LIMIT THEOREMS FOR EXTREME ORDER STATISTICS UNDER NONLINEAR NORMALIZATION

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1. Introduction.

In this paper we shall describe a stochastic model called maxima-scheme or $\mathcal{M}$-scheme: let $X_1, X_2, \ldots$ be a sequence of independent random variables (i.r.v.'s.) from which the random sequence $Z_n = max(X_1, \ldots, X_n)$ is formed. This scheme is a peculiar analog of the summation scheme when the semigroup operation $max(x, y)$ is regarded as an analog of the summing operation $x + y$.

Numerous facts related to the $\mathcal{M}$-scheme are gathered in the monograph by J.Galambos [2] . In particular, it is shown (theorem 3.10.2) that the class of the limit distributions for $(Z_n - a_n)/b_n$, where a_n and b_n are normalizing constants, $b_n > 0$, coincides with the class of the log-concare distribution functions. Galambos' theorem generalizes a well-known result due to Gnedenko [3] according to which in the case of independent identically distributed r.v.'s. only distributions of the types

$$\Phi_\alpha(x) = exp\{-x^{-\alpha}\}, \quad x \geq 0, \quad \alpha > 0$$

$$\Psi_\alpha(x) = exp\{-(-x)^{\alpha}\}, \quad x \leq 0, \quad \alpha > 0$$

$$\Lambda(x) = exp\{-e^{-x}\}, \quad x \in \mathbb{R}_1 ,$$

can appear as limit distributions in the $\mathcal{M}$-scheme.

The meaning of any limit theorem for a random sequence $\{Y_n\}_n$ is that it gives a sufficiently simple approximation to the distribution $P(Y_n < x)$. Let

$$P(\overset{-1}{G_n}(Y_n) < x) \xrightarrow[n]{w} P(Y < x),$$

where $G_n(x) = b_n x + a_n$, $\overset{-1}{G_n}$ is the inverse function of G_n and $\xrightarrow[n]{w}$ means weak convergence as $n \longrightarrow \infty$. If the limit distribution is continuous, then as a consequence we have the strong convergence, i.e.

$$\varepsilon_n = \sup_x |P(\overset{-1}{G_n}(Y_n) < x) - P(Y < x)|$$

$$:= \rho(\overset{-1}{G_n}(Y_n), Y) \xrightarrow[n]{} 0.$$

Since the metric ρ is invariant with respect to strongly monotone continuous transformations of r.v.'s., we have

$$\rho(Y_n, G_n(Y)) = \varepsilon_n \xrightarrow[n]{} 0,$$

i.e. we receive a uniform approximation to $P(Y_n < x)$ by means of some universal distribution of the r.v. Y.

Such a view-point to the limit theorems deprives the traditionaly used linear transformations of their exclusiveness. Actually, the transformation $G_n(x) = a_n |x|^{b_n} \operatorname{sign} x$, a_n, $b_n > 0$, will serve for constructing a simplified approximation if only one can prove a suitable limit theorem.

Thus it makes sense to extend the class of normalizing transformations $G_n(x)$. The present paper continues the Gnedenko-Galambos investigations in the area of limit theorems for the random sequence Z_n , preliminarily submitted to a strong monotone continuous transform $G_n(x)$.

2. Characterization of the class $\mathcal{ML}$ -laws.

Let $X_1, \ldots, X_n$ be i.r.v.'s. taking on values in $\mathbb{R}_1$. We shall denote the corresponding distribution functions (d.f.'s) by $F_{X_k}(x)$, $k = 1, \ldots, n$. Assume that there exists a sequence $\{G_n(x)\}_n$ of strongly monotone continuous transformations such that

$$P(G_n^{-1}[\max(X_1,\ldots,X_n)]<x) \xrightarrow[n]{w} P(X<x), \tag{1}$$

where X is a nondegenerate r.v. with a d.f. F_X .

Definition 1. The random sequence $\{X_n\}_n$ is said to satisfy the uniformity assumption (u.a.) with respect to $\{G_n(\cdot)\}_n$ if

$$\sup_{k\le n} \left[1-F_{X_k}(G_n(x)\right] \xrightarrow[n]{} 0 .$$

The u.a. is analogous to the assumption of uniform asymptotical negligibility of summands in the limit theorems for sums of i.r.v.'s. In both schemes the assumption means a closeness to the "semi-group unit", which is $-\infty$ for the $\mathcal{M}$-scheme, and 0 for the summation.

Theorem 1. Let the sequence $\{X_n\}_n$ satisfy the u.a. Then (1) is true iff

$$u(x) := \lim_{n\to\infty} \sum_{K=1}^{n} [1 - F_{X_K}(G_n(x))] < \infty . \quad (2)$$

Furthermore, $F_X(x) = \exp\{-u(x)\}$.

Consider the following condition

$$\lim_{n\to\infty} \sum_{K=1}^{m_n} [1 - F_{X_K}(G_n(x))] < \infty \quad (3)$$

for each sequence of integers $\{m_n\}_K$ such that

$$m_n < n, \quad m_n \xrightarrow[n]{} \infty, \quad \frac{m_n}{n} \xrightarrow[n]{} \lambda, \quad \lambda \in (0,1). \quad (*)$$

Definition 2. A nondegenerate d.f. F_X belongs to class $\mathcal{ML}$, if it is a weak limit of $P(G_n^{-1}[\max(X_1,\dots,X_n)] < x)$ under the u.a. and (3).

If the u.a. is satisfied, then condition (3) is equivalent to the following one:

$$\lim_n P[\max(X_1,\dots,X_{m_n}) < G_n(x)]$$

exists and is a nondegenerate d.f.

Condition (3) is essential for the d.f. F_X to belong to the class of self-decomposable laws, as it will be shown in Theorem 2.

Theorem 2. Let the sequence $\{X_n\}_n$ satisfy the conditions (1), u.a. and (3). Then for every λ, $\lambda \in (0,1)$, there exist a d.f. $F_\lambda(x)$ and a function $g_\lambda(x)$ such that:

$$F_X(x) = F_X(g_\lambda(x)) \cdot F_\lambda(x) . \tag{4}$$

Here $F_X(\cdot)$ is the limit d.f. in (1) and $g_\lambda(x)$ is determined by the following lemma.

Lemma 1. Under the conditions of Theorem 2 the limit

$$g_\lambda(x) := \lim_n G_{m_n}^{-1}[G_n(x)] \tag{5}$$

exists and satisfies the following functional equation

$$g_s[g_\lambda(x)] = g_{s.\lambda}(x), \quad s,\lambda \in (0,1), \tag{6}$$

at each continuity point x of $F_X(x)$.

If for each x the function $g_\lambda(x)$, considered as a function of λ, is solvable (i.e. each equation of the form $g_\lambda(x) = t$ for given x and t has a unique solution $\lambda = \bar{g}(t,x)$), then the solution of the functional equation (6) has the following form

$$g_\lambda(x) = h^{-1}[h(x) - \log \lambda] , \tag{7}$$

where $h(\cdot)$ is an invertible continuous function (see Theorem 20 in [1]).

Throughout this paper we shall consider only such normalizing transformations $G_n(x)$ that the limit function (5) is solvable with respect to λ .

Let us denote by „o" the composition of two functions, i.e. $f \circ g(x) = f[g(x)]$. Then we may write $F_X[g_\lambda(x)] = F_X \circ h^{-1}[h(x) - \log \lambda]$.

Lemma 2. Under the conditions of Theorem 2 the function $\log(F_X \circ h^{-1})(x)$ is concave.

Now, it is easy to prove that the function $F_\lambda(x)$ in Theorem 2 is a d.f., i.e. the limit F_X is self-decomposable in the sence that it may be represented in the form (4). It turns out that the converse is also true:

Theorem 3. Let a nondegenerate d.f. F_X have the decomposition

$$F_X(x) = F_X[g_\lambda(x)] \cdot F_\lambda(x)$$

for each $\lambda \in (0,1)$, where $F_\lambda(x)$ is a d.f. and $g_\lambda(x)$ is a solution of (6). Then F_X belongs to the class $\mathcal{ML}$. More precisely, there exist two sequences $\{G_n(x)\}_n$ (of strongly monotone continuous transformations) and $\{X_n\}_n$ (of i.r.v.'s) satisfying u.a. and (3) such that

$$P(\max(X_1,\ldots,X_n) < G_n(x)) \xrightarrow[n]{w} F_X(x).$$

The last two theorems imply that the class of $\mathcal{ML}$-laws coincides with the class of self-decomposable laws (in the sence of (4)).

The scheme of the proof of Theorem 3 is the following: $F_2(x)$ is a d.f.; this yields the log-concavity of $F_X \circ h^{-1}$, which leads to the existence of $\{G_n\}_n$ and $\{X_n\}_n$ such that (1) holds. This proof emphasizes the role of the log-concavity of $F_X \circ h^{-1}$ as a characteristic property of the class $\mathcal{ML}$.

3. The class $\mathcal{MS}$ of max-stable laws.

Let $X_1, \ldots, X_n$ be i.r.v.'s with a nondegenerate d.f. F. Denote by X a r.v. with the same d.f. F.

Definition 3. F is said to be max-stable if for each positive integer n there exists a strongly monotone continuous transformations $G_n(x)$ such that

$$G_n^{-1}[\max(X_1,\ldots,X_n)] \stackrel{d}{=} X$$

($\stackrel{d}{=}$ means coincidence in distribution), i.e. for each n holds the equality

$$F(x) = F^n[G_n(x)]. \tag{8}$$

Clearly, from Definition 3 it follows that each max-stable d.f. may be considered as the limit of the d.f. of the normalized maximum of i.i.d.r.v.'s. The converse is formulated as Corollary 1 of Theorem 4.

Suppose now that there exists a sequence $\{G_n(x)\}_n$ such that the weak convergence holds

$$G_n^{-1}[\max(X_1,\ldots,X_n)] \xrightarrow[n]{d} X,$$

where X is a nondegenerate r.v. with a d.f. H, i.e.

$$F^{n}[G_{n}(x)] \xrightarrow[n]{W} H(x). \tag{9}$$

The relation (9) means, as usual, that F belongs to the domain of attraction $\mathcal{O}(H)$ of the d.f. H.

Obviously, conditions (3) and u.a. of Section 2 are fulfilled in case of i.i.d.r.v.'s if (9) is assumed. The characteristic decomposition (4) of the limit distribution H is reduced here to the following functional equation

$$H(x) = H(g_{\lambda}(x)) \cdot H(g_{1-\lambda}(x)). \tag{10}$$

Now, Theorem 2 could be formulated in a more simple way:

Theorem 4. If the weak convergence (9) holds, then the limit distribution H has the form

$$H(x) = \exp\{-e^{-h(x)}\}, \tag{11}$$

where $h(x)$ is the invertible continuous function from (7).

Introduce the following notations:

$$r := \operatorname{rext} H = \sup\{x : H(x) < 1\},$$

$$\ell := \operatorname{lext} H = \inf H = \inf\{x : H(x) > 0\}.$$

Since $H(x)$ is a d.f., we have

$$\lim_{x\to r} h(x)=\infty\,, \quad \lim_{x\to \ell} h(x)=-\infty$$

(in case of nonidentically distributed i.r.v.'s the relation $\lim\limits_{x\to \ell} h(x) = c > -\infty$ is also possible).

Corollary 1. Each limit distribution (11) is max-stable.

Actually, consider independent copies $X_1, \ldots, X_n$ of a r.v. X with the d-f- $H(x) = \exp\{-e^{h(x)}\}$ and choose the normalizing transformations $G_n(x)$ as follows:

$$G_n(x) = h^{-1}[h(x)+\log n]\,.$$

Then $H[G_n(x)] = \exp\{-\frac{1}{n}\, e^{-h(x)}\}$, i.e. $H^n(G_n(x)] = H(x)$. Here the tail of the distribution H has the asymptotic behaviour:

$$e^{-h(x)} = -\log H(x) = -\log\,[1-(1-H(x))] =$$

$$= [1+o(1)]\,[1-H(x)]\,, \qquad x\to r\,.$$

Corollary 2. Each strongly monotone continuous distribution F is max-stable.

In fact, we have in this case

$$h(x) = -\log\log\frac{1}{F(x)}\,.$$

Now, consider a non-max-stable d-f- F which belongs to $\mathcal{O}(H)$

where H is a max-stable d.f. The construction of the normalizing transformations $G_n(\cdot)$ and the asymptotic of the tail of F are given in the following theorem.

Theorem 5. A nondegenerate d.f. F belongs to $\mathcal{O}(H)$ iff

$$1-F(x)=[1+o(1)]L(h(x))e^{-h(x)}, \text{ as } x \to \text{rext}\, F, \tag{12}$$

where $L(x)$ is a regularly varying function. The normalizing transformations can be chosen as

$$G_n(x)=h^{-1}\{h(x)+\log[n\ L(\log n)]\}. \tag{13}$$

This result is not surprising. Roughly speaking, the operation $max(X_1, \ldots, X_n)$ attracts the "mass" of the distribution to rext F. This explains why translation plays decisive role in construction of the normalizing transformations. For the same reason the asymptotic behaviour of the normalized maximum is completely determined by the behaviour of the tail of the distribution only for $x \longrightarrow$ rext F, and does not depend on the behaviour of F for $x \longrightarrow$ lext F.

Note here that the three distributions $\Phi_\alpha(x)$, $\Psi_\alpha(x)$ and $\Lambda(x)$ may be reduced to only one $\Lambda(h(x))$, where $h(x)$ is an invertible continuous function, since

$$\Phi_\alpha(x)=\Lambda(\alpha \log x), \quad \Psi_\alpha(x)=\Lambda(-\alpha \log(-x)).$$

We denote by $\mathcal{H}$ the class of invertible continuous functions on $\mathbb{R}_1$. Theorem 4 shows that by the normalization $G_n \in \mathcal{H}$ the class $\{\Phi_\alpha, \Psi_\alpha, \Lambda\}_{h(x)}$ of the three distribution types expands to the class $\{e^{-e^{-h(x)}}\}_{h \in \mathcal{H}}$, although all limit distributions are of the same double exponent type.

4. Application of main results.

A. Derivation of Gnedenko's distributions.

Under the linear normalization $G_n(x) = b_n x + a_n$, $b_n > 0$, we get

$$G_{m_n}^{-1} \cdot G_n(x) = \frac{b_n}{b_{m_n}} + \frac{a_n - a_{m_n}}{b_{m_n}},$$

$$\frac{m_n}{n} \xrightarrow[n]{} \lambda \in (0,1).$$

Then the form of the limit function g_λ is $g_\lambda(x) = \beta_\lambda x + \alpha_\lambda$ where

$$\beta_2 = \lim_n \frac{b_n}{b_{m_n}},$$

$$\alpha_\lambda = \lim \frac{a_n - a_{m_n}}{b_{m_n}}.$$

From these asymptotic relations the following equations can be obtained:

$$\beta_{\lambda s} = \beta_\lambda \cdot \beta_s, \quad \alpha_{\lambda s} = \alpha_s \beta_\lambda + \alpha_\lambda = \alpha_\lambda \beta_s + \alpha_s, \quad \lambda, s \in (0,1),$$

which yield two possible solutions (cf. [2], § 3.10):

i) $\beta_\lambda = 1$ and $\alpha_\lambda = k \log \lambda$

ii) $\beta_\lambda = \lambda^m$ and $\alpha_\lambda = k(\lambda^m - 1)$.

Thus, the concrete form of the limit distribution $H(x)$ depends on the parameters m and k.

The function $h(x)$ corresponding to $g_\lambda(x) = x + k \log \lambda$ can be obtained as a solution of the functional equation

$$h(x + k \log \lambda) = h(x) - \log \lambda ,$$

namely $h(x) = -\frac{x}{k} := h_1(x)$. Since for each x $g_\lambda(x) > x$ (a consequence of the representation (7)) then $k < 0$. The corresponding max-stable distribution $\exp\{-e^{-h_1(x)}\}$ obviously concides with $\Lambda(x)$ for $k = -1$.

The other possible form of $g_\lambda(x)$, namely $g_\lambda(x) = \lambda^m(x + k) - k$, leads to the functional equation

$$h(\lambda^m x - k) = h(x - k) - \log \lambda . \tag{14}$$

The inequality $g_\lambda(x) > x$ gives two other cases: $x + k \geqslant 0$ and $x + k \leqslant 0$. Let $x + k \geqslant 0$. In this case $m < 0$ and the function $h_2(x) = -\frac{1}{m}\log(x + k)$ is a solution of (14). For $k = 0$ the corresponding max-stable d.f. $\exp\{-e^{-h_2(x)}\}$ coincides with $\Phi_\alpha(x)$, $\alpha = -\frac{1}{m} > 0$.

Let now $x + k \leqslant 0$. In this case $m > 0$ and a solution of (14) is $h_3(x) = -\frac{1}{m}\log(-(x+k))$. If we set $\alpha = \frac{1}{m} > 0$ and $k = 0$, then the max-stable distribution $\exp\{-e^{-h_3(x)}\}$ coincides with $\Psi_\alpha(x)$.

Note that the parameter m determines the power α of the max-stable d.f. and k is connected with its support.

B. Description of $\mathcal{MS}$-laws for power normalization.

Let now $G_n(x)$ be of the power type, namely $G_n(x) = a_n |x|^{b_n} \operatorname{sign} x$ with $a_n, b_n > 0$. In this case

$g_\lambda(x) = d_\lambda |x|^{\beta_\lambda} \operatorname{sign} x$, where $\beta_\lambda = \lim_n \frac{b_n}{b_{m_n}}$,

$d_\lambda = \lim_n \left(\frac{a_n}{a_{m_n}}\right)^{b_{m_n}^{-1}}$, $\frac{m_n}{n} \xrightarrow[n]{} \lambda \in (0,1)$.

As in Subsection A we obtain two possibilities for the coefficients and , namely

i) $\beta_\lambda = 1$ and $d_\lambda = \lambda^k$

ii) $\beta_\lambda = \lambda^m$ and $d_\lambda = e^{k(\lambda^m - 1)}$,

where m and k are constants. In the case i) we have $g_\lambda(x) = \lambda^k x$ and this again leads to the known limit distributions $\Phi_\alpha(x)$ and $\Psi_\alpha(x)$, $\alpha = \alpha(k)$.

In the second case the function $g_\lambda(x) = \exp\{k(\lambda^m - 1)\}|x|^{\lambda^m} \operatorname{sign} x$ leads to the following functional equation

$$h[c|x|^{\lambda^m} \operatorname{sign} x] = h(cx) - \log\lambda, \quad c := e^{-k}. \tag{15}$$

For simplicity we suppose $c = 1$. According to the cases $x \geq 1$, or $0 \leq x \leq 1$, or $-1 \leq x \leq 0$, or $x \leq -1$, we obtain four solutions of (15). In addition, the sign of m is determined by the inequality $g_\lambda(x) > x$. Finally, for the corresponding max-stable distributions we obtain the following functions with $\alpha = \alpha(m) > 0$:

$$H_{1,\alpha}(x) = \exp\{-(\log x)^{-\alpha}\}, \quad x \geq 1$$

$$H_{2,\alpha}(x) = \exp\{-|\log x|^{\alpha}\}, \quad 0 \leq x \leq 1$$

$$H_{3,\alpha}(x) = \exp\{-|\log|x||^{-\alpha}\}, \quad -1 \leq x \leq 0$$

$$H_{4,\alpha}(x) = \exp\{-(\log|x|)^{\alpha}\}, \quad x \leq -1 .$$

Hence, the class of the max-stable laws for the power normalization $a_n |x|^{b_n}$ sign x contains six distribution types: $\Phi_\alpha(x)$, $\Psi_\alpha(x)$, $H_{1,\alpha}(x), \ldots, H_{4,\alpha}(x)$. Notice that the last four distributions are examples of distributions which are max-stable under power normalization, and not max-stable under linear normalization.

5. Proofs of the results.

Proof of Theorem 1. Set $Z_n = \max(X_1, \ldots, X_n)$, $F_n(x) := P(Z_n < x)$. Since $F_n(x) = \prod_{k=1}^{n} F_{X_k}(x)$, from u.a. and the Taylor expansion of the function $\log(1-z)$, $0 \leq z < 1$, it follows at once

$$\log F_n(G_n(x)) = \sum_{k=1}^{n} \log[1-(1-F_{X_k}(G_n(x)))] =$$

$$= -[1+O(1)] \sum_{K=1}^{n} [1-F_{X_K}(G_n(x))] ,$$

for $n \to \infty$.

Obviously, both convergences (1) and (2) either hold simultaneously or not, what to be proved.

Proof of Lemma 1. The conditions (3) and u.a. imply the convergence $\prod_{K=1}^{m_n} F_{X_K}(G_n(x)) \xrightarrow[n]{} S(n)$ where

$$S(x) := \lim_n P[Z_{m_n} < G_n(x)] =$$

$$= \lim_n P[G_{m_K}^{-1}(Z_{m_n}) < G_{m_n}^{-1}(G_n(x))] .$$

On the other hand, under assumption (1) we have

$$F_X(x) = \lim_n P[G_{m_n}^{-1}(Z_{m_n}) < x] .$$

If we use the corresponding modification of Khinchin's theorem (see Lemma 2.2.3 in [2]), then the last two equalities yield the existence of the limit functions (5) and the equality

$$S(x) = F_X(g_\lambda(x)) . \tag{16}$$

The transformations $\{ G_n(\cdot) \}_n$ are monotone and continuous and consequently such is $g_\lambda(x)$. In addition $g_\lambda(x)$ is nondecreasing in x because the function $F_X \circ g_\lambda(x)$,

$$F_X \circ g_\lambda(x) = \lim_n P[G_n^{-1}(Z_{m_n}) < x] ,$$

is a limit of nondecreasing functions. Let $s \in (0,1)$. Then by the definition of $g_\lambda(x)$ we have

$$g_{s,\lambda}(x) \underset{n}{\longleftarrow} G^{-1}_{m_n s} \circ G_n(x) =$$

$$= G^{-1}_{m_n s} \circ G_{m_n} \circ G^{-1}_{m_n} \circ G_n(x) \underset{n}{\longrightarrow} g_s \circ g_\lambda(x).$$

In Khinchin's theorem the limit normalizing function is uniquely determined, therefore

$$g_{s\lambda}(x) = g_s [g_\lambda(x)] ,$$

and this is exactly what was to be proved.

Proof of Lemma 2. Since

$$P[max(X_{m_n+1}, \ldots, X_n) < G_n(x)] =$$

$$= P[Z_n < G_n(x)]/P[G^{-1}_{m_n}(Z_{m_n}) < G^{-1}_{m_n} \circ G_n(x)] ,$$

then under the conditions of Theorem 2 there exists the limit function

$$F_\lambda(x) : =$$

$$= \lim_n P\ [\max(X_{m_n+1}, \ldots, X_n) < G_n(x)] = \frac{F_X(x)}{F_X(g_\lambda(x))} \cdot \tag{17}$$

As a limit of nondecreasing functions $F_\lambda(x)$ is also of the same type. Hence, for $x_1 < x_2$ we have the inequality

$$\frac{F_X(x_1)}{F_X(g_\lambda(x_1))} \leqslant \frac{F_X(x_2)}{F_X(g_\lambda(x_2))} \quad . \tag{18}$$

From (7) it is clear that for $\lambda \in (0,1)$, $g_\lambda(x) > x$. We choose $x_2 = g_\lambda(x_1)$. Then $x_1 = h^{-1}[\,h(x_2) + \log \lambda\,]$ and (18) turns into

$$(F_X \circ h^{-1})^2\,[\,h(x_2)] \geqslant$$

$$\geqslant (F_X \circ h^{-1})[\,h(x_2) + \log\lambda] \cdot (F_X \circ h^{-1})[\,h(x_2) - \log\lambda] \;.$$

Since $h(x_2) = \frac{1}{2}(h(x_2) + \log\lambda) + \frac{1}{2}(h(x_2) - \log\lambda)$,
then the last inequality means that the function $\log(F_X \circ h^{-1})$

is concave. This completes the proof.

Proof of Theorem 2. The decomposition (4) was already obtained by Lemma 1 and the formula (17). Now, we still have to show that $F_\lambda(x)$ is a d.f.. The properties

a) $0 \leq F_\lambda(x) \leq 1$,

b) $F_\lambda(x)$ is a nondecreasing in x function,

c) $\lim\limits_{x\to\infty} F_\lambda(x)=1$

are evident. Thus, we have to prove only

d) $\lim\limits_{x\to-\infty} F_\lambda(x) = 0$.

The property d) is obvious in the case $\operatorname{lext} F_X = l > -\infty$. Let us consider the case $\operatorname{lext} F_X = -\infty$. If $h(x) \to c > -\infty$ for $x \to -\infty$, then d) is obvious again. Hence, in the sequel we shall dwell on the case $h(x) \to -\infty$ for $x \to -\infty$. Then the relations

$$\log F_\lambda(x) = \log(F_X \circ h^{-1})[h(x)] - \log(F_X \circ h^{-1})[h(x) - \log\lambda] \leq$$

$$\leq (\log F_X \circ h^{-1})'[h(x) - \log\lambda] \cdot \log\lambda$$

hold. The first factor tends to $+\infty$ for $x \longrightarrow -\infty$, the second is negative. Thus, property d) follows at once, i.e. $F_\lambda(x)$ is a d.f. The proof is complete.

Proof of Theorem 3. Note that from the facts that $F_\lambda(x)$ is a d.f. and $g_\lambda(x) = h^{-1}[h(x) - \log\lambda]$ it follows that :

the function $(F_X \circ h^{-1})(x)$ is log-concave and nondecreasing,

$$\lim_{x\to\infty} h(x) = \infty .$$

We construct the desired sequences $\{G_n(\cdot)\}_n$ and $\{X_n\}_n$ as follows :

$$G_k(x) := h^{-1}[h(x)+\log k], F_{X_1}(x) := F_X(x),$$

$$F_{X_k}(x) := \frac{F_X \circ h^{-1}[h(x)-\log k]}{F_X \circ h^{-1}[h(x)-\log(k-1)]}$$

for all $k \geqslant 2$. We shall prove that the so defined function $F_{X_k}(x)$ is a d.f.:

a) $0 \leq F_{X_k}(x) \leq 1$, which is obvious.

b) $F_{X_k}(x)$ is nondecreasing. In fact, if $x_1 < x_2$, then $x_1 - \log k < x_1 - \log(k-1) < x_2 - \log(k-1)$ and there exist p, $q \in (0,1)$ such that $x_1 - \log(k-1) = p(x_1 - \log k) + q(x_2 - \log(k-1))$. We choose x_2 so that $p = q = \frac{1}{2}$, i.e. $x_2 = x_1 - \log\frac{k-1}{k}$. Then it is clear from the log-concavity of $F_X \circ h^{-1}$ that the inequality

$$(F_X \circ h^{-1})^2 \left[\tfrac{1}{2}(x_1 - \log k) + \tfrac{1}{2}(x_2 - \log(k-1))\right] \geqslant$$

$$\geqslant (F_X \circ h^{-1})[x_1 - \log k] \cdot (F_X \circ h^{-1})[x_2 - \log(k-1)]$$

holds. This is equivalent to

$$\frac{(F_X \circ h^{-1})[x_1 - \log(K-1)]}{(F_X \circ h^{-1})[x_2 - \log(K-1)]} \geqslant \frac{(F_X \circ h^{-1})[x_1 - \log K]}{(F_X \circ h^{-1})[x_1 - \log(K-1)]} .$$

But $x_1 - \log(K-1) = x_2 - \log K$ and it is obvious that $F_{X_K}(x_2) \geqslant F_{X_K}(x_1)$.

c) $\lim_{x \to \infty} F_{X_K}(x) = 1$, which is clear from the remark made at the beginning of the proof.

d) $\lim_{x \to -\infty} F_{X_K}(x) = 0$. Indeed, for $K = 1$ it is clear.

Let $\ell = \mathrm{lext}\, F_X$ be finite or infinite. If $h(x) \to c > -\infty$ for $x \to \ell$, then we determine d_n , $d_n > \ell$, from the asymptotic relation $h(x) \longrightarrow c + \log n$ for $x \to d_n$. Then

$$h^{-1}[h(x) - \log n] \to h^{-1}(c) = -\infty \qquad \text{for} \quad x \to d_n ,$$

$$h^{-1}[h(x) - \log(n-1)] \to h^{-1}(c + \log \frac{n}{n-1}) > -\infty , \quad x \to d_n .$$

Consequently, $F_{X_n}(x) \to 0$ for $x \to d_n$, i.e. $\mathrm{lext}\, F_{X_n} = d_n$.

Let now $h(x) \longrightarrow -\infty$ for $x \to \ell$. For $K \geqslant 2$ (as in the proof of Theorem 2) we have

$$\log F_{X_K}(x) =$$

$$= \log (F_X \circ h^{-1})[h(x) - \log \kappa] - \log (F_X \circ h^{-1})[h(x) - \log(\kappa-1)] \leq$$

$$\leq (\log F_X \circ h^{-1})'[h(x) - \log(\kappa-1)] \cdot \log \frac{\kappa-1}{\kappa} \longrightarrow -\infty, \ x \longrightarrow \ell.$$

Now we shall show that the weak convergence (1) is valid . For every n

$$P(Z_n < G_n(x)) = \prod_{\kappa=1}^{n} F_{X_\kappa}(G_n(x)) =$$

$$= \frac{(F_X \circ h^{-1})[h(x) + \log n] \cdot (F_X \circ h^{-1})[h(x) + \log \frac{n}{2}] \dots (F_X \circ h^{-1})[h(x) + \log \frac{n}{n}]}{(F_X \circ h^{-1})[h(x) + \log n] \dots (F_X \circ h^{-1})[h(x) + \log \frac{n}{n-1}]}$$

$$= F_X(x) .$$

In order to complete the proof we have still to show that the r.v.'s X_n , $n = 1, 2, \dots$ with corresponding d.f.'s $F_{X_n}(x)$ satisfy (3) and the u.a. The latter follows immediately from

$$F_{X_\kappa}(G_n(x)) = \frac{(F_X \circ h^{-1})[h(x) + \log \frac{n}{\kappa}]}{(F_X \circ h^{-1})[h(x) + \log \frac{n}{\kappa-1}]} .$$

Under the u.a., for each sequence $\{m_n\}_n$ satisfying (*), we get

$$-\sum_{k=1}^{m_k} [1 - F_{X_k}(G_n(x))] \sim \log \prod_{k=1}^{m_n} F_{X_k}(G_n(x)) .$$

The last product however converges:

$$\prod_{k=1}^{m_n} F_{X_k}(G_n(x)) = (F_X \circ h^{-1})[h(x) + \log \frac{n}{m_n}]$$

$$\xrightarrow[n\to\infty]{} (F_X \circ h^{-1})[h(x) - \log \lambda] = F_X(g_\lambda(x)).$$

This concludes the proof.

Proof of Theorem 4. As it was proved in Theorem 2 there exist the limits

$$\lim_n P[\max(X_1, \ldots, X_{m_n}) < G_n(x)] = H(g_\lambda(x))$$

$$\lim_n P[\max(X_{m_n+1}, \ldots, X_n) < G_n(x)] = H_\lambda(x),$$

where $H_\lambda(x)$ is a d.f. Now we obtain the form of $H_\lambda(x)$ in the case of i.i.d.r.v.'s :

$$P[\max(X_{m_n+1}, \ldots, X_n) < G_n(x)] =$$

$$= P\,[\,max(X_1,\dots,X_{n-m_n}) < G_{n-m_n}(G^{-1}_{n-m_n}\circ G_n(x))\,]$$

$$\xrightarrow[n]{} H(g_{1-\lambda}(x)).$$

The decomposition (4) of the limit distribution H is reduced to the functional equation (10). Putting $T(x) := H\circ h^{-1}(x)$ we can rewrite it as

$$T(h(x)) = T\,[\,h(x) - log\,\lambda\,]\;T[\,h(x) - log(1-\lambda)\,],$$

where $h(x)$ is the invertible continuous function from the representation (7) of $g_\lambda(x)$. Let us now put $T(log\,v) := W(v)$ for the new variable $v := exp\{h(x)\}$. This leads to the well known functional equation

$$W(v) = W\left(\frac{v}{\lambda}\right)\cdot W\left(\frac{v}{1-\lambda}\right),$$

whose solution has the form $W(v) = exp\left(\frac{a}{v}\right)$ where a is a negative constant. Thus, we get as a solution of (10)

$$H(x) = T(h(x)) = W(v) = exp\{-c\,e^{-h(x)}\}, \tag{19}$$

where $c = -a > 0$. If $h_* = h + c$, then

$$g_\lambda(x) = h_*^{-1}[h_*(x) - \log\lambda] = h^{-1}[h(x) - \log\lambda],$$

i.e. in the representation (19) we can put $c = 1$. Hence the proof is completed.

Proof of Theorem 5. On account of Theorem 1 the necessary and sufficient condition for the convergence $F^n(G_n(x)) \xrightarrow[n]{w} H(x)$ has the form

$$n[1 - F(G_n(x))] = [1 + o(1)]e^{-h(x)} \tag{20}$$

We put $y = G_n(x) = h^{-1}\{h(x) + \log[nL(\log n)]\}$. For G_n^{-1} we get $G_n^{-1}(y) = h^{-1}\{h(y) - \log[nL(\log n)]\}$.

It is easy to see, that the construction (13) is consistent with the convergence (5), i.e. for each sequence $\{m_n\}_n$ satisfying (*) we have

$$G_{m_n}^{-1} \circ G_n(x) = h^{-1}\{h(x) + \log\frac{n}{m_n} + \log\frac{L(\log n)}{L(\log m_n)}\}$$

$$\xrightarrow[n]{} h^{-1}\{h(x) - \log\lambda\} = g_\lambda(x),$$

because of the convergence

$$\frac{L(\log n)}{L(\log m_n)} \sim \frac{L(\log n)}{L(\theta_n \log n)} \xrightarrow[n]{} 1$$

where $\theta_n > 0$, $\theta_n \xrightarrow[n]{} 1$.

Since $h(x) \longrightarrow \infty$ for $x \longrightarrow \text{rext}\, F$,
$y = G_n(x) \to \text{rext}\, F$ for $x \longrightarrow \text{rext}\, F$ and also for $n \to \infty$.
Now for $y \longrightarrow \text{rext}\, F$ (20) becomes

$$1 - F(y) \sim \frac{1}{n}\, e^{-\{h(y) - \log[nL(\log n)]\}} \sim$$

$$\sim \frac{1}{n}\, \frac{L(\log[nL(\log n)]}{L(\log n)} \cdot e^{-\{h(y) - \log[nL(\log n)]\}} .$$

Here we used the asymptotic behaviour of the second term for $n \to \infty$:

$$\frac{L(\log n \left[1 + \frac{\log L(\log n)}{\log n}\right])}{L(\log n)} =$$

$$= \frac{L(\theta_n \cdot \log n)}{L(\log n)} \xrightarrow[n]{} 1 ,$$

where $0 < \theta_n \xrightarrow[n]{} 1$. Taking also into account that

$$L(\log[nL(\log n)]) \sim$$

$$\sim \quad L(h(y)+\log[nL(\log n)]) = L(h(y)),$$

we get finally:

$$1-F(y) \sim L(h(y)) \cdot e^{-h(y)} \qquad \text{for} \quad y \to \mathrm{rext}\, F\ .$$

Thus, the asymptotic equality (12) is true, which was to be established

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