Weak Asymptotic Results for t-Hill Estimator

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Abstract

Z. Fabian and M. Stehlik (2009) investigate a new estimator of extreme value index of a distribution function with regularly varying tail, the so called t-Hill estimator. We continue their work and obtain the limit distribution of this estimator, when the rank k of the upper order statistic is $o(n)$. Then we normalize the estimator properly and find its asymptotic normality.

Keywords: t-Hill estimator, Point estimation, Asymptotic properties of estimators.

Mathematics Subject Classification: 62F10, 62F12.

1 Introduction

Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be a sequence of i.i.d. random variables (r.v's) with distribution function (d.f.) F. We suppose that as $x \to \infty$, the survival function $\bar{F}(x) = 1 - F(x)$ satisfies

$$\overline{F}(x) \sim x^{-\alpha}.L(x), \tag{1}$$

where $\alpha > 0$ and $L(x)$ is a slowly varying function. This class of d.f. is denoted by $RV_{-\alpha}$. Let $\mathbf{X}_{(n,n)} \leq \mathbf{X}_{(n-1,n)} \leq ... \leq \mathbf{X}_{(1,n)}$ be the corresponding upper order statistics. In this paper we are concerned with one of the major problems in extreme value theory - estimation of the extreme value index $\frac{1}{\alpha}$.

Hill (1975) proposes the following Hill estimator

$$\frac{1}{k}\sum_{i=1}^{k} ln\frac{\mathbf{X}_{(i,n)}}{\mathbf{X}_{(k+1,n)}}, \quad k = 1, 2, ..., n-1.$$

The asymptotic normality of this estimator is investigated e.g. by E. Heausler et al. (1985). The Hill estimator it is not robust with respect to large observations. Fabian and Stehlik (2009) propose a robust and distribution sensitive Hill-like method for estimating $\frac{1}{\alpha}$, under condition (1). They define the following **t-Hill estimator**

$$\frac{1}{\widehat{\alpha}_{k,n}} := \frac{1}{H_{X,k,n}} - 1, \tag{2}$$

where

$$H_{X,k,n} := \frac{1}{k}\sum_{i=1}^{k} \frac{\mathbf{X}_{(k+1,n)}}{\mathbf{X}_{(i,n)}}, \quad k = 1, 2, ..., n-1 \tag{3}$$

and obtain its weak consistency when $k(n)$ is a sequence of integers with $k(n) \to \infty$ and $k(n)/n \to 0$ as $n \to \infty$.

In Section 2 of this paper we obtain the limit distribution of the t-Hill estimator, when the rank k of the upper order statistics is fixed and $n \to \infty$. In Section 3 we prove that it is asymptotically normal for large sample of Pareto distributed observations and $k = 1, 2, ..., n-1$. If $\bar{F} \in RV_{-\alpha}$, then (2) is asymptotically normal for $k(n) \to \infty$ and $k(n)/n \to 0$ as $n \to \infty$.

2 Limit distribution of t-Hill estimator

It this section we suppose that k is fixed and $k < n$.

Let $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n$ be i.i.d. uniformly distributed r.v's on $(0, 1)$ and let $\mathbf{U}_{(n,n)} \leq \mathbf{U}_{(n-1,n)} \leq ... \leq \mathbf{U}_{(1,n)}$ be their upper order statistics.

It is not difficult to check that

$$\left\{1 - \mathbf{U}_{(i,n)}, i = 1, 2, ..., n\right\} \stackrel{d}{=} \left\{\mathbf{U}_{(n-i+1,n)}, i = 1, 2, ..., n\right\} \tag{4}$$

and

$$\left\{\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}, i = 1, 2, ..., k\right\} \stackrel{d}{=} \left\{\mathbf{U}_{(k-i+1,k)}, i = 1, 2, ..., k\right\}. \tag{5}$$

Recall, the probability quantile transform states that

$$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \stackrel{d}{=} \left\{F^{\leftarrow}(U_{(i,n)}), i = 1, 2, ..., n\right\}, \tag{6}$$

where $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ are i.i.d. random variables with d.f. F and

$$F^{\leftarrow}(p) := inf\{x \in \mathbb{R} : F(x) \geq p\}, \quad p \in (0,1],$$

is the left-continuous inverse of F.

Theorem 1. Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be i.i.d. random variables with d.f. F and $\bar{F} \in RV_{-\alpha}$, $\alpha > 0$. For fixed k and $n \to \infty$

$$\frac{1}{\widehat{\alpha}_{k,n}} \overset{d}{\to} \left(\frac{1}{k}\sum_{i=1}^{k} \mathbf{U}_i^{\frac{1}{\alpha}}\right)^{-1} - 1.$$

Proof: The probability quantile transform (6), (4) and (1) entail

$$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \overset{d}{=} \left\{F^{\leftarrow}(\mathbf{U}_{(i,n)}), i = 1, 2, ..., n\right\}$$

$$\overset{d}{=} \left\{F^{\leftarrow}(1 - \mathbf{U}_{(n-i+1,n)}), i = 1, ..., n\right\} \overset{d}{=} \left\{\left(\frac{1}{\bar{F}}\right)^{\leftarrow}\left(\frac{1}{\mathbf{U}_{(n-i+1,n)}}\right), i = 1, ..., n\right\}$$

$$\overset{d}{=} \left\{\left(\mathbf{U}^{-1}_{(n-i+1,n)}\right)^{\frac{1}{\alpha}} L_1\left(\mathbf{U}^{-1}_{(n-i+1,n)}\right), \quad i = 1, 2, ..., n\right\}.$$

The reciprocal of a uniformly distributed r.v. is a.s. greater than one. Thus, the Karamata-representation for regularly varying functions implies
$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \overset{d}{=}$

$$\overset{d}{=} \left\{\left(\mathbf{U}^{-1}_{(n-i+1,n)}\right)^{\frac{1}{\alpha}} c(\mathbf{U}^{-1}_{(n-i+1,n)}) exp\{\int_1^{\mathbf{U}^{-1}_{(n-i+1,n)}} \frac{\varepsilon(t)}{t} dt\}, \quad i = 1, 2, ..., n\right\},$$

where $c(x) \to c_0 \in (0, \infty)$ as $x \to \infty$, $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varepsilon(t) \to 0$ as $t \to \infty$. Then for $H_{X,k,n}$ defined in (3) we have

$$H_{X,k,n} \overset{d}{=} \frac{1}{k}\sum_{i=1}^{k}\left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \frac{c(\mathbf{U}^{-1}_{(n-k,n)})}{c(\mathbf{U}^{-1}_{(n-i+1,n)})} exp\left(\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\, dx\right).$$

By (5), for $n \in \mathbb{N}$ and $k = 1, 2, ..., n$,

$$\frac{1}{k}\sum_{i=1}^{k}\left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \overset{d}{=} \frac{1}{k}\sum_{i=1}^{k} \mathbf{U}^{\frac{1}{\alpha}}_{(k-i+1,k)} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{U}_i^{\frac{1}{\alpha}}. \qquad (7)$$

If

$$\Delta_n := \left| H_{X,k,n} - \frac{1}{k}\sum_{i=1}^{k}\left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \right| \xrightarrow{\mathbb{P}} 0, \tag{8}$$

then in distribution

$$\lim_{n\to\infty} H_{X,k,n} = \lim_{n\to\infty} \frac{1}{k}\sum_{i=1}^{k}\left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}}$$

(cf. Th.4.1. of Billingsley (1977)).

We check (8). By the triangle inequality

$$\Delta_n \leq \frac{1}{k}\sum_{i=1}^{k}\left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \left| \frac{c(\mathbf{U}^{-1}_{(n-k,n)})}{c(\mathbf{U}^{-1}_{(n-i+1,n)})} exp\left(\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\,dx\right) - 1 \right|.$$

For $n \in \mathbb{N}$, $k = 1, 2, ..., n$ and $i = 1, 2, ..., k$, $0 \leq \left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \leq 1$ a.s. Thus,

$$0 \leq \Delta_n \leq \frac{1}{k}\sum_{i=1}^{k} \left| \frac{c(\mathbf{U}^{-1}_{(n-k,n)})}{c(\mathbf{U}^{-1}_{(n-i+1,n)})} exp\left(\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\,dx\right) - 1 \right|.$$

Now we have to show that the summands in the above expression converge in probability to zero for $n \to \infty$. The function $T(x, y) = x.y$ is continuous in the point $(x, y) = (1, 1)$. In order to use the continuity of composition we have to check the following two convergences

$$c(\mathbf{U}^{-1}_{(n-k,n)}).c^{-1}(\mathbf{U}^{-1}_{(n-i+1,n)}) \xrightarrow{\mathbb{P}} 1 \tag{9}$$

and

$$exp\left(\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\,dx\right) \xrightarrow{\mathbb{P}} 1, \quad n \to \infty. \tag{10}$$

The function $c : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $c(x) \to c_0 \in (0, \infty)$ as $x \to \infty$. $\mathbf{U}_{(n-i+1,n)} \xrightarrow{a.s.} 0$ and $\mathbf{U}_{(n-k,n)} \xrightarrow{a.s.} 0$ as $n \to \infty$ thus, (9) follows by continuity of $g(x, y) = \frac{x}{y}$ in $(x, y) = (c_0, c_0)$.

Consider (10). $\mathbf{U}_{(n-i+1,n)} \overset{a.s.}{\to} 0$ and $\mathbf{U}_{(n-k,n)} \overset{a.s.}{\to} 0$ as $n \to \infty$. In view of Karamata-representation for regularly varying functions $\varepsilon(t) \to 0$ as $t \to \infty$. Consequently a.s. for $\varepsilon_0 > 0$ there exists $n_{\varepsilon_0} \in \mathbb{N}$, such that for $n > n_{\varepsilon_0}$,

$$|\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\, dx| \leq \varepsilon_0 | \int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{1}{x}\, dx| = \varepsilon_0 \,|\, ln\, \frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\,|.$$

Let $\epsilon > 0$. By (5)

$$0 \leq P(|\int_{\mathbf{U}^{-1}_{(n-i+1,n)}}^{\mathbf{U}^{-1}_{(n-k,n)}} \frac{\varepsilon(x)}{x}\, dx| \geq \epsilon)$$

$$\leq P(\varepsilon_0 \,|\, ln\, \frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\,| \geq \epsilon) \leq P(\varepsilon_0 \,|\, ln\, \mathbf{U}_{(k-i+1,k)}| \geq \epsilon).$$

The random variable $ln\, \mathbf{U}_{(k-i+1,k)}$ does not depend on n and it is a.s. finite, hence for $\varepsilon_0 \to 0$ we obtain that (10) is satisfied. Thus, for $x > 0$,

$$\lim_{n\to\infty} P(\widehat{\alpha}^{-1}_{k,n} < x) = \lim_{n\to\infty} P(H_{X,k,n} \geq (1+x)^{-1})$$

$$= \lim_{n\to\infty} P(\frac{1}{k}\sum_{i=1}^{k} \left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k,n)}}\right)^{\frac{1}{\alpha}} \geq \frac{1}{1+x}) = P\left(\left(\frac{1}{k}\sum_{i=1}^{k} \mathbf{U}_i^{\frac{1}{\alpha}}\right)^{-1} - 1 < x\right).$$

□

Remark: If $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ are i.i.d. random variables with Pareto d.f.

$$F(x) = \begin{cases} 0 & , \quad x < 1 \\ 1 - x^{-\alpha} & , \quad x \geq 1 \end{cases}, \tag{11}$$

$\alpha > 0$, then in view of (6),

$$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \overset{d}{=} \left\{(1 - \mathbf{U}_{(i,n)})^{-1/\alpha}, i = 1, 2, ..., n\right\}.$$

Definition (2) and (7) give directly

$$\frac{1}{\widehat{\alpha}_{k,n}} \overset{d}{=} \left\{\frac{1}{k}\sum_{i=1}^{k} \left(\frac{1 - \mathbf{U}_{(i,n)}}{1 - \mathbf{U}_{(k+1,n)}}\right)^{\frac{1}{\alpha}}\right\}^{-1} - 1 \overset{d}{=} \left(\frac{1}{k}\sum_{i=1}^{k} \mathbf{U}_i^{\frac{1}{\alpha}}\right)^{-1} - 1.$$

3 Asymptotic normality of t-Hill estimator

In this section we consider the asymptotic distributions of the harmonic mean $X^*_{H,k(n),n} = H^{-1}_{X,k,n}$ and of the t-Hill estimator $\widehat{\alpha}^{-1}_{k(n),n} = X^*_{H,k(n),n} - 1$.

Proposition 2. Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be i.i.d. r.v's with d.f. $F(x) = 1 - x^{-\alpha}$, $x > 1$ and let Φ be the standard normal d.f. Then for $k(n) \to \infty$ and $k(n) < n$,

$$\lim_{n\to\infty} P\left(\frac{\sqrt{k(n)}(X^*_{H,k(n),n} - \frac{1+\alpha}{\alpha})}{\frac{1+\alpha}{\alpha}\sqrt{\frac{1}{\alpha(\alpha+2)}}} < x\right) = \Phi(x), \quad x \in \mathbb{R}. \tag{12}$$

Proof: As noted above

$$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \stackrel{d}{=} \left\{(1 - \mathbf{U}_{(i,n)})^{-1/\alpha}, i = 1, 2, ..., n\right\}.$$

Therefore

$$H_{X,k(n),n} \stackrel{d}{=} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \left(\frac{1 - \mathbf{U}_{(i,n)}}{1 - \mathbf{U}_{(k(n)+1,n)}}\right)^{\frac{1}{\alpha}} \stackrel{d}{=} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \mathbf{U}_i^{\frac{1}{\alpha}}.$$

It is easy to obtain that $E(\mathbf{U}^{\frac{1}{\alpha}}) = \alpha(\alpha + 1)^{-1}$ and $D(\mathbf{U}^{\frac{1}{\alpha}}) = \alpha(\alpha + 1)^{-2}(\alpha + 2)^{-1}$. In view of the CLT, as $k(n) \to \infty$

$$\frac{\sqrt{k(n)}(H_{X,k(n),n} - \frac{\alpha}{\alpha+1})}{\frac{1}{\alpha+1}\sqrt{\frac{\alpha}{\alpha+2}}} \stackrel{d}{=} \frac{\sqrt{k(n)}(\frac{1}{k(n)}\sum_{i=1}^{k(n)} \mathbf{U}_i^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha+1})}{\frac{1}{\alpha+1}\sqrt{\frac{\alpha}{\alpha+2}}}$$

are weakly asymptotically standard normal.

For $g(x) = \frac{1}{x}$, $g'(x) = -x^{-2}$ we apply the δ-method and obtain

$$\lim_{n\to\infty} P\left\{\frac{\sqrt{k(n)}(X^*_{H,k(n),n} - \frac{\alpha+1}{\alpha})}{-\frac{1}{\alpha+1}\sqrt{\frac{\alpha}{\alpha+2}}\frac{(\alpha+1)^2}{\alpha^2}} < x\right\} = \Phi(x), \quad x \in \mathbb{R}.$$

Now the symmetry of Φ entails (12). □

Corrolary: Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be i.i.d. random variables with Pareto d.f. F. Then for $k(n) \to \infty$ and $k(n) < n$,

$$\lim_{n\to\infty} P\left(\frac{\sqrt{k(n)}(\frac{1}{\widehat{\alpha}_{k(n),n}} - \frac{1}{\alpha})}{\frac{1+\alpha}{\alpha}\sqrt{\frac{1}{\alpha(\alpha+2)}}} < x\right) = \Phi(x), \quad x \in \mathbb{R}. \tag{13}$$

Theorem 3. Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ be i.i.d. r.v's with $\overline{F} \in RV_{-\alpha}, \alpha > 0$. If $k(n) \to \infty$ and $k(n)/n \to 0$ as $n \to \infty$, then (12) and (13) hold.

Proof: We have already seen that

$$\left\{\mathbf{X}_{(i,n)}, i = 1, 2, ..., n\right\} \stackrel{d}{=} \left\{\left(\frac{1}{\overline{F}}\right)^{\leftarrow}\left(\frac{1}{U_{(n-i+1,n)}}\right), i = 1, 2, ..., n\right\}.$$

Thus,

$$H_{X,k(n),n} \stackrel{d}{=} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{\left(\frac{1}{\overline{F}}\right)^{\leftarrow}\left(\frac{1}{\mathbf{U}_{(n-k(n),n)}}\right)}{\left(\frac{1}{\overline{F}}\right)^{\leftarrow}\left(\frac{1}{\mathbf{U}_{(n-k(n),n)}} \frac{\mathbf{U}_{(n-k(n),n)}}{\mathbf{U}_{(n-i+1,n)}}\right)}.$$

Since $\left(\frac{1}{\overline{F}}\right)^{\leftarrow}(x) \in RV_{1/\alpha}$, then for $\varepsilon > 0$ there exists $t_0(\varepsilon)$ such that for $t \geq t_0(\varepsilon)$ and $x \geq 1$,

$$(1-\varepsilon)x^{1/\alpha-\varepsilon} < \frac{\left(\frac{1}{\overline{F}}\right)^{\leftarrow}(tx)}{\left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t)} < (1+\varepsilon)x^{1/\alpha+\varepsilon} \tag{14}$$

(see de Haan (1970)).

By condition $k(n) \to \infty$ and $k(n)/n \to 0$ as $n \to \infty$, then $\mathbf{U}^{-1}_{(n-k(n),n)} \stackrel{a.s.}{\to} \infty$, $n \to \infty$. This together with (14) gives, that for $\varepsilon > 0$ there exists n_0, such that for all $n > n_0$, a.s.

$$H_{X,k(n),n} \geq \frac{1}{k(n)} \sum_{i=1}^{k(n)} \left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k(n),n)}}\right)^{\frac{1}{\alpha}+\varepsilon} \frac{1}{\varepsilon+1}.$$

Equality (5) entails

$$\sum_{i=1}^{k(n)} \left(\frac{\mathbf{U}_{(n-i+1,n)}}{\mathbf{U}_{(n-k(n),n)}}\right)^{\frac{1}{\alpha}+\varepsilon} \stackrel{d}{=} \sum_{i=1}^{k(n)} \mathbf{U}^{\frac{1}{\alpha}+\varepsilon}_{(k(n)-i+1,k(n))} \stackrel{d}{=} \sum_{i=1}^{k(n)} \mathbf{U}_i^{\frac{1}{\alpha}+\varepsilon}.$$

Then for $\varepsilon = k^{-1}(n)$ we have $\mathbf{U}^{\varepsilon} \to 1$ a.s. as $n \to \infty$ and

$$F_{X,k(n),n}(x) := P\left(\frac{H_{X,k(n),n} - \frac{\alpha}{\alpha+1}}{\frac{\sqrt{\alpha}}{(\alpha+1)\sqrt{\alpha+2}\sqrt{k(n)}}} < x\right)$$

$$\leq P\left(\frac{\frac{1}{k(n)}\sum_{i=1}^{k(n)}\mathbf{U}_i^{(\frac{1}{\alpha}+\frac{1}{k(n)})}-\frac{\alpha}{\alpha+1}}{\frac{\sqrt{\alpha}}{(\alpha+1)\sqrt{\alpha+2}\sqrt{k(n)}}} < x(1+\frac{1}{k(n)})+\frac{\alpha}{\frac{\sqrt{\alpha}}{\sqrt{\alpha+2}}\sqrt{k(n)}}\right).$$

The Central Limit Theorem implies $\limsup_{n\to\infty} F_{X,k(n),n}(x) \leq \Phi(x)$, for $x \in \mathbb{R}$. Analogously one concludes that $\liminf_{n\to\infty} F_{X,k(n),n}(x) \geq \Phi(x)$, $x \in \mathbb{R}$. Hence $\lim_{n\to\infty} F_{X,k(n),n}(x) = \Phi(x)$, $x \in \mathbb{R}$.

Again we apply the δ-method for $g(x) = \frac{1}{x}$, $g'(x) = -x^{-2}$ and obtain

$$\lim_{n\to\infty} P\left(\frac{\sqrt{k(n)}(X^*_{H,k(n),n}-\frac{\alpha+1}{\alpha})}{-\frac{1}{\alpha+1}\sqrt{\frac{\alpha}{\alpha+2}}\frac{(\alpha+1)^2}{\alpha^2}} < x\right) = \Phi(x).$$

The symmetry of Φ entails (12). Now (13) is an immediate consequence of (12). $\square$.

Remark: In view of Proposition 2 and its corollary when the observed random variable is Pareto distributed we do not need $k(n)$ to be infinitely small function of n. In this case it turns out that we do not need so large sample to have a good t-Hill estimator.

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