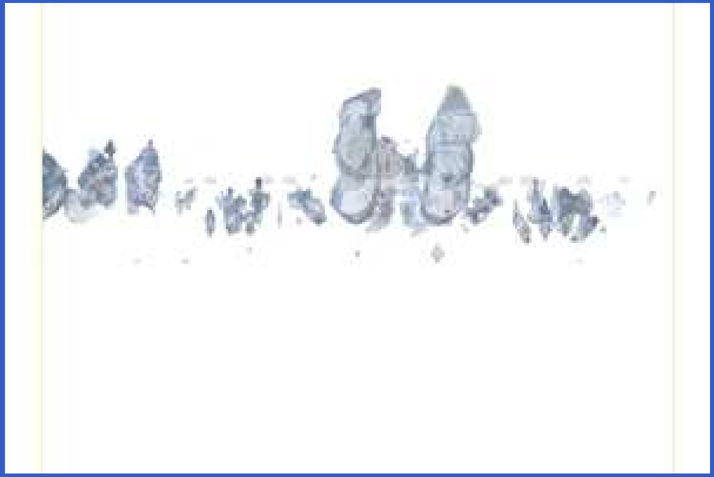




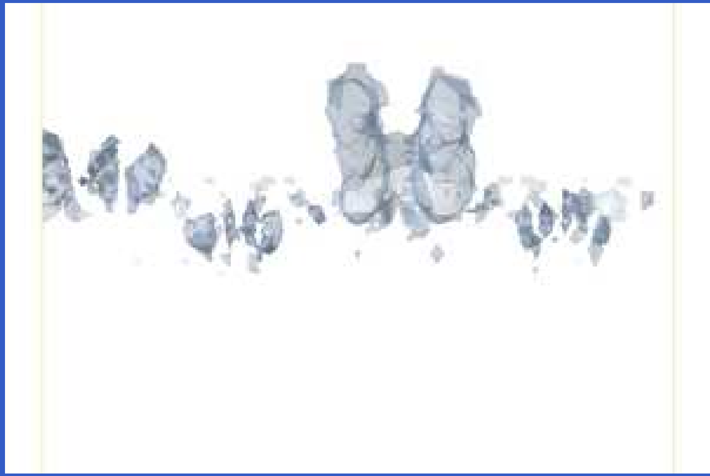
Vorticity: ω_1, ω_2, ω_3: t=1.0: $c_D = 0.22$

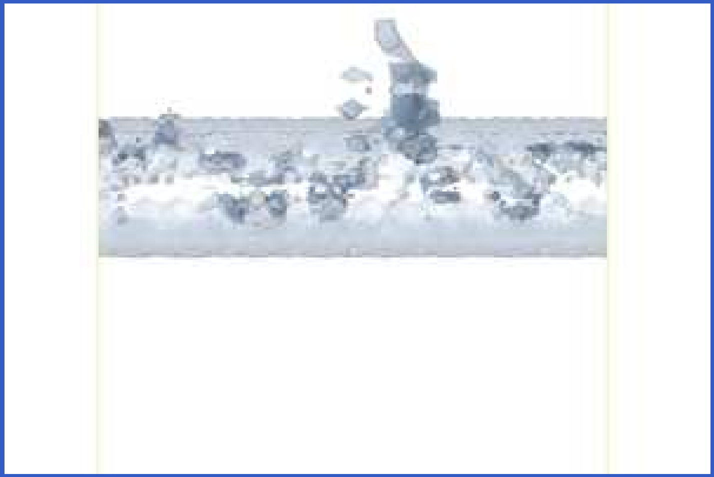


Vorticity: ω_1, ω_2, ω_3: t=1.25: $c_D = 0.25$

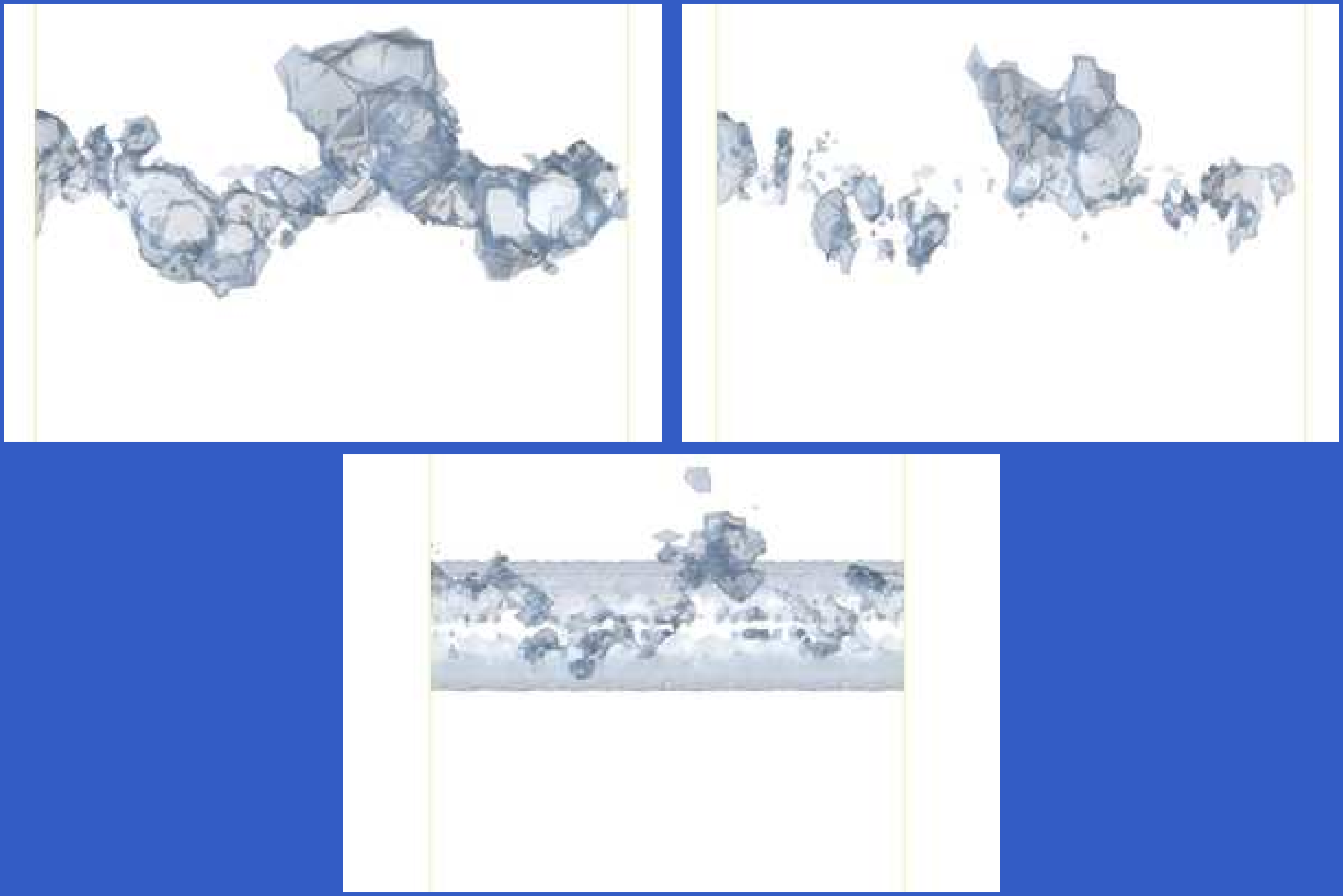


Vorticity: ω_1**,** ω_2**,** ω_3**: t=1.5:** $c_D = 0.28$

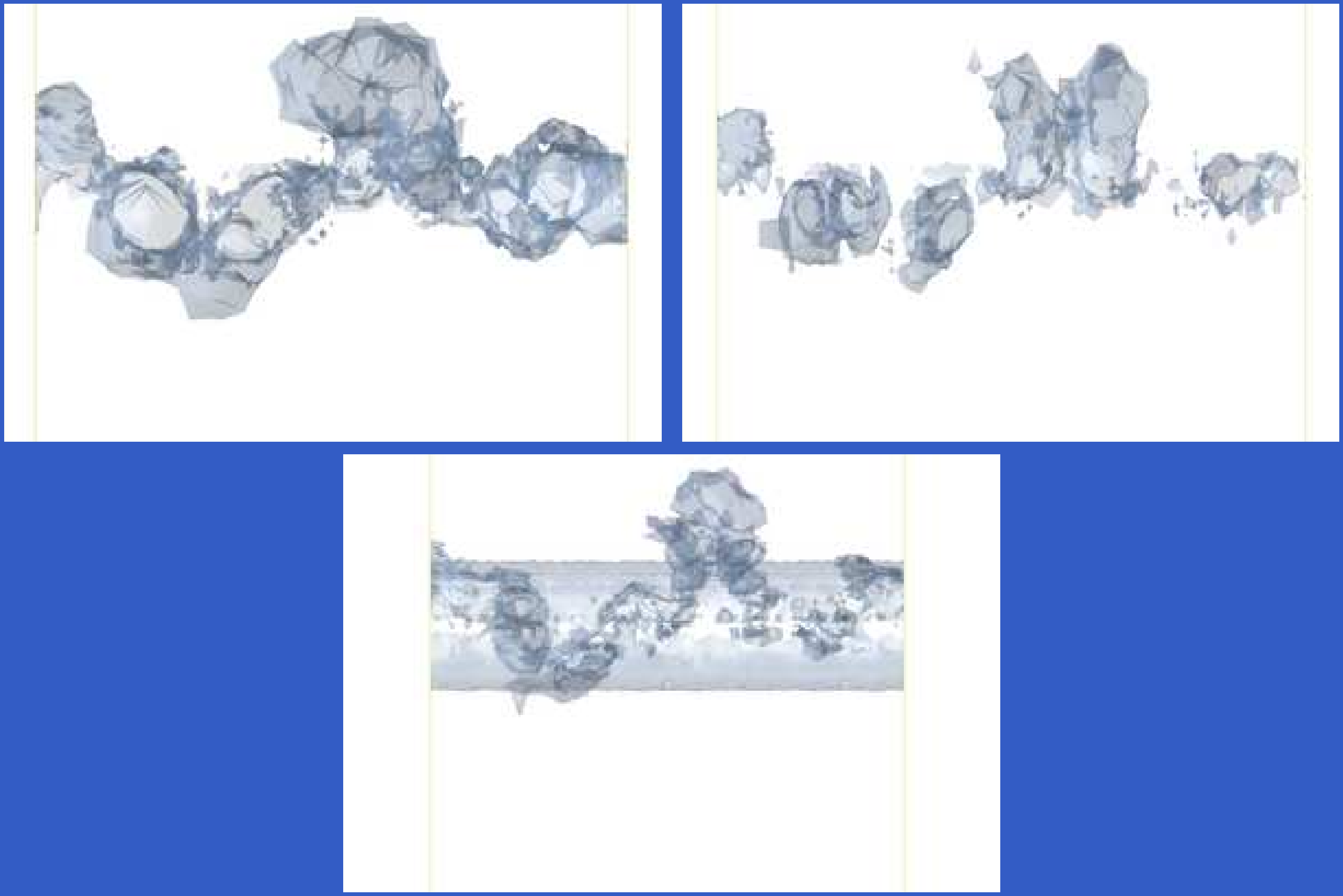




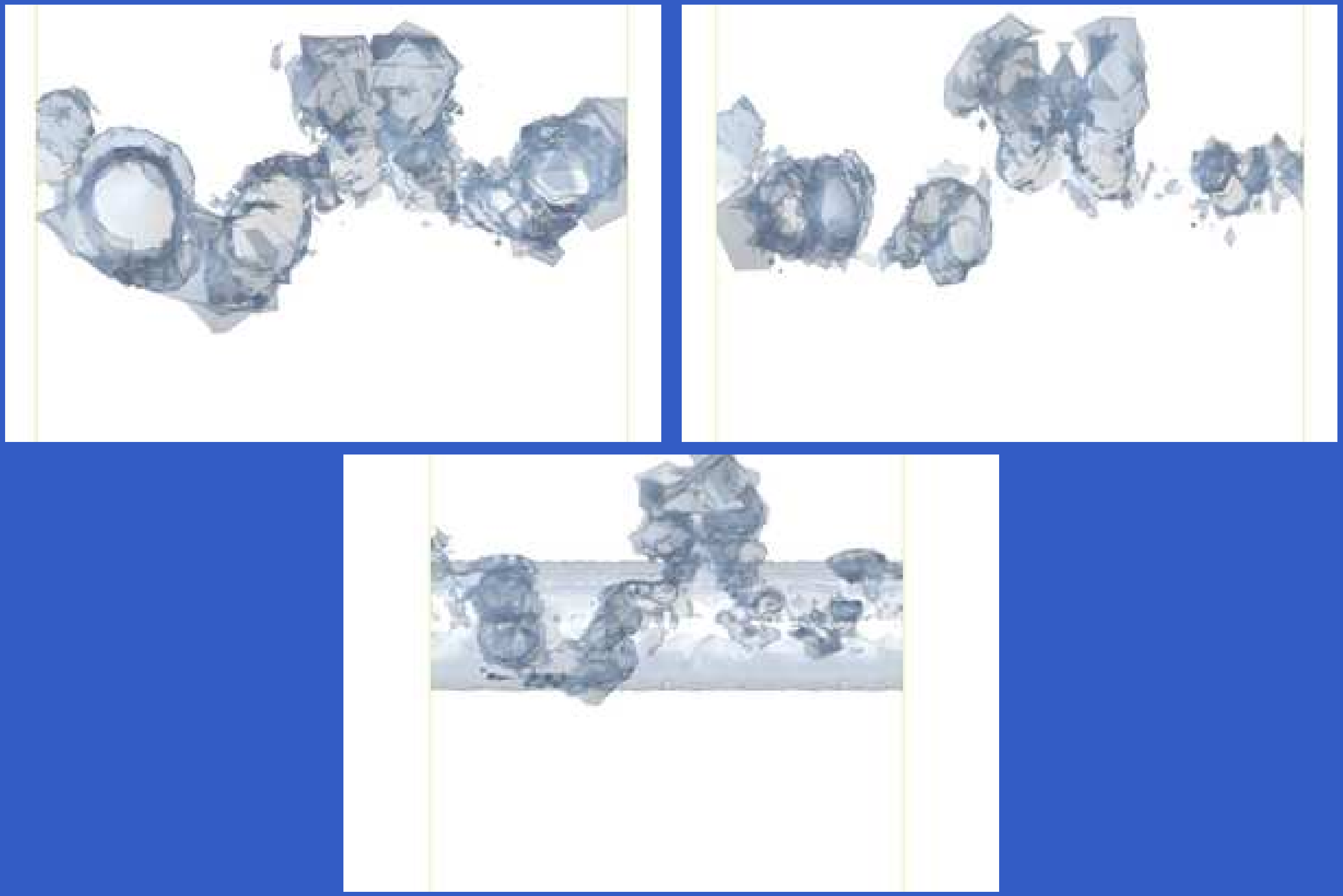
Vorticity: $\omega_1, \omega_2, \omega_3$: t=1.75: $c_D = 0.36$



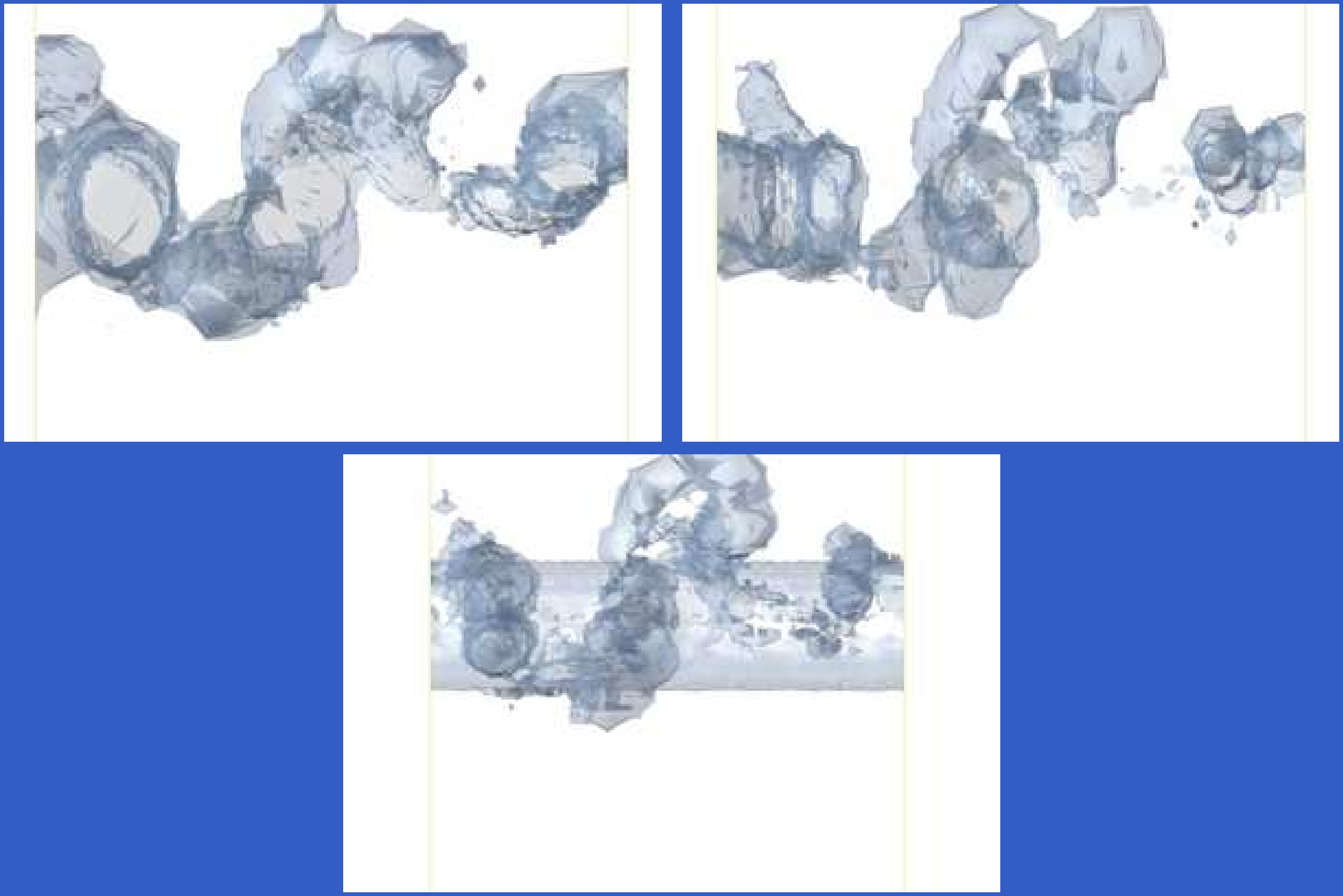
Vorticity: $\omega_1, \omega_2, \omega_3$: t=2.0: $c_D = 0.51$



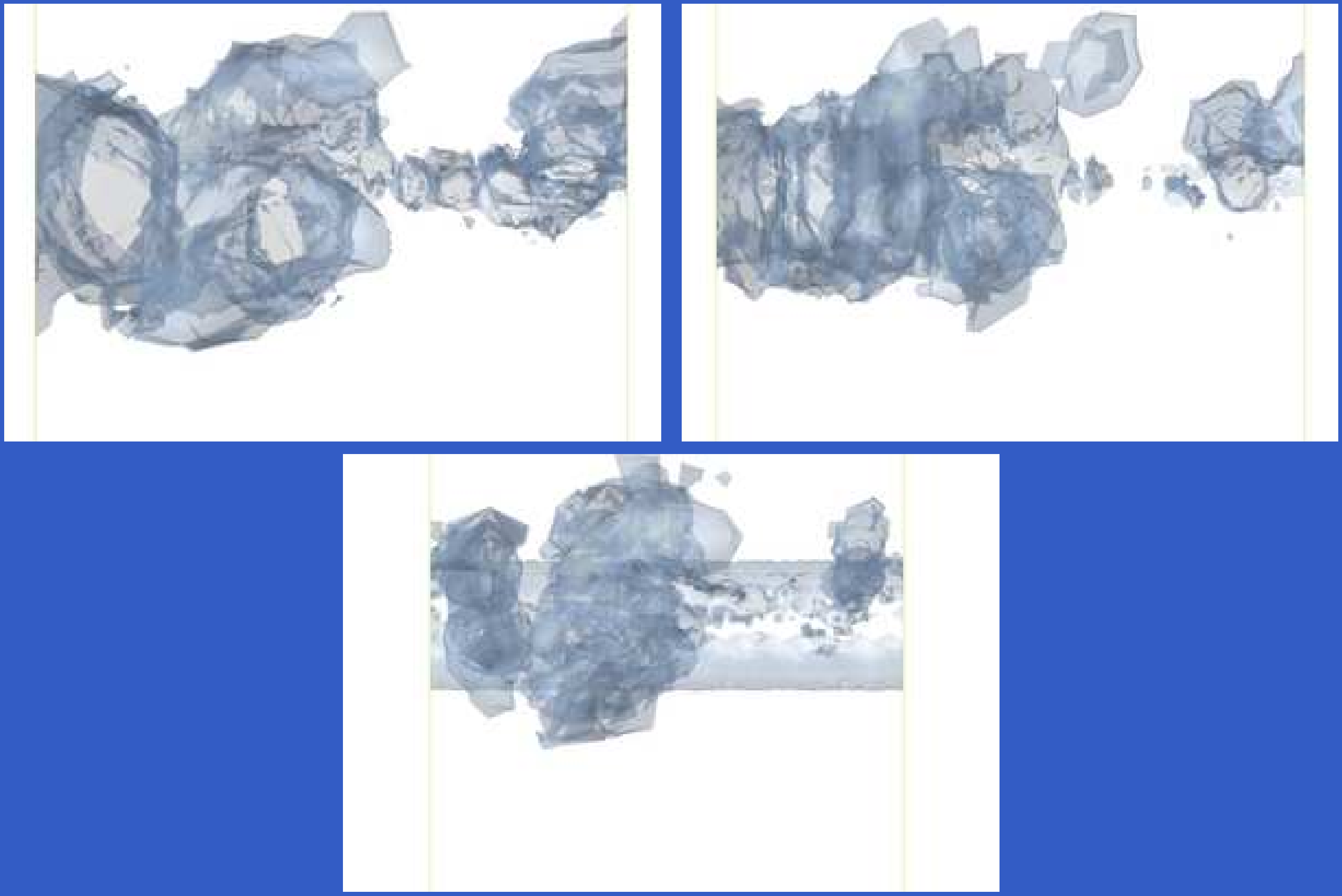
Vorticity: ω_1, ω_2, ω_3: t=2.25: $c_D = 0.78$



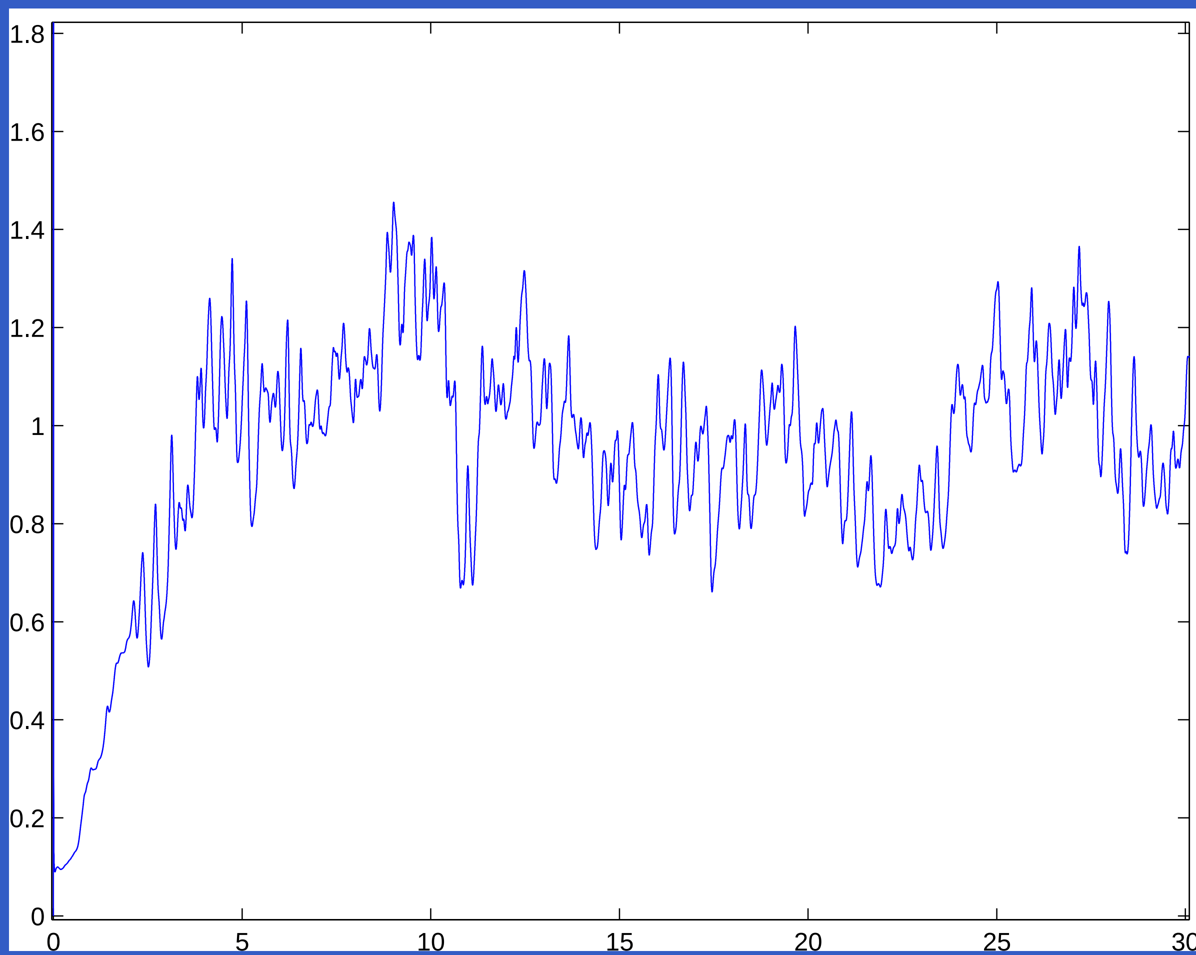
Vorticity: ω_1, ω_2, ω_3: t=2.5: $c_D = 1.14$



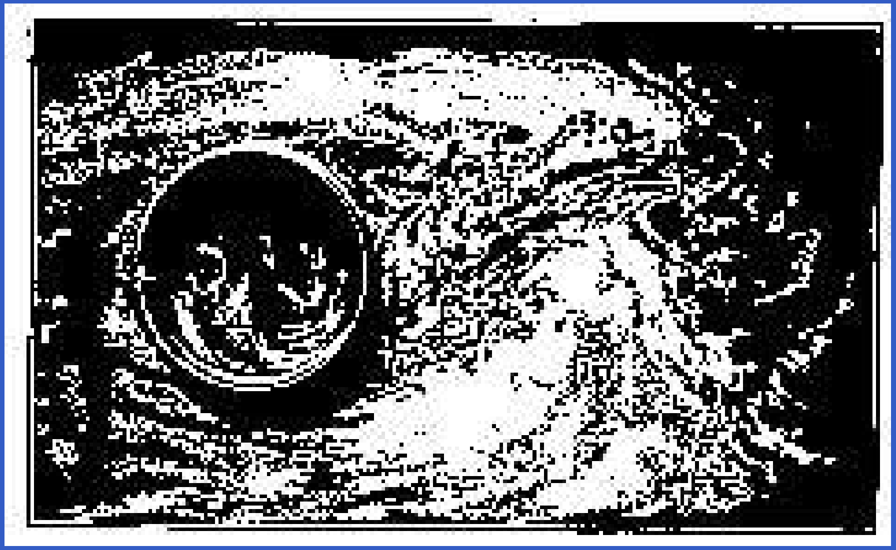
Vorticity: $\omega_1, \omega_2, \omega_3$: t=2.75: $c_D = 1.04$

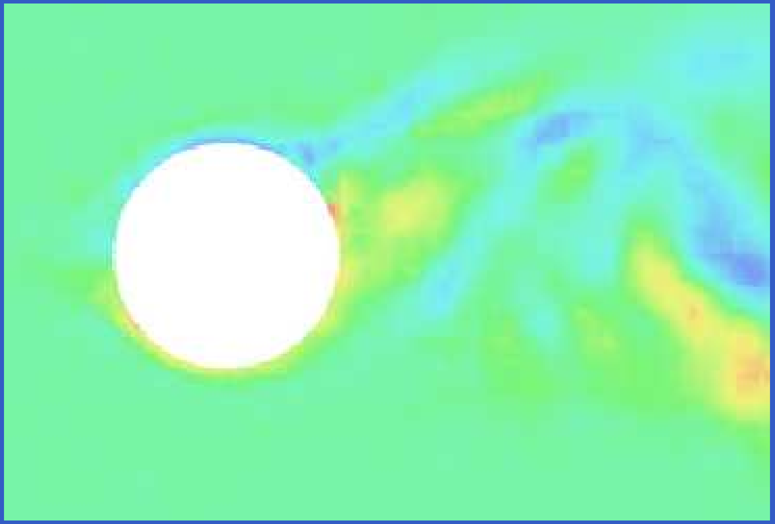


Drag Coefficient



PRANDTL EXP vs EULER/G2





Guadalupe Aug 20 1999

PROOF OF BLOWUP

(A) EG2 WELLPOSED DRAG: H^{-1} PERTURB

(B) POT SOL NOT WELLPOSED

(C) EG2 TURB NONSMOOTH:

$$\|R\|_0 \approx h^{-1/2}$$

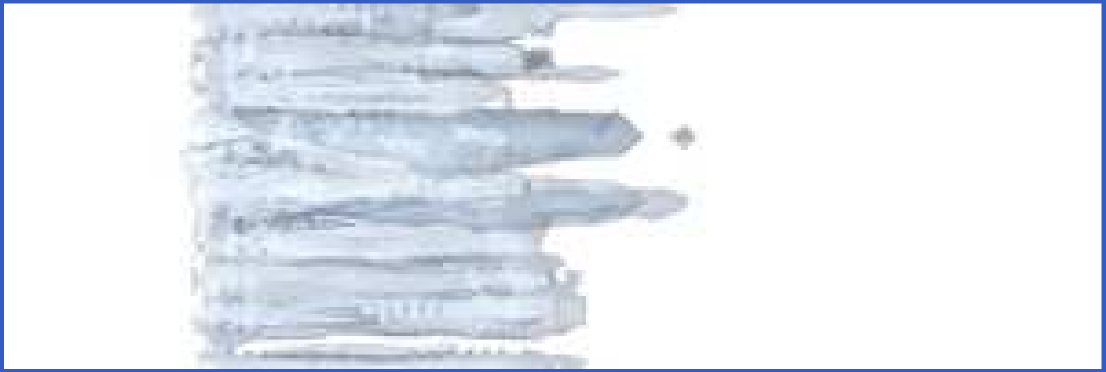
$$\|R\|_{-1} \approx h^{1/2}$$

(D) FINITE MESH SIZE ENOUGH

EG2 WELLPOSED

- COMPUTED S MODERATE SIZE!! NEW!!
- CANCELLATION/SMOOTH DATA IN DUAL!!
- EG2 SOL REPRESENTATIVE SOLUTION!!
- EG2 BLOWUP $\Rightarrow$ BLOWUP
- SOLVES D'ALEMBERT'S PARADOX

STREAMW VORTICITY AT SEP

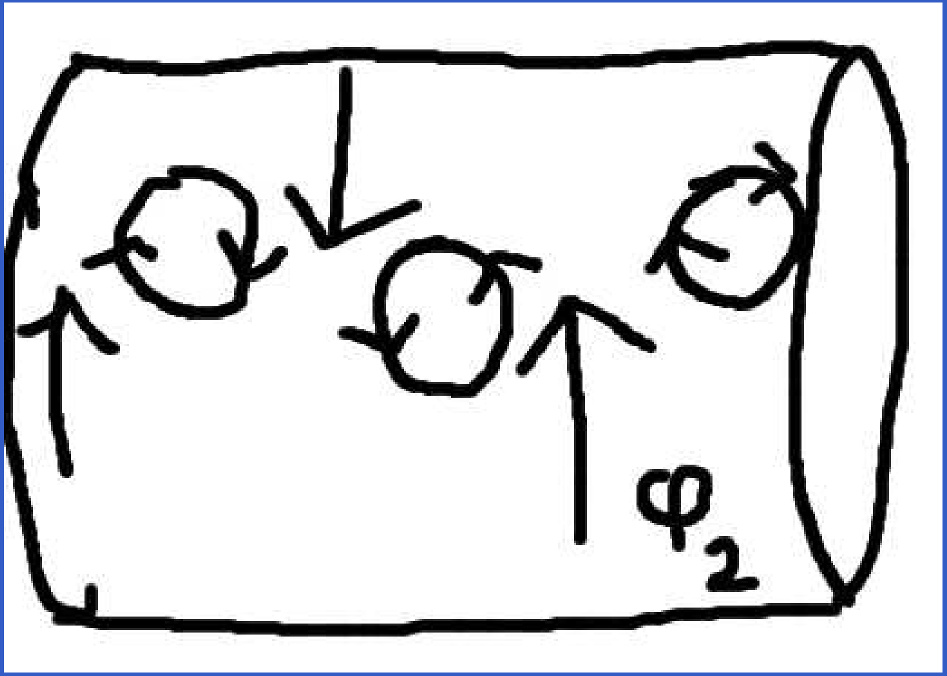


- LOW PRESSURE: DRAG
- WING: LIFT/DRAG

BLOWUP REAR SEPARATION

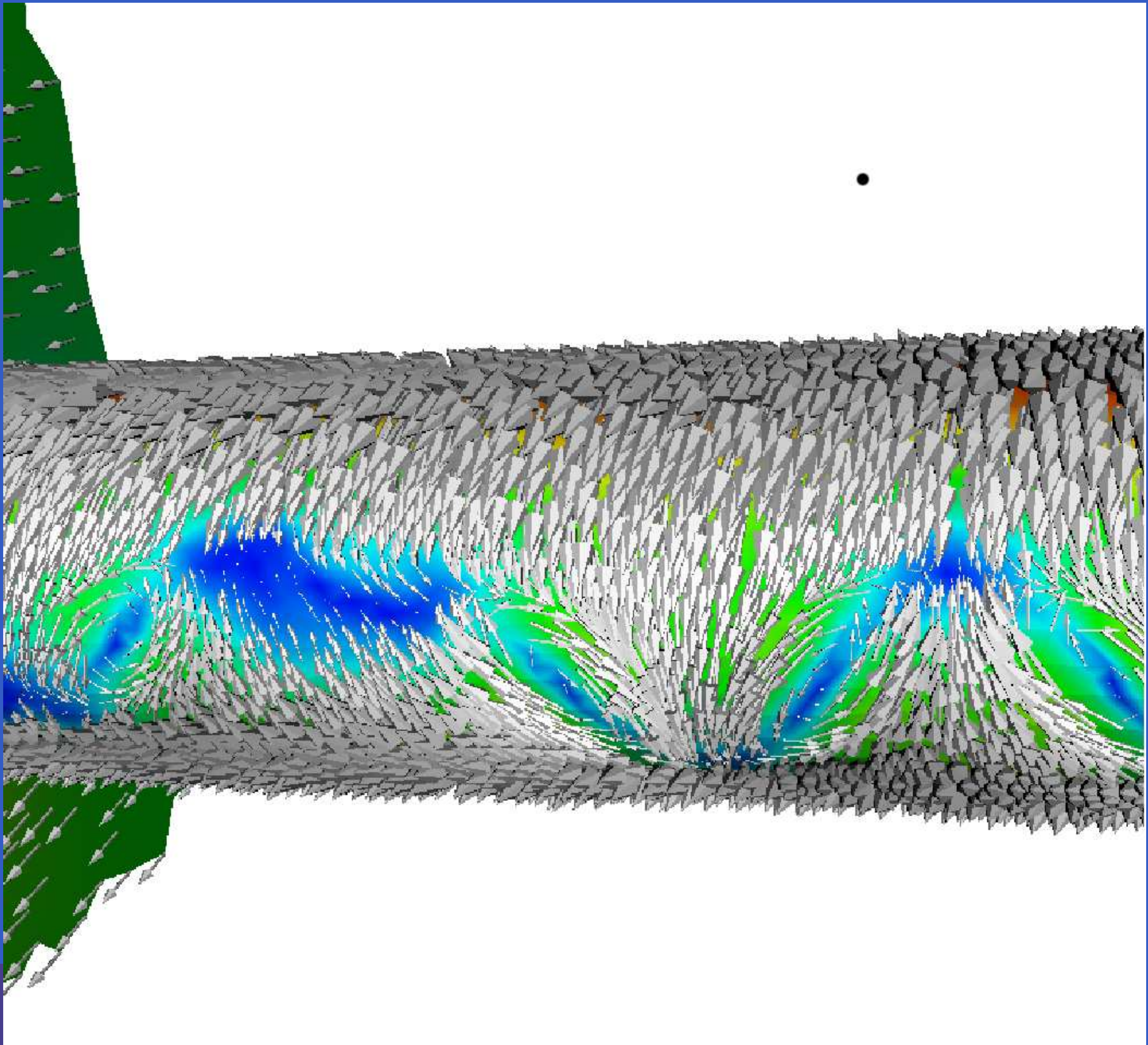
- POT FLOW in half-plane $\{x_1 > 0\}$:
- $u = (x_1, -x_2, 0)$
- LIN EQ: $\dot\varphi_2 - \varphi_2 = f_2$ RETARD
- $f_2 = f_2(x_3)$ oscill residual perturb
- $\varphi_2(t, x_3) = t\exp(t) f_2(x_3)$
- ω_1-vorticity: $\dot\omega_1 + x_1 \frac{\partial\omega_1}{\partial x_1} - \omega_1 = 0$, ACC
- INFLOW BC $\omega_1(\bar{x}_1, x_2, x_3) = \frac{\partial v_2}{\partial x_3} = t\exp(t)\frac{\partial f_2}{\partial x_3}$.
- Double exp growth $\exp(t) \Rightarrow$ BLOWUP

REAR SEPARATION

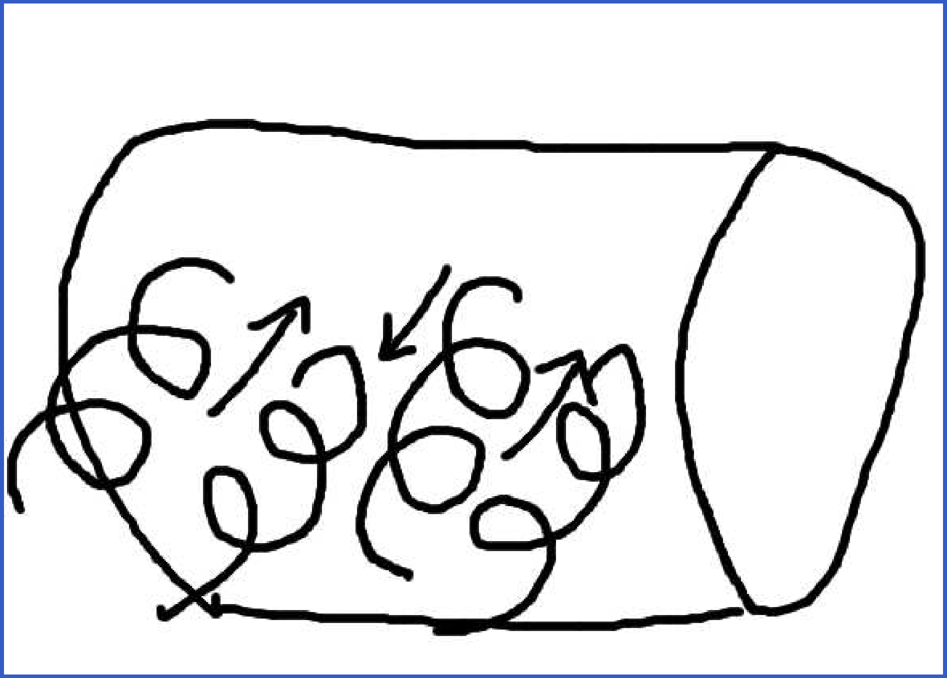


- DECENTERD OPPOSING VEL
- STREAMWISE VORTICITY ON SURFACE

SURFACE VORTICITY

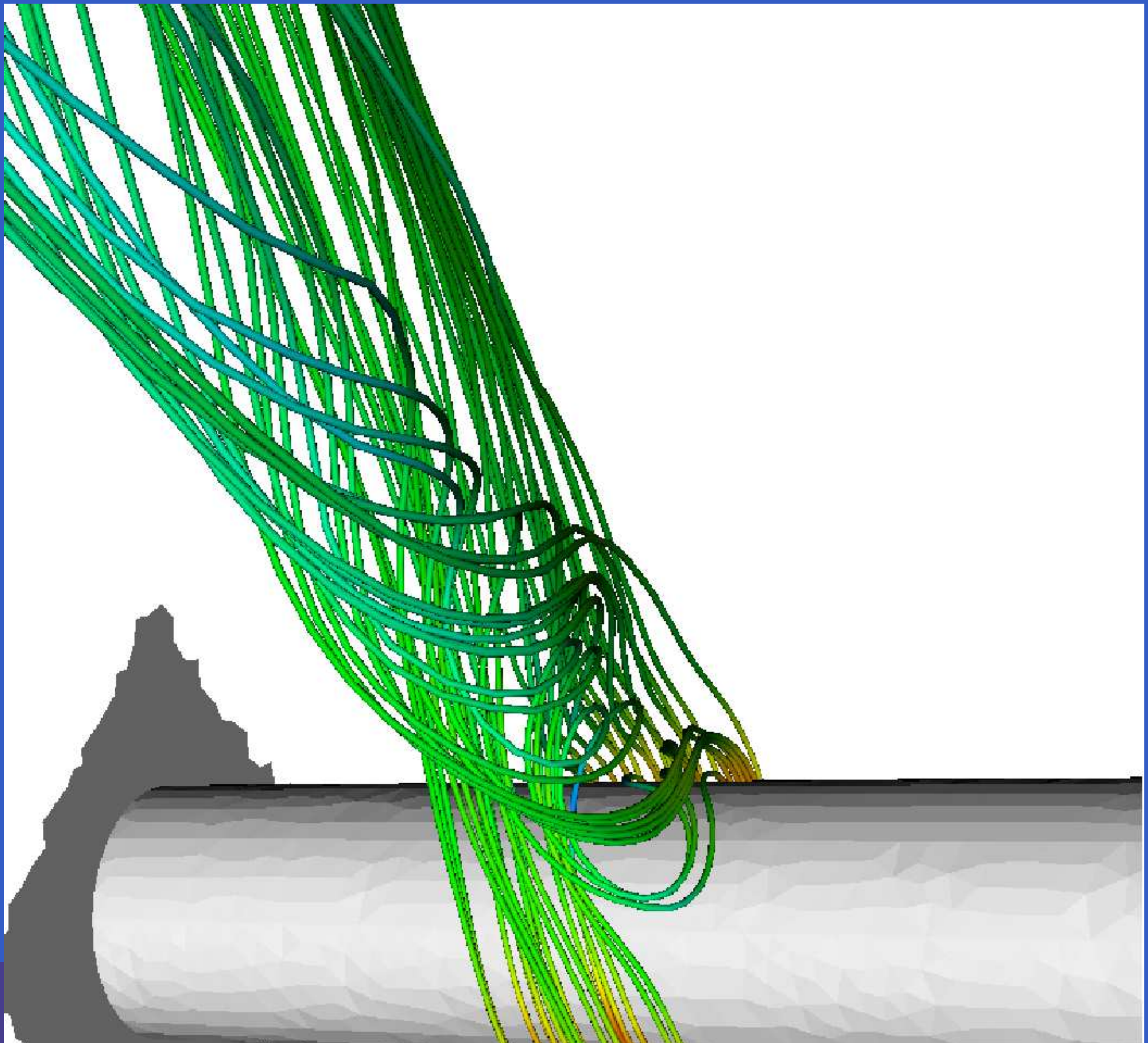


VORTEX STRETCHING

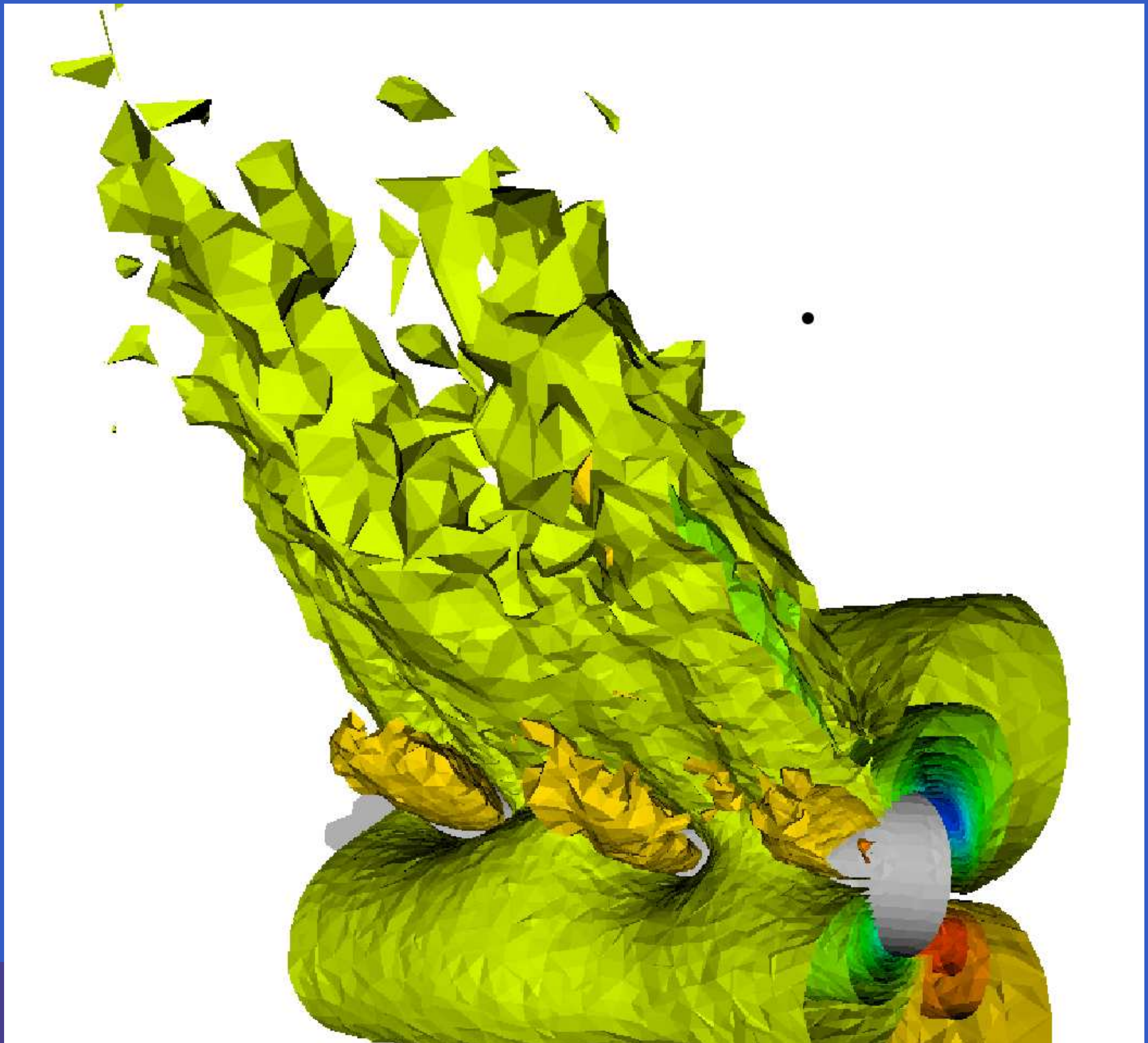


- VORTEX STRETCHING INTO WAKE
- OSC DIAGONAL PATTERN
- LOW PRESSURE!!

STREAMLINES



PRESSURE LEVEL SURFACES



CLASSICAL SEPARATION

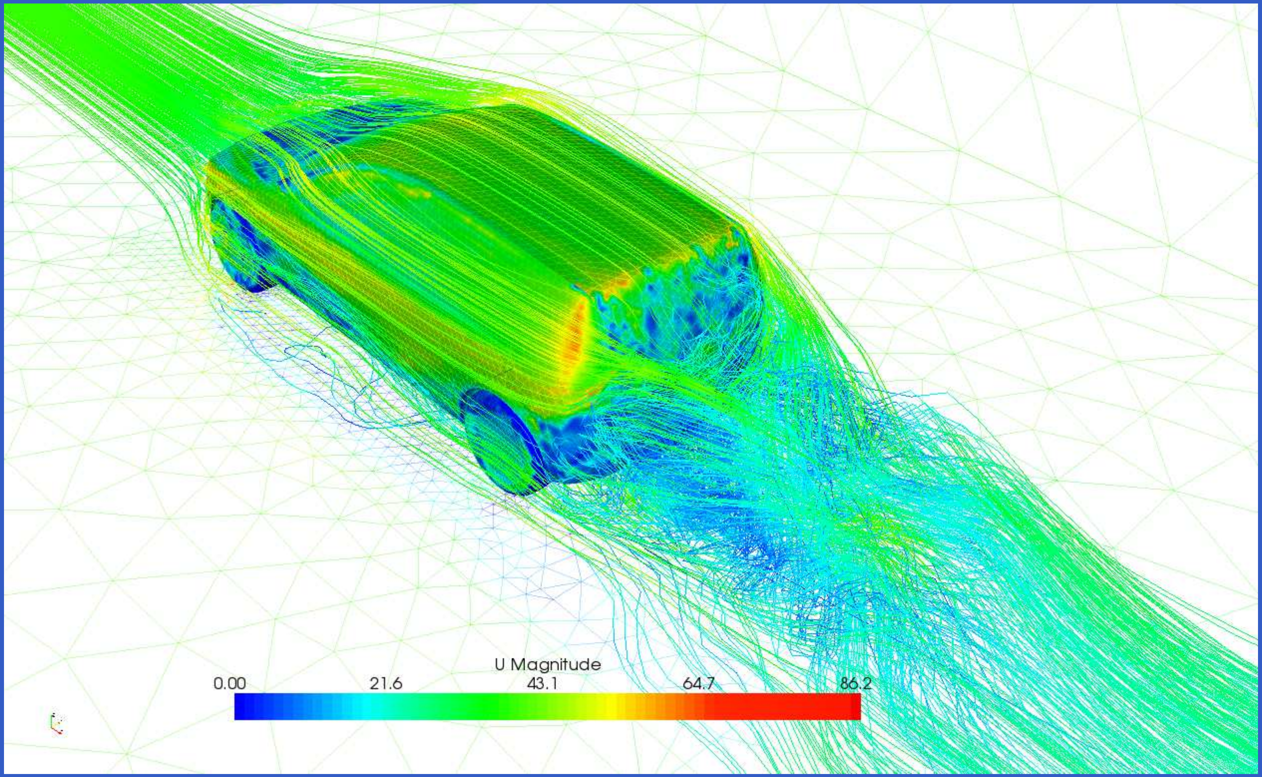
- PRANDTL: ADVERSE ∇P: $\frac{\partial P}{\partial \tau} < 0$
- TANGENT VEL $= 0 \Rightarrow$ NORMAL ∇P
- $= \frac{\partial P}{\partial n} = 0$
- DOES NOT STICK
- WRONG FOR LARGE REYNOLDS!!

NON-SEPARATION

$$\frac{\partial P}{\partial n} = \frac{U^2}{R}$$

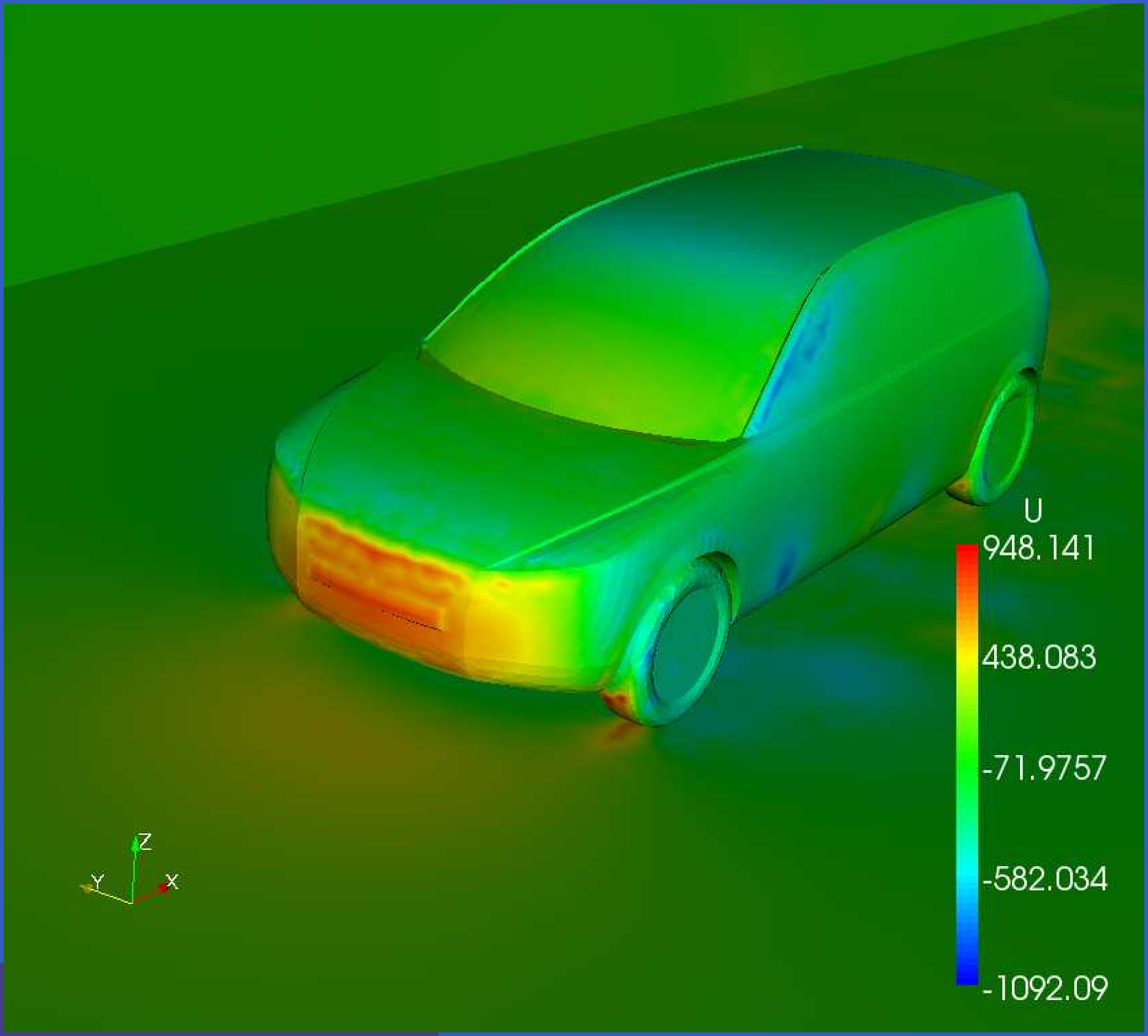
- SLIP $\Rightarrow \frac{\partial p}{\partial n} > 0$
- STICK: NON-SEPARATION
- TURBULENT BL $\approx$ SLIP/FRICTION
- EXPLAINS DRAG/LIFT FLYING
- NO TURBULENCE – NO FLYING

VOLVO CAR

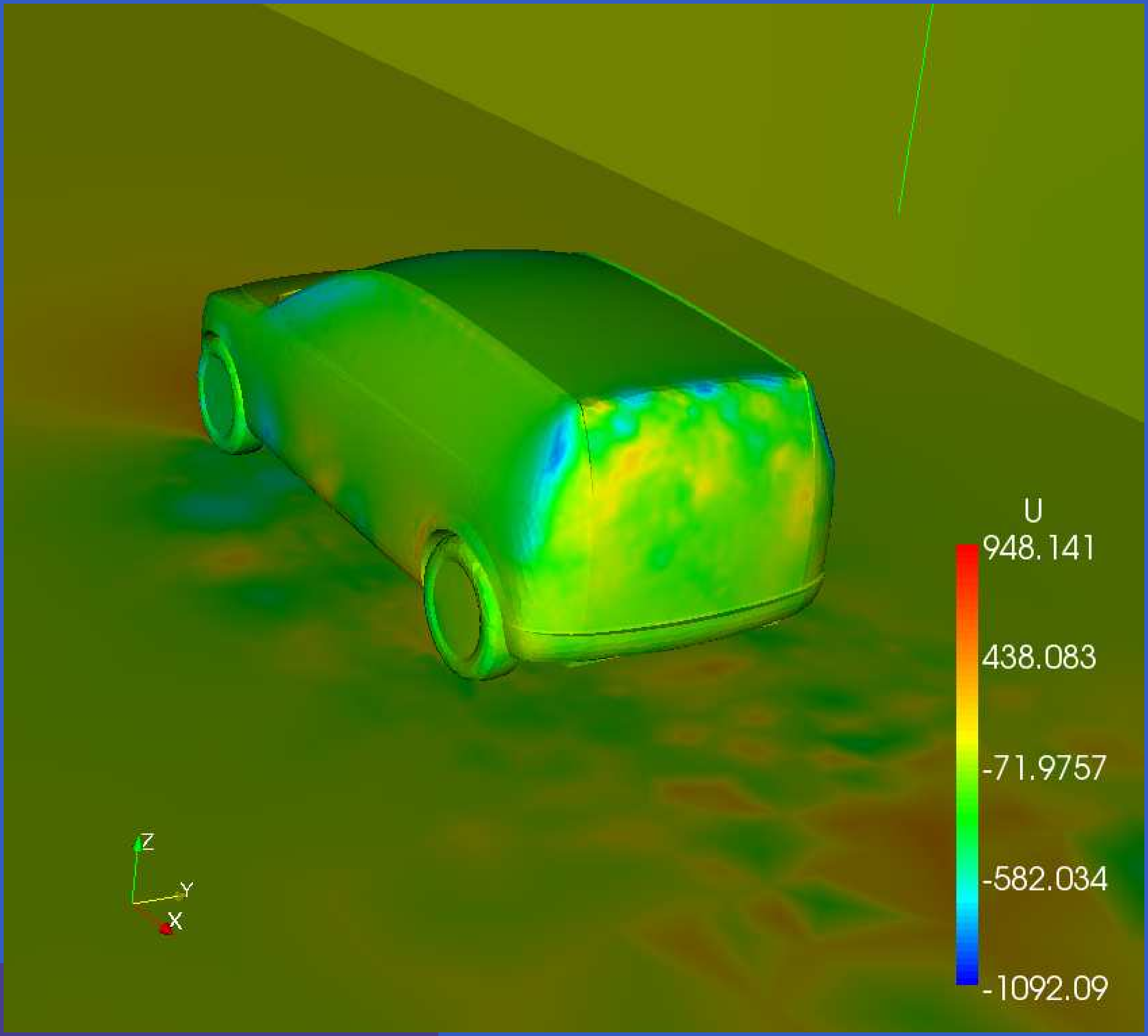


- DRAG ≈ 0.33.
- VIRTUAL WIND TUNNEL

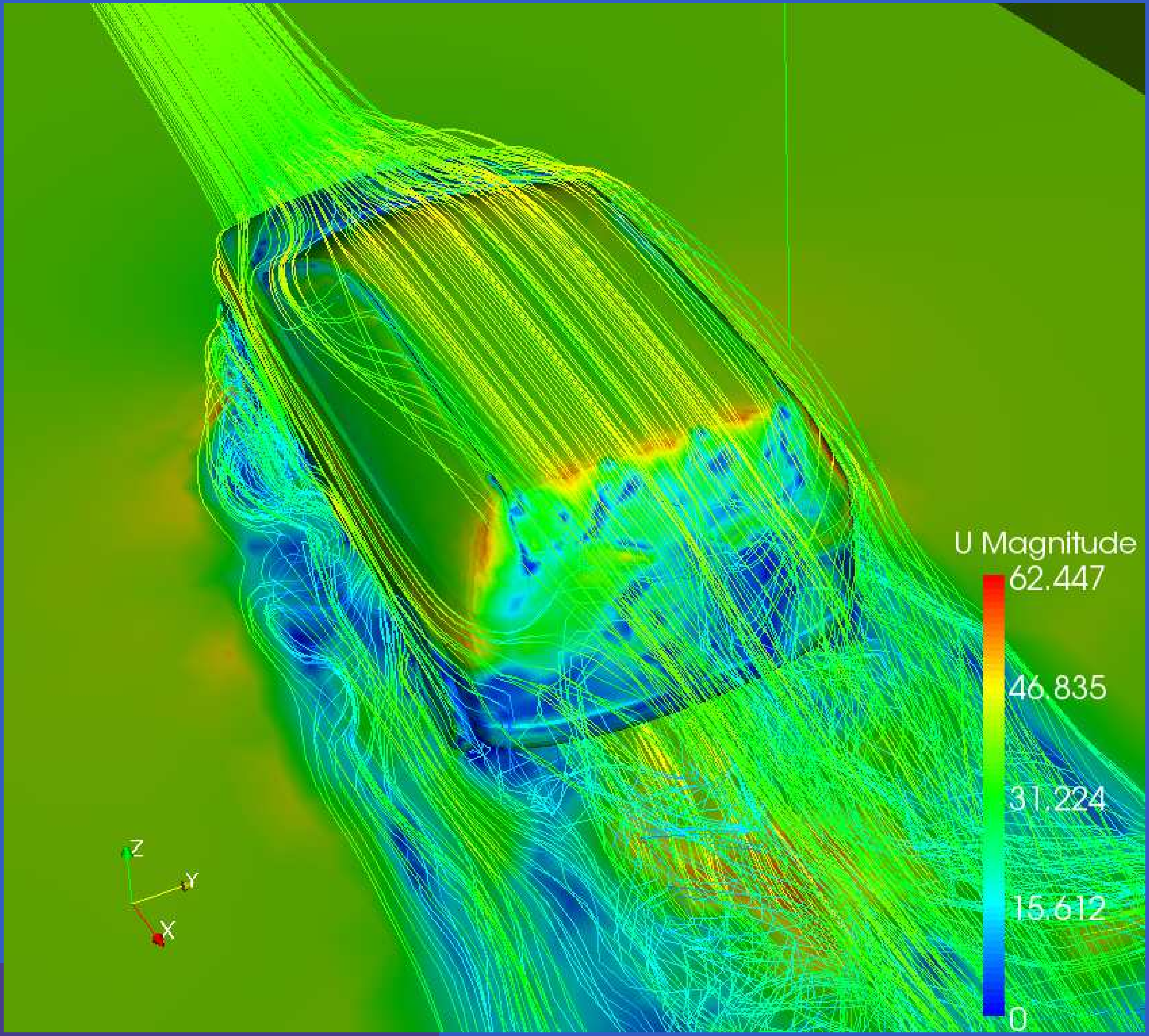
CAR: PRESSURE



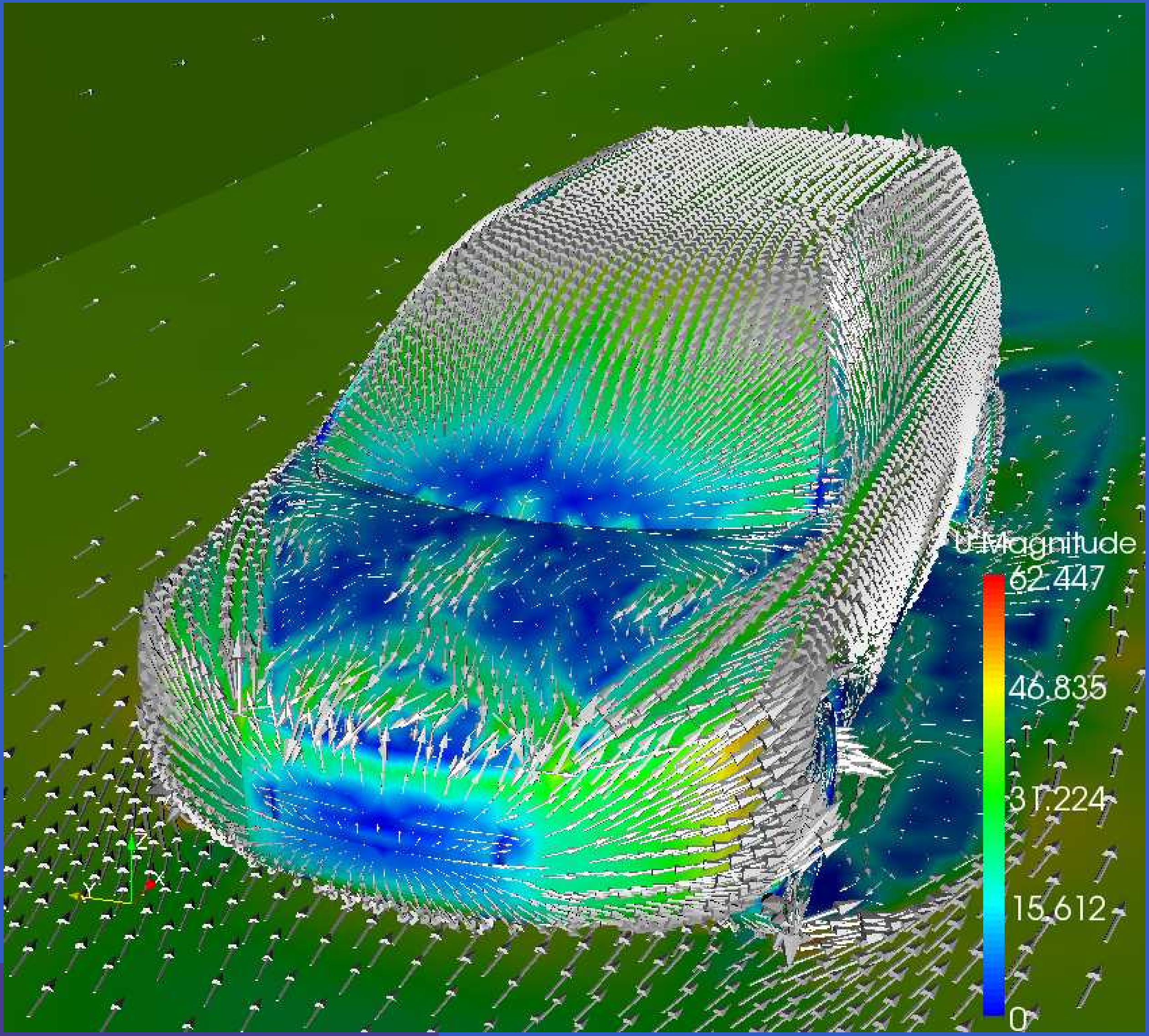
CAR: PRESSURE



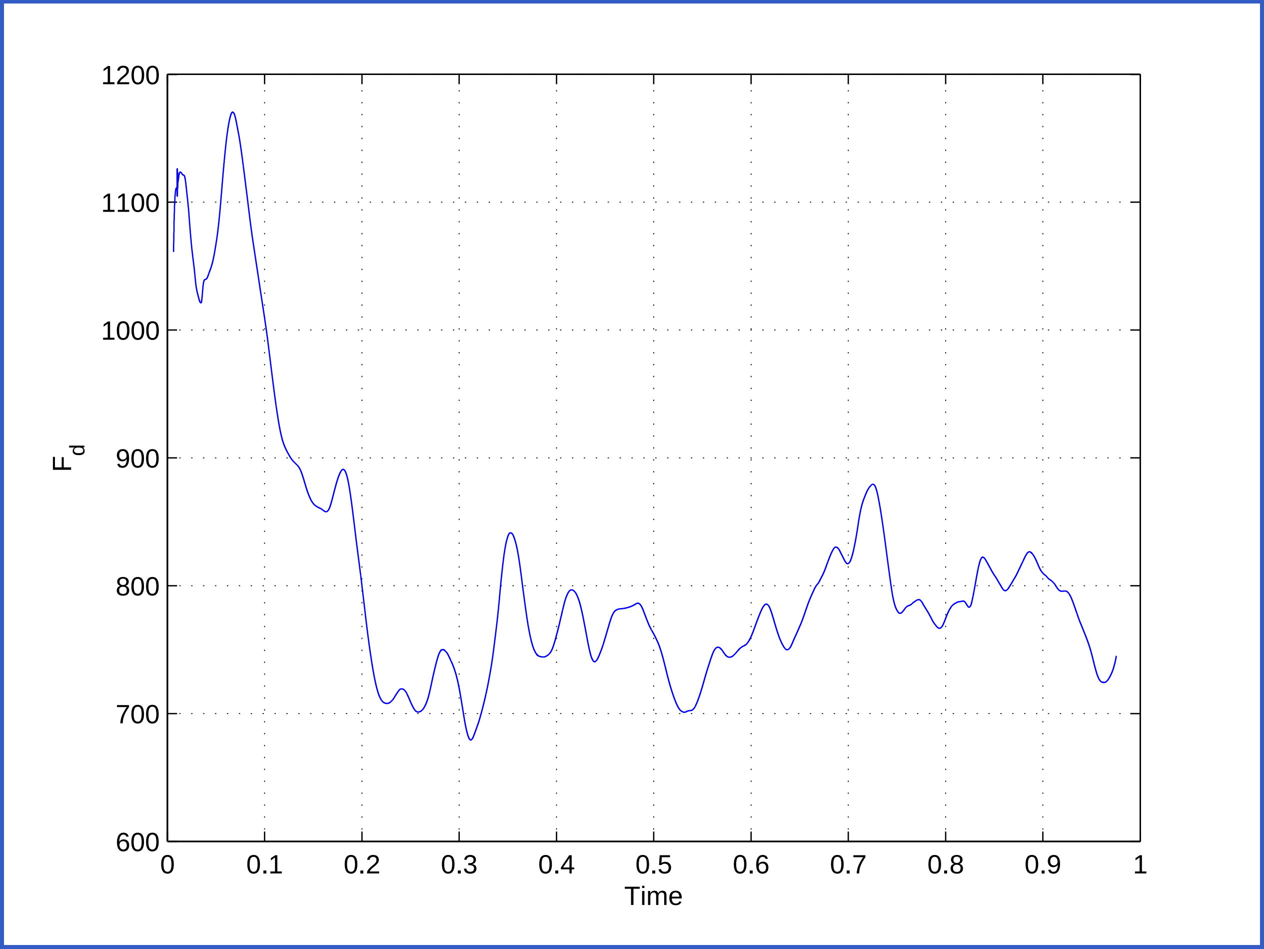
CAR: VELOCITY



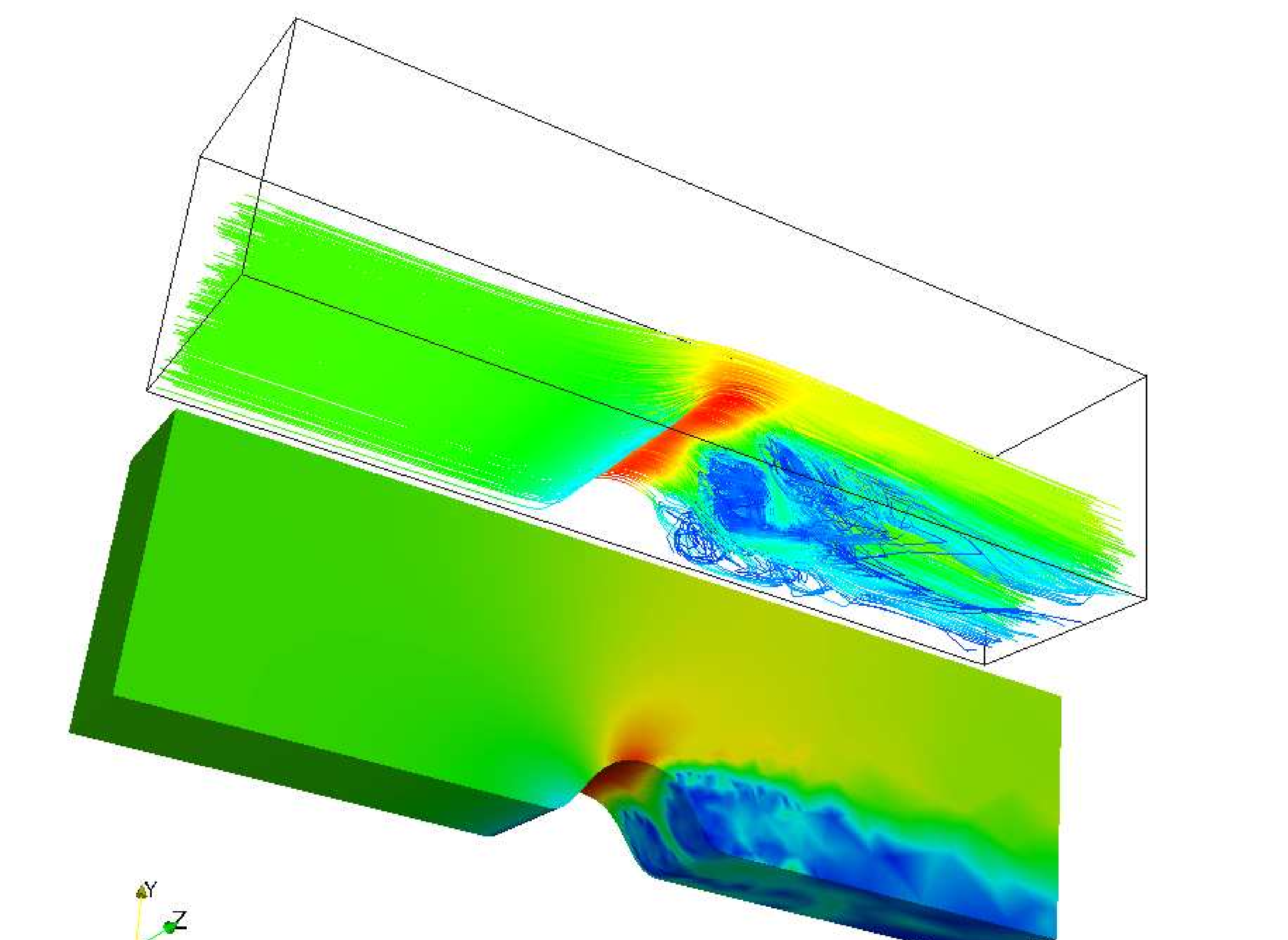
CAR: VELOCITY



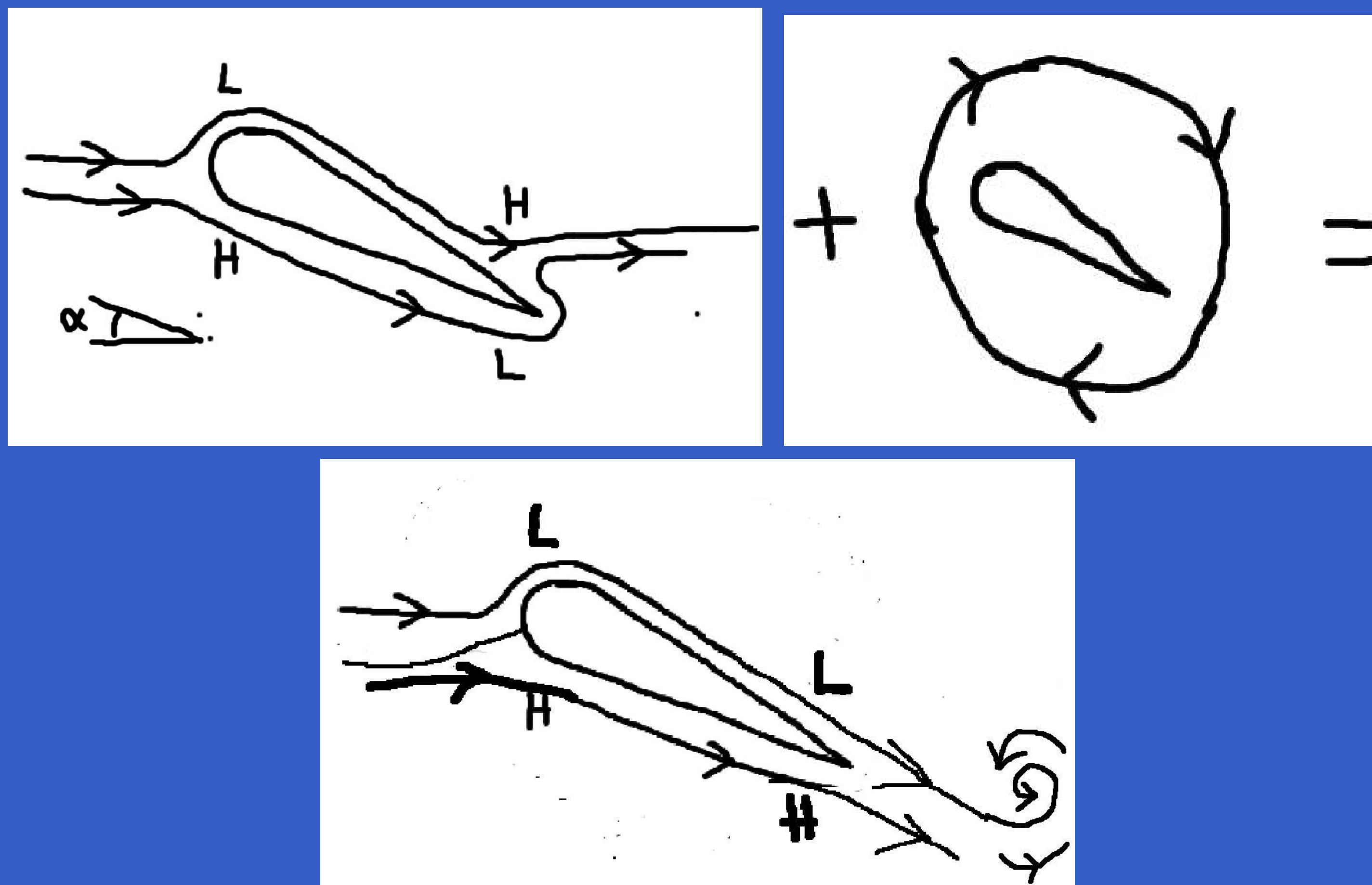
CAR: DRAG


1200
1100
1000
900
800
700
600
F_d
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time

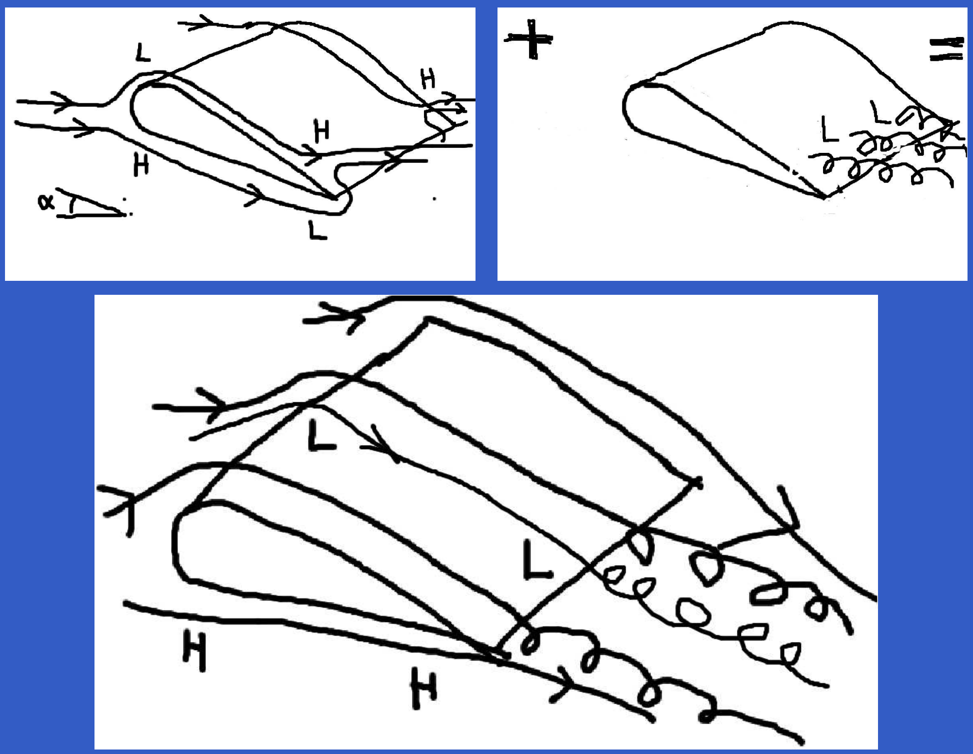
HILL SEPARATION



SECRET FLYING: KUTTA: WRONG



SECRET FLYING: NEW: CORRECT



- LIFT $=$ DOWNWASH
- LIFT $\Rightarrow$ DRAG

CONCLUSION

- BLOWUP OF EULER
- POT SOL $\Rightarrow$ TURB SOL WITH DRAG
- TRUE HYPERREALITY
- SOLVES D'ALEMBERT/CLAY/FLYING
- PUBLISHED IN BIT/JMFM/BOOK
- REACTION: MATH, FLUIDMECH, OTHER?