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SELF-SIMILAR EXTREMAL PROCESSES*

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Given an extremal process $X: [0,\infty) \to [0,\infty)^d$ with lower curve C and associated point process $N = \{(t_k, X_k): k \geq 0\}$, t_k distinct and X_k independent, given a sequence $\zeta_n = (\tau_n, \xi_n)$, $n \geq 1$, of time-space changes (max-automorphisms of $[0,\infty)^{d+1}$), we study the limit behavior of the sequence of extremal processes $Y_n(t) = \xi_n^{-1} \circ X \circ \tau_n(t) = C_n(t) \vee \max\{\xi_n^{-1} \circ X_k: t_k \leq \tau_n(t)\} \Longrightarrow Y$ under a regularity condition on the norming sequence ζ_n and asymptotic negligibility of the max-increments of Y_n. The limit class consists of self-similar (with respect to a group $\eta_\alpha = (\sigma_\alpha, L_\alpha)$, $\alpha > 0$, of time-space changes) extremal processes. By self-similarity here we mean the property $L_\alpha \circ Y(t) \stackrel{d}{=} Y \circ \sigma_\alpha(t)$ for all $\alpha > 0$. The univariate marginals of Y are max-self-decomposable. If additionally the initial extremal process X is assumed to have homogeneous max-increments, then the limit process is max-stable with homogeneous max-increments.

1. Introduction

An extremal process $Y: [0,\infty) \to [0,\infty)^d$ is a stochastic process with the following two properties:
(1) The sample paths are right-continuous increasing functions from the half line $[0,\infty)$ to the positive orthant $[0,\infty)^d$, called the time space and the state space, respectively.
(2) For any finite sequence of time points $0 = t_0 < \cdots < t_m$ there exist independent random variables (r.v.'s) $U_0, \ldots, U_m$ in $[0,\infty)^d$ such that

$$(Y(t_0), \ldots, Y(t_m)) \stackrel{d}{=} (U_0, U_0 \vee U_1, \ldots, U_0 \vee \cdots \vee U_m). \tag{1.1}$$

The probability distribution of an extremal process with independent max-increments U is completely determined by its distribution function (d.f.). The d.f. of an extremal process Y is the function $f: (0,\infty)^{d+1} \to [0,1]$,

$$f(t,x) = \mathbf{P}(Y(t) < x).$$

It is decreasing and right-continuous in t and increasing and left-continuous in x, thus lower semicontinuous.

We say that a sequence Y_n of extremal processes is convergent weakly in law to the extremal process Y with d.f. f, briefly $Y_n \Rightarrow Y$, if the sequence of r.v.'s $Y_n(t)$ converges in law to the r.v. $Y(t)$ for all t continuity points of f.

With an extremal process Y we associate a lower curve $C_Y: [0,\infty) \to [0,\infty)^d$, increasing and right-continuous, below which the sample functions of Y cannot pass. It is defined coordinatewise: $C^{(i)}(t)$ is the lower endpoint of the d.f. $F_t^{(i)}$ of the ith coordinate of the r.v. $Y(t)$, $i = 1, \ldots, d$. Any extremal process uniquely determines its lower curve.

The following two fundamental results for multivariate extremal processes are stated in [4].

THEOREM 1.1. Structure theorem. *Let $Y: [0,\infty) \to [0,\infty)^d$ be an extremal process with lower curve C. If the underlying probability space is sufficiently rich, there exists a consistent family of max-increments $U(s,t)$, $0 \leq s < t$, such that*
(1) $U(s,t) \geq C(t)$ *a.s., $s < t$;*
(2) $Y(t) = Y(s) \vee U(s,t)$ *a.s., $s < t$;*
(3) *for any finite sequence of time points $0 = t_0 < \cdots < t_m$, the $m+1$ vectors $Y(0), U(t_0,t_1), \ldots, U(t_{m-1},t_m)$ are independent.*

Thus, an extremal process is uniquely determined by a given family of max-increments. The converse is not always true: different families of max-increments may lead to the same extremal process. This phenomenon, called blotting, is studied in [4].

THEOREM 1.2. Decomposition theorem. *Let $Y: [0,\infty) \to [0,\infty)^d$ be an extremal process with lower curve C and a consistent family of max-increments. Then Y is the maximum of two independent extremal processes Y' and*

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Y'' with common lower curve C. The process Y' is generated by a Poisson point process N' whose mean measure does not change any instant space $S_t := \{t\} \times [0,\infty)^d$. The point process N'' associated with Y'' is the sum of a sequence of independent 0–1 point processes N_k on S_{t_k}, and t_k are distinct nonrandom time points. Both processes are independent.

Thus, if (T_k, X_k), $k \geq 1$, are points of the point process $N = N' + N''$, then

$$Y(t) = C_Y(t) \vee \max\{X_k:\ T_k \leq t\}$$

and we say that the point process N generates the extremal process Y. All realizations of the point process are assumed to be Radon measures on the open set $[0, C]^c = ([0,\infty) \times [0,\infty)^d) \setminus [0, C]$. Hence,

$$N([0,t] \times [0,x)^c) < \infty \quad \text{a.s. for} \quad t \geq 0, \quad x > C(t). \tag{1.2}$$

Consider arrays of the form $\{(t_{nk}, X_{nk}):\ k \geq 0\}$, $n \geq 1$, where X_{nk} are row-wise independent r.v.'s in $[0,\infty)^d$ and for each n the sequence of deterministic time points $0 = t_{n0} < t_{n1} < \ldots$ is strictly increasing to ∞. We transform an array into a sequence of extremal processes Y_n with lower curves $C_n(t)$ by setting

$$Y_n(t) = C_n(t) \vee \max\{X_{nk}:\ t_{nk} \leq t\}. \tag{1.3}$$

By virtue of (1.2), the maximum of the right-hand side of (1.3) is well defined. This fact allows us to preserve the notion of "triangular array" also for arrays generating sequences of extremal processes as above. The limit behavior of extremal processes generated by triangular arrays is studied, e.g., in [5, 6, 11, 13].

In this paper, we treat a particular case of triangular array with $X_{nk} = \xi^{-1} \circ X_k$ and $t_{nk} = \tau_n(t_k)$, where the mappings $\zeta_n = (\tau_n, \xi_n)$ are max-automorphisms of $[0,\infty)^{d+1}$. The point process $\{(t_k, X_k)\}$ is associated with an initial extremal process X. Now the partial extremal process Y_n in (1.3) has the form

$$Y_n(t) = \xi_n^{-1} \circ X \circ \tau_n(t). \tag{1.4}$$

Supposing that $Y_n \Rightarrow Y$, we are interested in the intrinsic properties of the limit class of extremal processes.

Recall that the max-automorphisms of the form $\zeta(t,x) = (\tau(t), \xi_1(x_1), \ldots, \xi_d(x_d))$ are continuous and strictly increasing in each component. They preserve the max-operation between extremal processes, i.e., $\zeta(X \vee Y) = \zeta(X) \vee \zeta(Y)$, and form a group with respect to the composition (cf. [2, 9]). Since τ is interpreted as the time change and ξ as the space change, we usually call ζ the time-space change.

Let F and G be d.f.'s on $\mathbf{R}^d$. We say that G belongs to type (F) if there is a max-automorphism L of $\mathbf{R}^d$ such that $G = F \circ L$.

The basic result in Sec. 2 states that the limit extremal process for (1.4) is self-similar in the sense that for all $t > 0$ there exists a space change $L_{\alpha(t)}$ such that

$$Y(t) \stackrel{\mathrm{d}}{=} L_{\alpha(t)} \circ Y(1), \tag{1.5}$$

where $\alpha\colon\ (0,\infty) \leftrightarrow (0,\infty)$ is strictly increasing.

The study of self-similar stochastic processes was initiated by Lamperti [7]. Self-similar extremal processes in a different framework (without the assumption of independence and with the use of affine normalization) are investigated in [8].

Equation (1.5) may also be interpreted as follows:

"All univariate marginals G_t, $t > 0$, of a self-similar extremal process $Y(t)$ are of the same type."

Under the assumptions of Sec. 2, it is shown that this type is max-self-decomposable. An analogous result for self-similar processes with additive increments was already proved by Sato [12] in 1991.

In Sec. 3, we assume additionally that the initial process X in (1.4) has homogeneous max-increments. Then the limit class SSHI of self-similar extremal processes with homogeneous max-increments coincides with the intersection of the max-stable extremal processes and the so-called (cf. [11]) G-extremal processes. The max-stable extremal processes are also studied in [3, 5, 6, 8, 11].

Above, we have defined extremal processes on the time-state space $[0,\infty) \times [0,\infty)^d$. In the same way, one defines extremal processes on $(-\infty,\infty) \times [-\infty,\infty)^d$ (by allowing mass at $-\bar{\infty}$, cf. [9]) or on $[0,1] \times [0,1]^d$, or on any other space homeomorphic to them.

2. Self-Similar Extremal Processes as Limiting

We start with an extremal process $X\colon [0,\infty) \to [0,\infty)^d$ with lower curve C_X, d.f. f, and let $N = \{(t_k, X_k)\colon k \le 0\}$ be the point process generating X by

$$X(t) = C_X(t) \vee \max\{X_k\colon t_k \le t\}.$$

Here X_k, $k \ge 0$, are independent r.v.'s in $[0,\infty)^d$ and the sequence $0 = t_0 < t_1 < \dots$ of deterministic time points increases to ∞. We assume that there exists a sequence $\zeta_n = (\tau_n, \xi_n)$ of max-automorphisms of $[0,\infty)^{d+1}$ such that the sequence of extremal processes

$$Y_n(t) = \xi_n^{-1} \circ X \circ \tau_n(t) = C_n(t) \vee \max\{\xi_n^{-1} \circ X_k\colon t_k \le \tau_n(t)\} \tag{2.1}$$

is convergent weakly in law to a nondegenerate extremal process Y, $Y_n \Rightarrow Y$, with lower curve C_Y and d.f. g, i.e.,

$$f_n(t,x) := f(\tau_n(t), \xi_n(x)) \xrightarrow{\text{w}} g(t,x)$$

or briefly

$$f_n = f \circ \zeta_n \xrightarrow{\text{w}} g. \tag{2.2}$$

(By a degenerate extremal process we understand here a deterministic one.) The lower curve of Y_n is $C_n(t) = \xi_n^{-1} \circ C_X(\tau_n(t))$, $t \ge 0$. The point process N_n in (2.1) with points

$$\{(t_{nk}, X_{nk})\colon k \ge 0\}, \qquad t_{nk} = \tau_n^{-1}(t_k), \qquad X_{nk} = \xi_n^{-1} \circ X_k, \tag{2.3}$$

form a triangular array of row-wise independent r.v.'s X_{nk} in $[0,\infty)^d$. We assume that the max-increments $U_n(s,t)$ of Y_n, $U_n(s,t) = \max\{X_{nk}\colon s < t_{nk} \le t\}$, $0 \le s < t$, are asymptotically negligible in the sense that they obey the following condition:

(AN) $$\max_{\{k\colon s<t_{nk}\le t\}} \mathbf{P}(X_{nk} \in [\vec{0}, x)^c) \longrightarrow 0, \quad n \to \infty,$$

for $(t,x) \in A_Y$, where the set A_Y is determined by its instant sections $A_Y^t = [C_Y(t), \vec{\infty}) \setminus \{C_Y(t)\}$. As is known, in this case the limit extremal process Y is max-id if $Y(0)$ is max-id. (The class of multivariate max-id extremal processes is discussed, e.g., in [4].) Consequently, the d.f. g of the limit extremal process Y is positive on the open set $\operatorname{int} A_Y$ above the lower curve C_Y, hence the family of max-increments is uniquely determined (cf. [4]).

We are interested in characterizing the class max-L of the possible limit extremal processes for sequences of type (2.1) or, equivalently, the class of limit d.f.'s in (2.2), under the (AN)-condition.

By (2.2), for n large enough $\zeta_n\colon \{0 < g < 1\} \to \{0 < f < 1\}$. As a coordinate-wise mapping, ζ_n acts on rectangles in $(0,\infty)^{d+1}$. The smallest rectangle S containing the set $\{0 < g < 1\}$ we call the max-support of g. Denote $q := \inf S$, $w := \sup S$, and the interior of S by $\operatorname{int} S$. Without loss of generality, we assume that $q = \vec{0}$, so $C_Y(0) = \vec{0}$, and $w = (\infty, \vec{w})$, where $\vec{w} = (0,\infty)^d$. Hence,

$$Y\colon [0,\infty) \longrightarrow [\vec{0}, \vec{w}] \quad \text{if} \quad \vec{w} < \vec{\infty},$$

$$Y\colon [0,\infty) \longrightarrow [0,\infty)^d \quad \text{otherwise}.$$

Further, ad hoc we assume ζ_n increasing in n for normalizing increasing maxima.

To characterize the class max-L using general max-automorphisms as above is a difficult problem for which the necessary theoretical background seems to be not yet prepared (e.g., the convergence-to-type theorem does not hold in its classical form). Here we tackle the study of the class max-$L(\mathcal{R})$ under the use of regular norming sequences $\{\zeta_n\}$.

Definition. A sequence $\{\zeta_n\}$ of time-space changes is referred to as regular on an increasing subset $B \subseteq [0,\infty)^{d+1}$ (in the sense that $z_1 \in B$ and $z_2 > z_1$ imply $z_2 \in B$) if for each $\alpha \in (0,1]$ there is a time-space change η_α such that for $m_n \sim \alpha n$ and $n \to \infty$ we have

$$\zeta_n^{-1} \circ \zeta_{m_n}(t,x) \longrightarrow \eta_\alpha(t,x), \quad (t,x) \in B. \tag{2.4}$$

In addition, the correspondence $\alpha \leftrightarrow \eta_\alpha$ is one-to-one.

Thus, we assume that the norming sequence ζ_n in (2.2) is regular on the max-support S of the limit d.f. g. By virtue of (2.4), the family $\{\eta_\alpha\colon \alpha \in (0,1]\}$ can be embedded in a one-parameter group $\{\eta_\alpha\colon \alpha \in (0,\infty)\}$, with

$$\eta_\alpha^{-1} = \eta_{\alpha^{-1}}, \qquad \eta_\alpha \circ \eta_\beta = \eta_{\alpha\beta}, \qquad \eta_1 = \text{id}. \tag{2.5}$$

(Here id is the identical mapping.)

Now, for $m_n < n$, $m_n \sim n\alpha$, where $\alpha \in (0,1)$, $t > 0$, and $I_n(t) := \{k:\ \tau_{m_n}(t) < t_k \le \tau_n(t)\}$, let us decompose the extremal process Y_n in (2.1) as

$$Y_n(t) = \xi_n^{-1} \circ X \circ \tau_n(t) = \xi_n^{-1} \circ X \circ \tau_{m_n}(t) \vee \max\{\xi_n^{-1} \circ X_k:\ k \in I_n(t)\}. \tag{2.6}$$

Substituting here

$$Z_{n,m_n}(t) := C_n(t) \vee \max\{\xi_n^{-1} \circ X_k:\ k \in I_n(t)\},$$

we can express $Y_n(t)$ in two equivalent forms:

$$Y_n(t) = Y_n(\tau_n^{-1} \circ \tau_{m_n}(t)) \vee Z_{n,m_n}(t) = (\xi_n^{-1} \circ \xi_{m_n}) \circ Y_{m_n}(t) \vee Z_{n,m_n}(t).$$

Transition to the weak limit (along a subsequence if necessary) and the regularity condition (2.4) with $\eta_\alpha := (\sigma_\alpha, L_\alpha)$, $\alpha \in (0,1)$, supply two expressions of the limit extremal process:

$$Y \stackrel{d}{=} Y \circ \sigma_\alpha \vee Z_\alpha \stackrel{d}{=} L_\alpha \circ Y \vee Z_\alpha. \tag{2.7}$$

Here $Z_{n,m_n} \Rightarrow Z_\alpha$. Equivalently, the d.f. g of Y satisfies two functional equations:

$$g(t,x) = g(\sigma_\alpha(t), x) g_\alpha(t,x) = g(t, \mathbf{L}_\alpha^{-1}(x)) g_\alpha(t,x). \tag{2.7a}$$

Here g_α is the d.f. of the extremal process Z_α.

Thus, both expressions of $\xi_n^{-1} \circ X \circ \tau_{m_n}$ lead to the following characterization of the class max-$L(\mathcal{R})$:

$$Y \circ \sigma_\alpha \stackrel{d}{=} L_\alpha \circ Y \tag{2.8}$$

or

$$g(\sigma_\alpha(t), x) = g(t, L_\alpha^{-1}(x)). \tag{2.8a}$$

Below, we gather the properties intrinsic for this class of extremal processes.

Definition. An extremal process Y is referred to as self-similar with respect to a one-parameter group $\eta_\alpha = (\sigma_\alpha, L_\alpha)$ of time-space changes if it satisfies (2.8) for all $\alpha \in (0,\infty)$.

From this point of view, above we have proved

Proposition 2.1. *The limit extremal process Y is self-similar.*

The family $\{\eta_\alpha\}$ is defined by (2.4) for $\alpha \in (0,\infty)$. For $\alpha \to 0$ and $\alpha \to \infty$, we impose the following natural boundary conditions on $\eta_\alpha(t,x) = (\sigma_\alpha(t), L_\alpha^{(1)}(x_1), \dots, L_\alpha^{(d)}(x_d))$:

(BC) $\sigma_\alpha(t) \longrightarrow 0$ for $\alpha \to 0$, $\sigma_\alpha(t) \longrightarrow \infty$ for $\alpha \to \infty$,
$L_\alpha^{(i)}(x_i) \longrightarrow 0$ for $\alpha \to 0$, $L_\alpha^{(i)}(x_i) \longrightarrow w_i$ for $\alpha \to \infty$,

where 0 and ∞ are fixed points of σ_α, 0 and w_i are fixed points of $L_\alpha^{(i)}$, $i = 1, \dots, d$, and $(t,x) \in S$.

Lemma 2.1. *The one-to-one correspondence $a \leftrightarrow \eta_\alpha$, $\alpha \in (0,\infty)$, is strictly increasing, hence continuous.*

Indeed, let us assume that $\eta_{\alpha_1} \ge \eta_{\alpha_2}$ for $\alpha_1 < \alpha_2$. Then $\eta_r(z) \ge z$, where $r = \alpha_1/\alpha_2 < 1$ and consequently $\eta_{r^n}(z) \ge z$, $\forall n > 1$, which violates (BC).

Hence, $\{\eta_\alpha:\ \alpha \in (0,\infty)\}$ is a continuous one-parameter group (briefly, c.o.g.). Now put $t = 1$ and $\sigma_\alpha(1) = s$ in (2.8) and observe that

$$Y(s) \stackrel{d}{=} L_{\alpha(s)} \circ Y(1), \tag{2.9}$$

where $\alpha(s)$ is a solution of $\sigma_\alpha(1) = s$. Moreover, this solution is unique because of Lemma 2.1.

Denote by $G_s(\cdot) = g(s,\cdot)$ the d.f. of the univariate process marginals. We have

Proposition 2.2. *For every $s > 0$, $G_s \in$ type(G_1). Furthermore, for each pair $s,t > 0$,*

$$Y(s) \stackrel{d}{=} L_{\alpha(s,t)} \circ Y(t), \tag{2.9a}$$

where $\alpha(s,t) = \alpha(s)/\alpha(t)$.

One of the consequences of (2.9) is the following property:

PROPOSITION 2.3. *The limit extremal process Y is stochastically continuous at all $t > 0$. At $t = 0$, Y may jump to the upper boundary of S.*

Proof. Let $s_n \uparrow t$, $t > 0$. Then for x, a continuity point of $g(t, x)$, we have

$$g(t-0, x) = \lim g(s_n, t) = \lim g(t, L_{\alpha(t)/\alpha(s_n)}(x)) = g(t, x), \quad n \to \infty,$$

since $x_n := L_{\alpha(t)/\alpha(s_n)}(x) \downarrow x$ for $n \to \infty$. Moreover, for $\alpha \to 0$ we have

$$g(0, x) = \lim g(\sigma_\alpha(1), x) = \lim g(1, L_{\alpha^{-1}}(x)) = g(1, \vec{w}).$$

(Here we have used the lower semicontinuity of g.) Hence

$$\mathbf{P}(Y(0) < x) = \mathbf{P}(Y(1) < \vec{w}) \quad \text{for} \quad \vec{0} < x < \vec{w}.$$

Obviously, if G_1 does not allow mass at the upper boundary, i.e., if $\mathbf{P}(Y(1) < \vec{w}) = 1$, then $Y(0) = 0$ a.s. and Y is stochastically continuous at all $t \geq 0$. In the case $\mathbf{P}(Y(1) < \vec{w}) = p < 1$, then $Y(0) = 0$ with probability p and jumps to the upper boundary of S with probability $1 - p$, i.e., $\mathbf{P}(Y(0) \in [\vec{0}, \vec{w})^c) = 1 - p$. Now, the functional equation (2.9) for $s = 0$ and

$$Y(0) \stackrel{d}{=} \lim L_\alpha \circ Y(1), \quad \alpha \to 0,$$

supply the last part of the statement.

PROPOSITION 2.4. *The lower curve C_Y is continuous.*

Indeed, $L_{\alpha(t)}\colon\ (C_Y(1), \vec{w}) \leftrightarrow (C_Y(t), \vec{w})$. So, all lower vertices $C_Y(t)$ of G_t lie on the same orbit of $L_{\alpha(t)}$ through $C_Y(1)$.

The limit extremal process Y is max-id. By Theorem 1 in [4], Y is Poisson, i.e., it is generated by a Poisson point process N. Now Proposition 2.3 and decomposition Theorem 1.2 determine N (but not uniquely, because of the blotting phenomenon: two point processes N_1 and N_2 on $[\vec{0}, C_Y]^c$ that coincide on the set A_Y above the lower curve C_Y, but differ on $[\vec{0}, C_Y]^c \setminus A_Y$, generate the same extremal process Y).

PROPOSITION 2.5. *The point process N associated with the limit extremal process Y is Poisson. It is the sum of a Poisson point process N' with mean measure not changing instant spaces $S_t := \{t\} \times [\vec{0}, \vec{w}]$, $t > 0$, and a 0–1 point process $N_0 = \{(0, Y(0))\}$, where $Y(0)$ is max-id.*

Let us return to the decomposition (2.7). The extremal process Z_α is max-id, too, since it is limiting for a triangular array with the (AN)-condition. It has the same lower curve C_Y. Now the functional equation

$$g(t, x) = g(t, L_\alpha^{-1}(x)) g_\alpha(t, x)$$

can be interpreted as follows.

PROPOSITION 2.6. *For all $t > 0$, the univariate marginals $G_t(\cdot) = g(t, \cdot)$ of the limit extremal process Y are max-self-decomposable with respect to the semigroup $\{L_\alpha^{-1}\colon\ \alpha \in (0, 1]\}$ of space changes, i.e.,*

$$G_t(x) = G_t(L_\alpha^{-1}(x)) G_{t,\alpha}(x). \tag{2.10}$$

The component $G_{t,\alpha}(x) = g_\alpha(t, x)$ is max-id. The max-self-decomposability with respect to a one-parameter semigroup of max-automorphisms of $\mathbf{R}^d$ is discussed in [9]. Such a d.f. G is continuous everywhere except maybe on the boundary of the support. One consequence of (2.10) is the inequality $x < L_\alpha^{-1}(x)$, i.e., the mapping L_α is contracting for $\alpha \in (0, 1)$. Analogously, from the first equation in (2.7a) we conclude that $\sigma_\alpha(t)$ for $\alpha \in (0, 1)$.

Let us denote the invariant (or symmetric) group of g by

$$\operatorname{Inv}(g) := \{\text{time-space changes} \quad \eta \quad \text{of} \quad [0, \infty)^{d+1}\colon\ g \circ \eta = g\}.$$

The force of characteristic equation (2.8a), also written as

$$g(t, x) = g(\sigma_\alpha(t), L_\alpha(x)),$$

is stressed by the next statement.

PROPOSITION 2.7. $\operatorname{Inv}(g)$ *contains a c.o.g.* $\{\eta_\alpha\colon\ \alpha \in (0, \infty)\}$.

As is known, the compactness of $\mathrm{Inv}(g)$ is necessary and sufficient for the application of the convergence-to-type theorem in limit relation (2.2).

We have already observed that every extremal process $Y \in$ max-$L(\mathcal{R})$ is self-similar and all its increments $U(s,t)$, $0 \le s < t$, are max-id (since Y is max-id). The converse statement is also true: any self-similar extremal process (with max-id increments) is limiting for a sequence $Y_n = \xi_n^{-1} \circ X \circ \tau_n$, where the norming sequence is regular and the max-increments of Y_n are asymptotically negligible. To see this we need the following two statements.

Lemma 2.2. *Let Y_n, $n \ge 0$, be extremal processes with d.f.'s f_n. If $Y_n \Rightarrow Y_0$ and Y_0 is stochastically continuous, then the sequence Y_n is asymptotically continuous, i.e., the sequence of d.f.'s f_n satisfies the condition*

(AC) $$\max_{0<t\le c}[f_n(t-0) - f_n(t)] \longrightarrow 0, \quad n \to \infty,$$

for all $c > 0$.

Proof. Indeed, Y_0 stochastically continuous and $Y_n \Rightarrow Y_0$ imply

$$f_n(t-0) - f_n(t) \longrightarrow f_0(t-0) - f_0(t) = 0, \quad \forall t > 0, \quad n \to \infty.$$

Both conditions (AC) and (AN) are closely related, as the following theorem states.

Theorem 2.1. *Assume that $Y_n \Rightarrow Y_0$. If the sequence Y_n is asymptotically continuous, then it has asymptotically negligible max-increments $U_n((s,t])$ for $0 \le s < t$. The converse holds under an additional continuity assumption on the limit process: $Y_0(t-0) \ge C_0(t)$ a.s. for $t > 0$. This condition is automatically fulfilled if the lower curve C_0 of Y_0 is continuous.*

Proof. Denote the d.f. of U_n by H_n. The max-increments $U_n((s,t])$, $0 \le s < t$, are asymptotically negligible if and only if

(AN)′ $$H_{n,t}(x) = \mathbf{P}(U_n(t) \in [\vec{0}, x)) \longrightarrow 1, \quad n \to \infty,$$

for $t > 0$ and $\forall x > C_0(t)$. On the other hand, by the decomposition theorem,

$$Y_n(t) = Y_n(t-0) \vee U_n(t).$$

Thus, condition (AN)′ means that

$$H_{n,t}(x) = \frac{f_n(t,x)}{f_n(t-0,x)} \longrightarrow 1, \quad n \to \infty, \tag{2.11}$$

for $t > 0$ and $x > C_0(t)$.

The sequence Y_n is asymptotically continuous if and only if the asymptotic relation (2.11) holds for all $t > 0$ and $x \in [0,\infty)^d$. Obviously critical values are $x \in (C_0(t-0), C_0(t))$ for which (2.11) may not be fulfilled. This case is avoided by the additional assumption $Y_0(t-0) \ge C_0(t)$ a.s.

Note that Y_n asymptotically continuous and $Y_n \Rightarrow Y_0$ does not imply $U_n(0) \to C_0(0)$. Now we can prove the main statement of this section.

Theorem 2.2. *The class max-$L(\mathcal{R})$ coincides with the class of extremal processes which are self-similar with respect to a c.o.g. $\{\eta_\alpha\colon\ \alpha \in (0,\infty)\}$ of time-space changes, satisfying the (BC)-condition.*

Proof. We have still to show that if Y is self-similar, then $Y \in$ max-$L(\mathcal{R})$. The self-similarity condition implies that the extremal process Y is stochastically continuous at $t > 0$, its lower curve C_Y is continuous, and its d.f. g satisfies the functional equation $g(\sigma_\alpha(t), L_\alpha(x)) = g(t,x)$. Let $N = \{(t_k, Y_k)\colon\ k \ge 0\}$ be the point process generating Y by

$$Y(t) = C_Y(t) \vee \max\{Y_k\colon\ 0 \le t_k \le t\}.$$

Here $t_0 = 0$ and $Y_0 = Y(0)$. Define $t_{nk} = \sigma_n^{-1}(t_k)$, $X_{nk} = L_n^{-1} \circ Y_k$ for $\alpha = n$, $n \ge 1$, and observe that $t_{n0} = 0$, $X_{n0} \stackrel{\mathrm{d}}{=} Y(0)$, so the sequence

$$Y_n(t) := C_n(t) \vee Y(0) \vee \max\{X_{nk}\colon\ 0 < t_{nk} \le t\} = L_n^{-1} \circ Y \circ \sigma_n(t) \stackrel{\mathrm{d}}{=} Y(t)$$

is trivially convergent. Here the norming sequence η_n is regular.

The r.v. $Y(0)$ is max-id, since $Y(0) = \lim Y(t_n)$, $t_n \downarrow 0$. Put $n(t) := \{k\colon\ t_{nk} \le t\}$ and observe that $n(t) \to \infty$ for $n \to \infty$. Then for all $n \ge 1$,

$$Y(0) \stackrel{\mathrm{d}}{=} Y_{n0} \vee \cdots \vee Y_{n,n(t)},$$

where Y_{nk} are independent identically distributed (i.i.d.) copies of $Y(0)$. Define $X'_{n0} = Y_{n0}$, $X'_{nk} = Y_{nk} \vee X_{nk}$ for $1 \le k \le n(t)$. By Lemma 2.2 and Theorem 2.1, the max-increments of Y_n over intervals $(s,t]$, $0 \le s < t$, are asymptotically negligible. So are the r.v. Y_{nk}. Hence, Y belongs to the class max-$L(\mathcal{R})$, since

$$Y(t) \stackrel{\mathrm{d}}{=} Y_n(t) = C_n(t) \vee \max\{X'_{nk}\colon\ 0 \le t_{nk} \le t\}.$$

As a matter of fact, both conditions (2.5) and (BC) determine the analytical form of the time-space changes η_α on S, as the following lemma claims.

Lemma 2.3. *The continuous one-parameter group $\{\eta_\alpha\colon\ \alpha \in (0,\infty)\}$ of time-space changes of $[0,\infty)^{d+1}$, $\eta_\alpha\colon S \leftrightarrow S$, satisfying the boundary conditions* (BC), *can be expressed on S in the form*

$$\eta_\alpha(z) = h^{-1}(h(z) + ec\log\alpha), \tag{2.12}$$

where $e = (1,\dots,1) \in \mathbf{R}^{d+1}$, $c > 0$, and $h\colon S \leftrightarrow (-\infty,\infty)^{d+1}$ is a continuous and strictly increasing coordinatewise mapping.

The proof of this lemma is a modification of Theorem 20 in [1]. Expression (2.12) means that there exists a time-space change $h\colon S \leftrightarrow \mathbf{R}^{d+1}$ so that, in the new coordinates $z' = h(z)$, the one-parameter group $\eta'_\alpha = h\circ\eta_\alpha$ is a simple translation along the diagonal in $\mathbf{R}^{d+1}$, i.e.,

$$\eta'_\alpha(z) = z' + e\theta(\alpha)$$

with $\theta(\alpha) = c\log\alpha \in (-\infty,\infty)$. Denote the translation group along the diagonal by $D_r(z) := z + er$, $z \in \mathbf{R}^{d+1}$, $r \in \mathbf{R}^1$. Note that $D_rD_s = D_{d+s}$, $D_0 = \mathrm{id}$, $D_r^{-1} = D_{-r}$.

Definition. An extremal process $Y:\ (-\infty,\infty) \to [-\infty,\infty)^d$ with d.f. g is called *diagonal* if for all $r \in \mathbf{R}^1$, $g\circ D_r = g$

In other words, diagonal means self-similar with respect to the translation group.

Since $g\circ\eta_\alpha(z) = g\circ h^{-1}(z' + e\theta)$, in fact Theorem 2.2 claims

"The class max-$L(\mathcal{R})$ consists of all extremal processes Y related by a time-space change $h\colon S \leftrightarrow \mathbf{R}^{d+1}$ to a diagonal process M, i.e., $Y \overset{d}{=} h^{-1}\circ M$."

3. Self-Similar Extremal Processes with Homogeneous Max-Increments

Here we consider the same stochastic model as in Sec. 2 with one additional condition: the initial extremal process X has homogeneous max-increments, i.e., the associated increment process

$$U_X(s,t) = C_X(t) \vee \max\{X_k\colon\ s < t_k \le t\}, \quad 0 \le s < t,$$

satisfies

$$U_X(s,t) \overset{d}{=} U_X(0, t-s).$$

Then the limit extremal process Y (in addition to the fact that it is self-similar) has some additional properties. Our next goal is to state them.

Consider the partial extremal process $Y_n(t) = \xi_n^{-1}\circ X\circ\tau_n(t)$. For arbitrary s, $0 \le s < t$, let $m_n = m_n(s)$ be a subsequence of integers such that $\tau_n^{-1}\circ\tau_{m_n}(t) \to s > 0$. Then the decomposition

$$Y_n(t) = Y_n(\tau_n^{-1}\circ\tau_{m_n}(t)) \vee \max\{\xi_n^{-1}\circ X_k\colon\ \tau_n^{-1}\circ\tau_{m_n}(t) < \tau_n^{-1}(t_k) \le t\}$$

supplies the following equation for the limiting extremal process Y:

$$Y(t) \overset{d}{=} Y(s) \vee Y(t-s).$$

On the other hand,

$$Y(t) = Y(s) \vee U_Y(s,t) \quad \text{a.s.}$$

by the structure theorem. The family $\{U_Y(s,t)\}$, $U_Y(s,t) \ge C_Y(t)$ a.s., of the max-increments of Y is uniquely determined since Y is max-id. Let $H_{s,t}$ be the d.f. of $U_Y(s,t)$. Comparing the last two equations for $Y(t)$, we observe

$$U_Y(s,t) \overset{d}{=} C_Y(t) \vee Y(t-s)$$

or equivalently

$$H_{s,t}(x) = G_t(x)/G_s(x) = G_{t-s}(x). \tag{3.1}$$

The d.f. g of the limit process Y satisfies the following functional equation for $x > C_Y(t)$:

$$g(t,x) = g(s,x)g(t-s,x), \quad s < t. \tag{3.2}$$

The solution of (3.2) is well known, namely,

$$g(t,x) = G^t(x),$$

where $G(x) = \mathbf{P}(Y(1) < x)$ and G is a max-id d.f. on $[0,\infty)^d$. Thus

$$\mathbf{P}(U_Y(s,t) < x) = G^{t-s}(x). \tag{3.3}$$

Now the self-similarity of Y, namely $Y(t) = L_{\alpha(t)} \circ Y(1)$, implies

$$G^t(x) = G(\mathbf{L}_t^{-1}(x)) \tag{3.4}$$

for all $t > 0$, where $\{\mathbf{L}_t := L_{\alpha(t)}, t > 0\}$ is a c.o.g. The functional equation (3.4) is characteristic for the class of max-stable d.f.'s (cf. [9]). Thus we have

Proposition 3.1. *All univariate marginals of Y belong to the same type, and this type is max-stable with respect to the one-parameter group $\{\mathbf{L}_t, t > 0\}$ of space changes.*

Corollaries. 1. $\mathbf{P}(Y(0) = 0) = 1$. *Indeed, $\mathbf{L}_t \to C_Y(0)$ for $t \to 0$ and $x \in \{0 < G < 1\}$, and we have assumed that $C_Y(0) = 0$. The left-hand side of (3.4) equals 1 for $t = 0$.*
2. $Y(as) \stackrel{d}{=} \mathbf{L}_a \circ Y(s)$, $\forall a > 0$.
3. Y is stochastically continuous for all $t \geq 0$.

In view of (3.1) and (3.3), we conclude that

$$H_{s,t}(x) = \frac{g(t-s,x)}{g(0,x)} = H_{0,t-s}(x).$$

Thus, we state

Proposition 3.2. *The limit extremal process has homogeneous max-increments.*

Hence, normalization with regular sequences and transition to the weak limit preserve the homogenity property of the initial process X.

Proposition 3.3. *The finite-dimensional distributions (f.d.d.) of Y are of the form*

$$\mathbf{P}(Y(t_1) < x_1, \ldots, Y(t_k) < x_k) = G^{t_1}(x_1)G^{t_2-t_1}(x_2)\ldots G^{t_k-t_{k-1}}(x_k)$$

for $0 < t_1 < \cdots < t_k$, $x_1 < \cdots < x_k$, and $G(x) = \mathbf{P}(Y(1) < x)$.

In [11], extremal processes with these f.d.d. are called G-extremal processes. We denote their class by $\mathcal{R}$.

Note that the type of an extremal process is determined by the type of its max-increments. In general, the type of the univariate marginals of an extremal process does not determine the type of the process itself, e.g., given that $G_t(x) = \mathbf{P}(Y(t) < x)$ is max-id $\forall t > 0$, we cannot claim that the quotient

$$H_{s,t}(x) = \mathbf{P}(U_Y(s,t) < x) = \frac{G_t(x)}{G_s(x)}$$

(hence the process Y) is max-id, too. (Recall that a max-id d.f. may have indecomposable components, cf. [10].) But in our case (3.3) and (3.4) mean that the increment process is max-stable, too.

Proposition 3.4. *The type of the limit extremal process Y is uniquely determined by the type of the univariate marginals, namely, Y is max-stable (briefly, $Y \in MS$).*

This means that for all integers n there exist i.i.d. extremal processes $Y_1, \ldots, Y_n$, copies of Y, and a space change L_n such that

$$Y \stackrel{d}{=} L_n^{-1}(Y_1 \vee \cdots \vee Y_n)$$

(cf. [3] and [5]).

Consider the functional equation (3.4) once more. Another consequence of it is the next property.

Proposition 3.5. *The lower curve C_Y of the limit extremal process Y is constant, i.e., $C_Y(t) \equiv C_Y(1) = \inf\{G > 0\}$.*

Denote the class of possible limit extremal processes for triangular arrays described in this section by SSHI. We have observed that every $Y \in$ SSHI is a self-similar extremal process with homogeneous max-id increments. Propositions 3.2 and 3.4 stress the inclusion SSHI $\subset MS \cap \mathcal{R}$.

The converse observation is also true. Indeed, let Y be a max-stable extremal process with homogeneous max-increments and d.f. g. Hence $\forall t > 0$, $g(t,x) = G^t(x) = G(\mathbf{L}_t^{-1}(x))$, where G is the d.f. of the r.v. $Y(1)$ and $\inf\{G > 0\} =: q \geq 0$. Define a r.v. $X \overset{d}{=} Y(1)$ and let $X_1, \ldots, X_n$ be i.i.d. copies of X. Put $t_{nk} := k/n$, $X_{nk} := \mathbf{L}_n^{-1} \circ X_k$. Then the triangular array $\{(t_{nk}, X_{nk})\}$ generates a sequence of extremal processes Y_n,

$$Y_n(t) = q \vee \max\{\mathbf{L}_n^{-1} \circ X_k : \ k \leq nt\},$$

which is convergent weakly in law to the initial extremal process Y, namely,

$$\mathbf{P}(Y_n(t) < x) = G^{[nt]}(\mathbf{L}_n(x)) \sim G^t(x) = \mathbf{P}(Y(t) < x). \tag{3.5}$$

The partial extremal process Y_n can also be expressed as

$$Y_n(t) = \mathbf{L}_n^{-1} \circ X \circ \tau_n(t),$$

where $\tau_n(t) = nt$ and $X(t) := q \vee \max\{X_k : \ k \leq t\}$. The extremal process X has homogeneous max-increments $U_X(s,t)$ (since X_k are i.i.d.r.v.), and its d.f. f is

$$f(t,x) = \mathbf{P}(X(t) < x) = G^{[t]}(x).$$

Obviously, the norming sequence $\zeta_n = (\tau_n, \mathbf{L}_n)$ is regular. Further, the stochastic continuity of the limit extremal process Y in (3.5) guarantees the asymptotic continuity of the sequence Y_n, which implies the (AN)-condition for the max-increments of Y_n. Consequently, the process Y belongs to the class SSHI, and so we have established the following property, characteristic for the limit class.

Proposition 3.6. *The class SSHI coincides with the class of all self-similar extremal processes with homogeneous max-increments. Thus,* SSHI $= \mathcal{R} \cap MS$.

Example. Let Y be an extremal process with d.f.

$$g(t,x) = \begin{cases} 0, & \text{for } x \leq 0, \\ \exp\{-t/x^\gamma\}, & \text{for } x > 0,\ \gamma > 0. \end{cases}$$

Obviously, $g(t,x) = g(\alpha t, \alpha^H x)$, where $H = 1/\gamma$. Thus, Y is self-similar with respect to the c.o.g. η_α with $\sigma_\alpha(t) = \alpha t$, $L_\alpha(x) = \alpha^H x$. Further, $g_t(x) = (e^{-x^{-\gamma}})^t$, $g_1(x) = \Phi_\gamma(x)$, i.e., $Y \in$ SSHI.

We complete this section with the following nonsurprising result.

Proposition 3.7. *Let X be univariate extremal process with d.f. f and homogeneous max-increments. Suppose that there exists a nondegenerate d.f. G and a regular norming sequence $\{\zeta_n = (\tau_n, \xi_n)\}$ of time-space changes such that*

$$\zeta_n \circ f(1, x_0) \longrightarrow G(x_0), \quad n \to \infty, \tag{3.6}$$

for a continuity point $x_0 \in \{0 < G < 1\}$. If G is max-stable with respect to the group $\{L_s : \ s > 0\}$ of the limiting space changes, then there exists an extremal process $Y \in$ SSHI so that

$$\xi_n^{-1} \circ X \circ \tau_n \Longrightarrow Y \quad \text{and} \quad \mathbf{P}(Y(1) < \cdot) = G(\cdot).$$

Proof. By the regularity of the norming sequence for any $s > 0$ there exists a subsequence $\{n_s\} \subset \{n\}$ such that $\tau_n^{-1} \circ \tau_{n_s}(1) \to s$ and $\xi_n^{-1} \circ \xi_{n_s}(x_0) \to L_s(x_0)$. Then

$$\xi_n^{-1} \circ X \circ \tau_n(s) \sim \xi_n^{-1} \circ X \circ \tau_{n_s}(1) = \xi_n^{-1} \circ \xi_{n_s}(\xi_{n_s} \circ X \circ \tau_{n_s}(1)).$$

Consequently, for $n \to \infty$ we have

$$\zeta_n \circ f(s, x_0) \xrightarrow{\mathrm{w}} G(L_s(x_0)) = G^s(x_0) := g(s, x_0).$$

For the weak convergence to a max-stable d.f. G, it is sufficient to have convergence in one point $x_0 \in \{0 < G < 1\}$. This fact is proved in [15] using linear normalization, but it is also true if one uses max-automorphisms. Now the f.d.d. given by Proposition 3.3 determine an extremal process Y with d.f. g, which has the properties claimed in the statement.

In the multivariate case, let χ be a set which intersects every orbit of the group $\{L_s : s > 0\}$ in exactly one point. The statement remains true if we suppose (3.6) to hold for all $x_0 \in \chi$.

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