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ON LIMIT LAWS FOR CENTRAL ORDER STATISTICS UNDER POWER NORMALIZATION

E. I. Pancheva, A. Gacovska-Barandovska

Smirnov (1949) derived four limit classes of distributions for linearly normalized central order statistics. In this paper we investigate the possible limit distributions of the k-th upper order statistics with central rank using regular power norming sequences and obtain twelve limit classes.

Keywords: k-th upper order statistic, Central rank, Power normalization, Regular norming sequence.

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1. Introduction

Below $\{k_n\}$ is a sequence of integers such that $\frac{k_n}{n} \to \theta \in (0,1)$ and $X_{k_n,n}$ is the k_n-th upper central order statistic (u.c.o.s.) from a sample of iid rv's $X_1, ..., X_n$ with a continuous df F, thus

$$X_{n,n} < ... < X_{k_n,n} < ... < X_{1,n}.$$

We denote by GMA the group of all max-automorphisms on $\mathbb{R}$ with respect to the composition "$\circ$". They are strictly increasing continuous mappings and hence they preserve the max-operation "$\vee$" in the sense that $L(X \vee Y) = L(X) \vee L(Y)$. Let Φ denote the standard normal df and $\overline{F} = 1 - F$. In Pancheva and Gacovska (2013), abbreviated here as PG'13, the authors have proved the following

Theorem 1. *Let H be a non-degenerate df, $\frac{k_n}{n} \to \theta \in (0,1)$ and $\{G_n\}$ be a sequence of norming mappings in GMA. Then*

$$F_{k_n,n}\left(G_n(x)\right) := P\left({G_n}^{-1}(X_{k_n,n}) < x\right) \xrightarrow[n]{w} H(x) \tag{1}$$

if and only if

$$\sqrt{n}\cdot\frac{\theta-\overline{F}\left(G_n(x)\right)}{\sqrt{\theta(1-\theta)}} \xrightarrow[n]{w} \tau(x) \tag{2}$$

where $\tau(x)$ is a non-decreasing function uniquely determined by the equation

$$H\left(x\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\tau(x)} e^{-\frac{x^2}{2}}\,dx = \Phi\left(\tau(x)\right). \tag{3}$$

Let $\mathcal{A}$ be the group of all affine transformations on $\mathbb{R}$, $\alpha(x) = ax+b,\ a>0,\ b$ real and $\mathcal{P}$ be the group of all power transformations on $\mathbb{R}$, $p(x) = b|x|^a sign(x)$ with a and b positive. We denote by $l_H = \inf\{x : H(x) > 0\}$ the left endpoint of the support of H and by $r_H = \sup\{x : H(x) < 1\}$ the right endpoint.

Definition 1. *A sequence $\{G_n\} \subset GMA$ is a regular norming sequence on $(l_H, r_H)\times(0,\infty)$ if for all $t\in(0,\infty)$ there exists $g_t(x)\in GMA$ such that*

$$\lim_n G^{-1}_{[nt]}\circ G_n(x) = g_t(x), \quad x\in(l_H, r_H) \tag{4}$$

and the correspondence $t \to g_t(x)$ is continuous 1-1 mapping. The last means that $s\neq t$ implies $g_s\neq g_t$, for $s,t>0$.

Example 1. a) A norming sequence $\{\alpha_n\}\subset\mathcal{A}$ is regular iff $\forall t>0$ and $n\to\infty$

$$\frac{a_n}{a_{[nt]}} \to A_t > 0, \qquad \frac{b_n - b_{[nt]}}{a_{[nt]}} \to B_t \in \mathbb{R},$$

thus

$$\alpha^{-1}_{[nt]}\circ\alpha_n(x) = \frac{a_n}{a_{[nt]}}x + \frac{b_n-b_{[nt]}}{a_{[nt]}} \to A_t x + B_t.$$

b) A norming sequence $\{p_n\}\subset\mathcal{P}$ is regular iff $\forall t>0$ and $n\to\infty$

$$\frac{a_n}{a_{[nt]}} \to A_t > 0, \qquad \left(\frac{b_n}{b_{[nt]}}\right)^{1/a_{[nt]}} \to B_t > 0,$$

then

$$p_{[nt]}^{-1} \circ p_n(x) = \left(\frac{b_n |x|^{a_n}}{b_{[nt]}} \right)^{\frac{1}{a_{[nt]}}} sign(x) \to B_t |x|^{A_t} sign(x).$$

Definition 2. *A df F belongs to θ-normal domain of attraction of H (briefly θ-NDA(H)) if (1) holds and* $(\frac{k_n}{n} - \theta)\sqrt{n} \to 0$.

A basic result to characterize the class of limit df's in (1) is the following theorem proved in PG'13.

Theorem 2. *If a df H has θ-NDA with respect to a regular norming sequence* $\{G_n\} \subset GMA$*, then its corresponding function* $\tau(x)$ *satisfies for* $t \in (0, \infty)$ *the following functional equation*

$$\sqrt{t} \cdot \tau(x) = \tau\left(g_t(x)\right), \quad x \in C\left(\tau\right). \tag{5}$$

Here $C(\tau)$ means the set of all continuity points of τ. Solving (5) with respect to τ, given that $g_t \in \mathcal{A}$, Smirnov (1949) derived four different classes of limit df's $H = \Phi \circ \tau$, namely

1. $H_1(x) = \begin{cases} 0, & x < 0 \\ \Phi\left(cx^{\alpha}\right), & x \geq 0 \end{cases}$; $c > 0$, $\alpha > 0$.

2. $H_2(x) = \begin{cases} \Phi\left(-c|x|^{\alpha}\right), & x < 0 \\ 1, & x \geq 0 \end{cases}$; $c > 0$, $\alpha > 0$.

3. $H_3(x) = \begin{cases} \Phi\left(-c_1|x|^{\alpha}\right), & x < 0 \\ \Phi\left(c_2 x^{\alpha}\right), & x \geq 0 \end{cases}$; $c_1, c_2 > 0$, $\alpha > 0$.

4. $H_4(x) = \begin{cases} 0, & x < l_H \\ \frac{1}{2}, & l_H \leq x < r_H \\ 1, & x \geq r_H \end{cases}$.

Solving (5) , given that $g_t \in GMA$, Pancheva and Gacovska (2013) obtained 13 possible types of limit df's in (1). The aim of this note is to find the possible solutions of (5) with respect to τ if $g_t \in \mathcal{P}$ and to list the corresponding limit laws $H = \Phi \circ \tau$ in (1).

Let us first make precise the notion "type(H)" for a non-degenerate df H. We say that $G \in g$-type(H) if there exists a mapping $\phi \in GMA$ such that $G = H \circ \phi$ (where g stands for general). In order to distinguish the cases $\phi \in \mathcal{A}$ and $\phi \in \mathcal{P}$ we define two subsets of the set g-type(H), namely

$\alpha - type(H) = \{G : \exists \alpha \in \mathcal{A},\ G(x) = H(\alpha(x))\}$,
$p - type(H) = \{G : \exists p \in \mathcal{P},\ G(x) = H(p(x))\}$.
In section 2 we make a short overview of the properties of τ and g_t, needed further on and proved in PG'13. Then in section 3 we obtain 12 different p-types of limit laws. In section 4 we discuss the results.

2. Preliminaries

In this section we cite without proof some of the main results of PG'13 needed in section 3 below.

A. Properties of $\tau(x)$:
We introduce two important subsets of the interval $S = (l_H, r_H)$, namely $\mathcal{D} = \{x : \tau(x) \neq 0,\ \tau(x) \neq -\infty,\ \tau(x) \neq +\infty\}$ and $I = \{x : \tau(x) = 0\}$, $\mathcal{D} = S \backslash I$. Functional equation (3) , $H(x) = \Phi(\tau(x))$, implies:

1. $\tau(x)$ is a non-decreasing function such that $\tau(x) = -\infty$ for $x < l_H$ and $\tau(x) = +\infty$ for $x > r_H$. For $x \in I$ by definition $\tau(x) = 0$, hence $H(x) = 1/2$. Defining the median of H by $m_H = \sup\{x : H(x) < 1/2\}$, we reach uniqueness of the median.

2. $\tau(x)$ is negative for $x \in (l_H, m_H)$ and $\tau(x) \geq 0$ for $x \in (m_H, r_H)$.

Further, as a direct consequence of functional equation (5), $\sqrt{t} \cdot \tau(x) = \tau\left(g_t(x)\right)$, we observe that for any fixed x that belongs to the interior of $\mathcal{D}$ the whole trajectory $\Gamma_x = \{g_t(x) : t \in (0, \infty)\}$ belongs to $\mathcal{D}$.

Now, it is not difficult to check the following

Proposition 1. *$\tau(x)$ is continuous and strictly increasing on the interior of $\mathcal{D}$.*

B. Properties of $g_t(x)$
Observe that functional equation (5) gives no information for the value of g_t in the boundary cases $t = 0$ and $t = \infty$. For thoroughly determining the trajectory of g_t through a point x in the interior of $\mathcal{D}$ we suppose that the following boundary conditions hold.

BC 1. $\lim_{t\to 0} g_t(x) = m_H$.
BC 2. $\lim_{t\to\infty} g_t(x) = \begin{cases} l_H, & x < m_H, \\ r_H, & x > m_H \end{cases}$

By (5), these conditions are fulfilled if τ is everywhere continuous. Hence, as a consequence of the boundary conditions, every trajectory Γ_x starts from m_H and increases to r_H if $m_H < x < r_H$, or decreases to l_H if $l_H < x < m_H$. Furthermore, limit relation (4) entails that the family $\{g_t : t \in (0, \infty)\}$ forms a continuous one-parameter group with respect to composition, with the half-group property

$$g_t \circ g_s = g_{ts} \tag{6}$$

and identity element $g_1,\ g_1(x) = x$. Then $g_t^{-1} = g_{1/t}$.

Denote by Supp$H$ the support of the df H. For the subset $I \subset S$ we state the following possibilities:

Proposition 2. *Either $I = (m_H, r_H)$ or $I = \{m_H\}$. In the first case $l_H = m_H$ and SuppH $= [m_H, r_H)$ where H is the two jumps distribution with jump high 1/2 , and in the second case SuppH $= (l_H, r_H),\ l_H < m_H < r_H$.*

P r o o f. Obviously the left endpoint of the interval I is m_H. Assume the interval is $I = (m_H, a)$, $m_H < a < r_H$. For arbitrary $x \in I$, by the functional equation (5) it follows that $\tau(g_t(x)) = 0$ for all $t > 0$. Hence $g_t(x) < a$ for all $t > 0$ and this contradicts the boundary condition BC2. Thus $I = (m_H, r_H)$.
We still have to prove that $l_H = m_H$. For arbitrary fixed $x \in I$, limit relation (2) implies
$\sqrt{n}\left[\theta - \overline{F}(G_n(x))\right] \longrightarrow 0,\ n \to \infty.$
Let us assume that there exists a unique solution x_θ of $\overline{F}(x) = \theta$. This assumption is in fact not restrictive, since we can always transform the initial model so that x_θ falls into an interval where the continuous df F is strictly increasing. Now the limit relation above can be rewritten as
$\sqrt{n}\left[\overline{F}(x_\theta) - \overline{F}(G_n(x))\right] \longrightarrow 0,\ n \to \infty,\ x \in I$
and it is equivalent to
i) $G_n(x) \longrightarrow x_\theta$ for F- almost all $x \in I$ and $n \to \infty$.
Consequently for $\varepsilon > 0$ and $n > n_0(\varepsilon)$
$x_\theta - \varepsilon < G_n(x) < x_\theta + \varepsilon.$
On the other hand, as known from the theory of central u.o.s
ii) $X_{k_n,n} \xrightarrow{a.s.} x_\theta, n \to \infty,$
implying $P(x_\theta - \varepsilon < X_{k_n,n} < x_\theta + \varepsilon) \longrightarrow 1.$
Moreover, both sequences $\{G_n\}$ and $\{X_{k_n,n}\}$ are comparable in the sense that
iii) $G_n^{-1}(X_{k_n,n}) \xrightarrow{d} Y$, where Y has df H.
The latter is equivalent to

$P(G_n^{-1}(X_{k_n,n}) < x) \longrightarrow \Phi \circ \tau(x) = \frac{1}{2}$, for $x \in I$.
On the base of i) and ii), for arbitrary $\varepsilon > 0$ one may choose $n_0 = n_0(\varepsilon)$, $\varepsilon_1 = \varepsilon_1(\varepsilon)$, $\varepsilon_2 = \varepsilon_2(\varepsilon)$, and $\varepsilon_i \to 0$ as $\varepsilon \to 0$, such that for $n > n_0$

$P\left(G_n(x - \varepsilon_1) \leq X_{k_n,n} < G_n(x + \varepsilon_2)\right) \geq \frac{1}{2}$.

Indeed, by i) we obtain $G_n^{-1}(x_\theta - \varepsilon) < x < G_n^{-1}(x_\theta + \varepsilon)$. Hence one may choose $\varepsilon_1 = \varepsilon_1(\varepsilon)$, $\varepsilon_2 = \varepsilon_2(\varepsilon)$ such that

$x - \varepsilon_1 < G_n^{-1}(x_\theta - \varepsilon) < x < G_n^{-1}(x_\theta + \varepsilon) < x + \varepsilon_2$.

Then by ii)

$P\left(G_n(x - \varepsilon_1) < X_{k_n,n} < G_n(x + \varepsilon_2)\right) \geq P(x_\theta - \varepsilon < X_{k_n,n} < x_\theta + \varepsilon) \longrightarrow 1$.

Hence for n large enough, $n > n_0(\varepsilon)$ we have

$P\left(G_n(x - \varepsilon_1) < X_{k_n,n} < G_n(x + \varepsilon_2)\right) \geq \frac{1}{2}$.

Applying iii) we conclude that

$\frac{1}{2} \leq P\left(G_n(x - \varepsilon_1) < X_{k_n,n} < G_n(x + \varepsilon_2)\right)$
$= P(G_n^{-1}(X_{k_n,n}) < x + \varepsilon_2) - P(G_n^{-1}(X_{k_n,n}) < x - \varepsilon_1)$
$\longrightarrow \frac{1}{2} - \Phi(\tau(x - \varepsilon_1))$, since $x + \varepsilon_2 \in I$.

It is clear that $\Phi(\tau(x - \varepsilon_1)) = 0$ hence $x - \varepsilon_1 < l_H$.
Consequently $m_H \leq x < l_H + \varepsilon_1$ for ε_1 arbitrary small. The last statement is a contradiction, hence $l_H = m_H$. $\square$

The first case in proposition 2 we call "singular" since the support of H consists of the two endpoints only: l_H and r_H. Further on, we consider only the non singular case $I = \{m_H\}$.

Proposition 3. *The three points l_H, m_H, r_H are the only possible fixed points of the continuous one-parameter group $\{g_t : t \in (0, \infty)\}$.*

Assume the functional equation (5) holds, $\sqrt{t} \cdot \tau(x) = \tau\left(g_t(x)\right)$. Obviously it implies:

a) $\tau(x) < 0$ for $x \in (l_H, m_H)$, thus g_t is decreasing in t. Moreover g_t is repulsive for $t \in (0, 1)$, i.e. $g_t(x) > x$, and g_t is attractive for $t > 1$, i.e. $g_t(x) < x$.

b) $\tau(x) > 0$ for $x \in (m_H, r_H)$, thus g_t is increasing in t. Moreover g_t is attractive for $t \in (0, 1)$ and g_t is repulsive for $t > 1$.

Our next aim is to solve the half - group property (6) rewritten as functional equation $g(t, g(s, x)) = g(ts, x)$.

Proposition 4. *Let $\{g_t : t \in (0, \infty)\}$ be a continuous one-parameter group in GMA. If $g_t : (l_H, r_H) \to (l_H, r_H)$ satisfies a) and b), then there exist continuous* *and strictly increasing mappings* $h : (l_H, m_H) \to (-\infty, \infty)$ *and* $l : (m_H, r_H) \to (-\infty, \infty)$ *such that for* $t > 0$*:*

$$g_t(x) = \begin{cases} h^{-1}\left(h(x) - \log t\right), & x \in (l_H, m_H) \\ l^{-1}\left(l(x) + \log t\right), & x \in (m_H, r_H) \end{cases} .$$

Next we substitute the explicit form of $g_t(x)$ in (5) and solve it with respect to $\tau(x)$. Denote $S_1 = (l_H, m_H)$ and $S_2 = (m_H, r_H)$.

Proposition 5. *Let* $\{g_t : t \in (0, \infty)\}$ *be the continuous one-parameter group from Proposition 4. Suppose* τ *satisfies* $\sqrt{t} \cdot \tau(x) = \tau(g_t(x))$ *for* $t \in (0, \infty)$ *and* $x \in (l_H, r_H)$*, given that* $\tau(x) > 0$ *on* S_2 *and* $\tau(x) < 0$ *on* S_1*. Then:*

$$\tau(x) = \begin{cases} \tau_1(x) = -c_1 e^{-h(x)/2}, \ c_1 > 0, \ \text{on } S_1 \\ \tau_2(x) = c_2 e^{l(x)/2}, \ \ c_2 > 0, \ \text{on } S_2 \end{cases} .$$

After obtaining the explicit form of $\tau(x)$ we now state the characterization theorem for the limit distribution $H = \Phi \circ \tau$.

Theorem 3. *The non-degenerate df* H *in the limit relation (1) may take one of the following four explicit forms:*

1. $$H_1(x) = \begin{cases} 0, & x < m_H = l_H \\ \frac{1}{2}, & x = m_H = l_H \\ \Phi\left(\tau_2(x)\right), & x \in (m_H, r_H) \\ 1, & x \geq r_H \end{cases}$$

2. $$H_2(x) = \begin{cases} 0, & x \leq l_H \\ \Phi\left(\tau_1(x)\right), & x \in (l_H, m_H) \\ \frac{1}{2}, & x = m_H = r_H \\ 1, & x > m_H = r_H \end{cases}$$

3. $$H_3(x) = \begin{cases} 0, & x \leq l_H \\ \Phi\left(\tau_1(x)\right), & x \in (l_H, m_H) \\ \Phi\left(\tau_2(x)\right), & x \in (m_H, r_H) \\ 1, & x \geq r_H \end{cases}$$

4. $H_4(x) = \begin{cases} 0, & x < l_H \\ \frac{1}{2}, & x \in (l_H, r_H) \\ 1, & x \geq r_H. \end{cases}$

Note, in theorem 3 above, we use the notion "form" of H having in mind "limit class" H.

3. Power normalization

In this section we suppose that there exists a sequence $G_n(x) = b_n|x|^{a_n} sign(x),\ a_n$ and b_n positive, to normalize the upper c.o.s. $X_{k_n,n}$ in theorem 1 so that

$$F_{k_n,n}(\ b_n|x|^{a_n} sign(x)\ ) \xrightarrow{w} H(x),\ n \to \infty\ . \tag{7}$$

We observe that

$$G_n^{-1}(y) = \left(\frac{|y|}{b_n}\right)^{1/a_n} sign(y)\ ,$$

thus

$$G_{[nt]}^{-1} \circ G_n(x) = \left(\frac{b_n}{b_{[nt]}}\right)^{1/a_{[nt]}} |x|^{a_n/a_{[nt]}} \cdot sign(x)\ .$$

Hence, if $\forall t > 0$

$$\frac{a_n}{a_{[nt]}} \to \alpha_t > 0, \quad (\frac{b_n}{b_{[nt]}})^{1/a_{[nt]}} \to \beta_t > 0\ ,$$

then $\{G_n\}$ is a regular sequence and

$$G_{[nt]}^{-1} \circ G_n(x) \to g_t(x) = \beta_t|x|^{\alpha_t} sign(x) \in GMA. \tag{8}$$

Proposition 6. *If $\{G_n\}$ is a regular norming sequence in $\mathcal{P}$, then the family $\{g_t : t \in (0,\infty)\}$ defined in (8) forms a continuous one-parameter group with respect to composition with identity element g_1, $g_1(x) = x$.*

P r o o f. Obviously $\alpha_1 = \beta_1 = 1$, so g_1 is the identical mapping. Then $g_t^{-1} = g_{1/t}$. We have to check the half-group property $g_t \circ g_s = g_{ts}\ \forall t, s > 0$. Indeed:

$$g_t(g_s(x)) = \beta_t\beta_s^{\alpha_t}|x|^{\alpha_s\alpha_t} sign(x)\ ,$$

$$g_{ts}(x) = \beta_{ts}|x|^{\alpha_{ts}} sign(x) \ .$$

Hence (6) is true if equivalently

(9) $$\beta_{ts} = \beta_t.\beta_s^{\alpha_t} = \beta_t^{\alpha_s}.\beta_s \quad \text{and} \quad \alpha_{ts} = \alpha_t.\alpha_s$$

hold and this is easy to be seen:

$$\alpha_t\alpha_s \leftarrow \frac{a_n}{a_{[nt]}} \frac{a_n}{a_{[ns]}} = \frac{a_n}{a_{[nts]}} \frac{a_{[nts]}}{a_{[nt]}} \frac{a_n}{a_{[ns]}} \to \alpha_{ts} \ ,$$

$$\beta_{ts} \leftarrow \left(\frac{b_n}{b_{[nts]}}\right)^{1/a_{[nts]}} = \left[\left(\frac{b_n}{b_{[ns]}}\right)^{1/a_{[ns]}}\right]^{a_{[ns]}/a_{[nts]}} \cdot \left(\frac{b_{[ns]}}{b_{[nts]}}\right)^{1/a_{[nts]}} \to \beta_s^{\alpha_t}\beta_t$$

□

Let us look at (9) as functional equations for α_t and β_t. There are two possibilities, namely

i) $\alpha_t = 1$ and $\beta_t = t^b \ , \ b \neq 0$.

or

ii) $\alpha_t = t^a \ , a \neq 0$ and $\beta_t = e^{c^*(1-t^a)} \ , \ c^*$ real.

Consequently, in case i) the mapping $g_t(x) = t^b x$ is affine without translation and in case ii) $g_t(x) = c|\frac{x}{c}|^{t^a} sign(x) \ , \ c = e^{c^*} > 0$, is a power mapping.

We consider separately both cases.

Case i). $g_t(x) = t^b x, \ b \neq 0$.

The fixed points of $\{g_t : t \in (0, \infty)\}$ are the points $-\infty, \ 0, \ +\infty$. On the other hand $l_H, \ m_H, \ r_H$ are the only possible fixed points, thus there are three possibilities for the Supp$H$:

$(-\infty = l_H, r_H = m_H = 0]$, $[0 = l_H = m_H, r_H = +\infty)$ and $(-\infty = l_H, r_H = +\infty)$.

Case i_1**)**. Let Supp H $= (-\infty = l_H, r_H = m_H = 0]$.

Here $\tau < 0$ and $g_t(x) = -t^b|x|$ is decreasing in t , hence $b > 0$. There is a continuous and strictly increasing $h : (l_H, m_H) \leftrightarrow (-\infty, +\infty)$ such that $g_t(x) = h^{-1}(h(x) - \log t)$. The solution of the functional equation $h(-t^b|x|) = h(x) - \log t, \ x < 0$ is $h(x) = -\frac{1}{b}\log|x|$ and consequently $\tau_1(x) = -c_1 e^{-h(x)/2} = -c_1|x|^{1/2b}$, c_1 some positive constant, and

$$H_1(x) = \Phi(-c_1|x|^{\alpha}), \ \alpha = \tfrac{1}{2b} > 0, \ c_1 > 0, \ x \leq 0.$$

Case i_2**)**. Now Supp H $= [l_H = m_H = 0, r_H = \infty)$.

Here $\tau_2(x) = c_2 e^{l(x)/2} > 0$, $g_t(x) = l^{-1}(l(x) + \log t) = t^b x$, $l : (m_H, r_H) \to (-\infty, +\infty)$. The functional equation $l(t^b x) = l(x) + \log(x)$ entails the solution $l(x) = \frac{1}{b} \log x$. Thus $0 < \tau_2(x) = c_2 e^{l(x)/2} = c_2 x^\alpha$, $\alpha = \frac{1}{2b}$, $c_2 > 0$, $x > 0$ and $g_t(x)$ is increasing in t, hence $b > 0$, attractive for $t \in (0, 1)$ and repulsive for $t > 1$. Consequently,

$$H_2(x) = \Phi(c_2 x^\alpha),\ \alpha > 0,\ c_2 > 0,\ x \geq 0$$

has a jump 1/2 at $l_H = m_H = 0$.

Case i_3**)**. Take Supp H $= (l_H = -\infty, r_H = +\infty)$.

In the same way as above one obtains for some positive constants c_3 and c_3^*

$$H_3(x) = \begin{cases} \Phi(-c_3|x|^\alpha), & x < 0 \\ \Phi(c_3^* x^\alpha), & x \geq 0 \end{cases}$$

i.e. H_3 is continuous and strictly increasing on $(-\infty, +\infty)$, $m_H = 0$, $H_3(0) = 1/2$.

Remark. In case i) the affine mapping $g_t(x) = t^b x =: A_t x + B_t$ has coefficients $A_t \neq 1$ and $B_t = 0$. Thus the two jumps Smirnov's distribution does not appear here as limiting df.

Case ii). $g_t(x) = c|\frac{x}{c}|^{t^a} sign(x)$, $c > 0$, $a \neq 0$.
We observe that:

1) the fixed points of the group $\{g_t(x) : t \in (0, \infty)\}$ are the 5 points: $-\infty,\ -c,\ 0,\ +c,\ +\infty$;

2) the equation chain $\tau(x) = \tau(c|\frac{x}{c}|^{t^a} sign(x)) = 0$ implies that the median $m_H \in \{-c,\ 0,\ +c\}$;

3) in view of Proposition 3 the support of the limit df H may be one of the 7 intervals $(-\infty, -c),\ (-c, 0),\ (0, c),\ (c, \infty),\ (-\infty, 0),\ (-c, c),\ (0, \infty)$;

4) in view of functional equation (5) $g_t(x)$ is repulsive for $x \in (l_H, m_H),\ t \in (0, 1)$ and also for $x \in (m_H, r_H),\ t > 1$; $g_t(x)$ is attractive for $x \in (m_H, r_H),\ t \in (0, 1)$ and also for $x \in (l_H, m_H),\ t > 1$; $g_t(x)$ is decreasing in t if $x \in (l_H, m_H)$ and increasing in t if $x \in (m_H, r_H)$.

Having in mind these properties we consider the different cases for Supp$H$ separately. Assume below $c = 1$ for simplicity, without loss of generality.

Case ii_1**).** Supp$H = (l_H = -\infty, m_H = r_H = -1]$ and $g_t(x) = -|x|^{t^a}$.
The boundary condition $m_H = \lim_{t\to 0} -|x|^{t^a} = -1$ implies $a > 0$. Recall, for $x \in (l_H, m_H)$ $g_t(x) = h^{-1}(h(x) - \log t)$, so we have to solve the functional equation $h(-|x|^{t^a}) = h(x) - \log t$. It has the solution $h(x) = -\frac{1}{a} \log\log|x|$ where

$h:(-\infty,-1)\to(-\infty,+\infty)$ is strictly increasing. Then
$\tau_4(x)=-c_4e^{-h(x)/2}=-c_4(\log|x|)^{\alpha}\ ,\ \alpha=\frac{1}{2a}>0\ ,\ c_4>0$
and the corresponding limit df has the explicit form

$$H_4(x)=\Phi(-c_4(\log|x|)^{\alpha})\ ,\ x\le -1\ ,\ H_4(-1)=\Phi(0)=1/2.$$

Case ii_2**).** Supp$H=(-1,0)$ and $g_t(x)=-|x|^{t^a}$.
For the median $m_H=\lim_{t\to 0}-|x|^{t^a}$ we obtain $m_H=-1$ for $a>0$ and $m_H=0$ for $a<0$. Thus this case gives rise to two different limit df's H_5 and H_6:

a) Supp$H_5=[l_H=m_H=-1,\ r_H=0)$ and $g_t(x)=-|x|^{t^a},\ a>0$.
Recall, that $\tau(x)>0$ in the interval (m_H,r_H) and $g_t(x)=l^{-1}(l(x)+\log t)$, $l:(-1,0)\to(-\infty,+\infty)$. Now the solution of the functional equation $l(-|x|^{t^a})=l(x)+\log t\ ,\ a>0$ is $l(x)=\frac{1}{a}\log|\log|x||$. Hence
$\tau_5(x)=c_5e^{l(x)/2}=c_5|\log|x||^{\alpha}\ ,\ \alpha=\frac{1}{2a}>0\ ,\ c_5>0$
and the corresponding limit df has the explicit form

$$H_5(x)=\Phi(c_5|\log|x||^{\alpha})\ ,\ x\in(-1,0)\ ,\ H_5(-1)=\Phi(0)=1/2.$$

b) Supp$H_6=(l_H=-1,\ m_H=r_H=0]$ and $g_t(x)=-|x|^{t^a},\ a<0$.
In the interval $(l_H=-1,\ m_H=0)$ $\tau<0$ and $g_t(x)=h^{-1}(h(x)-\log t)$. We have to solve the functional equation $h(-|x|^{t^a})=h(x)-\log t$ where $a<0$ and $x\in(-1,0)$. Its solution is $h(x)=\frac{1}{|a|}\log|\log|x||$. We observe that indeed $h:(-1,0)\to(-\infty,+\infty)$ is strictly increasing. Then
$\tau_6(x)=-c_6e^{-h(x)/2}=-c_6|\log|x||^{-\alpha}\ ,\ \alpha=\frac{1}{2|a|}>0\ ,\ c_6>0$
and correspondingly

$$H_6(x)=\Phi(-c_6|\log|x||^{-\alpha})\ ,\ x\in(-1,0)\ ,\ H_6(0)=\Phi(0)=1/2.$$

Case ii_3**).** Supp$H=(0,1)$ and $g_t(x)=x^{t^a}$.
For the median $m_H=\lim_{t\to 0}x^{t^a}$ we obtain $m_H=1$ for $a>0$ and $m_H=0$ for $a<0$. Thus also this case gives rise to another two different limit df's H_7 and H_8:

a) Supp$H_7=[l_H=m_H=0,\ r_H=1)$ and $g_t(x)=x^{t^a},\ a<0$.
In this interval $\tau(x)>0$ and $g_t(x)=l^{-1}(l(x)+\log t)$ is increasing in t. As a solution of the functional equation $l(x^{t^a})=l(x)+\log t$ we obtain $l(x)=-\frac{1}{|a|}\log|\log x|$. Then

$\tau_7(x) = c_7 e^{l(x)/2} = c_7 |\log x|^{-\alpha} > 0\ ,\ \alpha = \frac{1}{2|a|} > 0\ ,\ c_7 > 0$
and

$$H_7(x) = \Phi(c_7 |\log x|^{-\alpha})\ ,\ x \in (0,1)\ ,\ H_7(0) = \Phi(0) = 1/2\ ,\ H_7(1) = \Phi(\infty) = 1.$$

b) Supp$H_8 = (l_H = 0,\ m_H = r_H = 1]$ and $g_t(x) = x^{t^a},\ a > 0$.
Here $\tau < 0$ and $g_t(x) = h^{-1}(h(x) - \log t)$. The solution of the functional equation $h(x^{t^a}) = h(x) - \log t$ is now $h(x) = -\frac{1}{a}\log|\log x|$. Then
$\tau_8(x) = -c_8 e^{-h(x)/2} = -c_8 |\log x|^{\alpha}\ ,\ \alpha = \frac{1}{2a} > 0\ ,\ c_8 > 0$
and

$$H_8(x) = \Phi(-c_8 |\log x|^{\alpha}),\ x \in (0,1),\ H_8(0) = \Phi(-\infty) = 0,\ H_8(1) = \Phi(0) = 1/2.$$

Case ii_4**).** Supp$H = [1, \infty)$ and $g_t(x) = x^{t^a}$.
The median $m_H = \lim_{t\to 0} x^{t^a}$ in this case is $m_H = 1$, hence $a > 0$. In the interval $(l_H = m_H = 1,\ \infty)$ $\tau(x)$ is positive, hence $g_t(x) = l^{-1}(l(x) + \log t)$ and we have to solve $l(x^{t^a}) = l(x) + \log t$ for $x > 1,\ a > 0,\ t > 0$. The solution is $l(x) = \frac{1}{a}\log(\log x)$. Then
$\tau_9(x) = c_9 e^{l(x)/2} = c_9 (\log x)^{\alpha} > 0,\ \alpha = \frac{1}{2a} > 0,\ c_9 > 0,$
and

$$H_9(x) = \Phi(c_9 (\log x)^{\alpha}),\ x \in (1, \infty),\ H_9(1) = \Phi(0) = 1/2,\ H_9(\infty) = \Phi(\infty) = 1.$$

Case ii_5**).** Supp$H = (-\infty, 0)$ and $g_t(x) = -|x|^{t^a}$.
Here $m_H = \lim_{t\to 0} -|x|^{t^a} = -1$, hence $a > 0$. For $x \in (-\infty, -1)$ $\tau(x)$ is negative, thus $g_t(x) = h^{-1}(h(x) - \log t)$ where $h : (l_H, m_H) \to (-\infty, \infty)$. For $x \in (-1, 0)$ $\tau(x)$ is positive and so $g_t(x) = l^{-1}(l(x) + \log t)$ where $l : (m_H, r_H) \to (-\infty, \infty)$.
The solutions of the functional equations $h(-|x|^{t^a}) = h(x) - \log t,\quad x \in (-\infty, -1)$ and $l(-|x|^{t^a}) = l(x) + \log t,\quad x \in (-1, 0)$ are: $h(x) = -\frac{1}{a}\log\log|x|$, resp. $l(x) = \frac{1}{a}\log|\log|x||$. Then
$\tau_{10}(x) = -c_{10} e^{-h(x)/2} = -c_{10} (\log|x|)^{\alpha}$ for $x \le -1,\ c_{10} > 0,\ \alpha = \frac{1}{2a} > 0$
and
$\tau_{10}(x) = c_{10}^* e^{l(x)/2} = c_{10}^* |\log|x||^{\alpha}$ for $x \in (-1, 0),\ c_{10}^* > 0$.
Consequently

$$H_{10}(x) = \begin{cases} \Phi(-c_{10}(\log|x|)^{\alpha}), & x \in (-\infty, -1) \\ \Phi(c_{10}^{*}|\log|x||^{\alpha}), & x \in (-1, 0) \ . \end{cases}$$

Case ii_6**).** Supp$H = (-1, 1)$ and $g_t(x) = |x|^{t^a} sign(x)$.
In this case $l_H = -1 < m_H = 0 < r_H = 1$ and $g_t(x)$ has to be decreasing in t for $x \in (l_H, m_H)$ and increasing in t for $x \in (m_H, r_H)$. Hence $a < 0$. For $x \in (-1, 0)$ $\tau(x) < 0$ and $g_t(x) = h^{-1}(h(x) - \log t)$. For $x \in (0, 1)$ $\tau(x) > 0$ and $g_t(x) = l^{-1}(l(x) + \log t)$. We have to solve the functional equations

$h(-|x|^{t^a}) = h(x) - \log t, \quad x \in (-1, 0)$

and

$l(x^{t^a}) = l(x) + \log t, \quad x \in (0, 1).$

The solutions are respectively $h(x) = \frac{1}{|a|}\log|\log|x||$ and $l(x) = -\frac{1}{|a|}\log|\log x|$. Denote $\alpha = \frac{1}{2|a|} > 0$. Consequently:

$\tau_{11}(x) = -c_{11}e^{-h(x)/2} = -c_{11}|\log|x||^{-\alpha}$ for $x \in (-1, 0)$ $c_{11} > 0$,

and

$\tau_{11}(x) = c_{11}^{*}e^{l(x)/2} = c_{11}^{*}|\log x|^{-\alpha}$ for $x \in (0, 1),\ c_{11}^{*} > 0$.

Finally we obtain the explicit form of the limit distribution H_{11}, namely

$$H_{11}(x) = \begin{cases} \Phi(-c_{11}|\log|x||^{-\alpha}), & x \in (-1, 0) \\ \Phi(c_{11}^{*}|\log x|^{-\alpha}), & x \in (0, 1). \end{cases}$$

Case ii_7**).** Supp$H = (0, \infty)$ and $g_t(x) = x^{t^a}$.
Again, from the monotonicity properties of $g_t(x)$ we conclude that $a > 0$ and $m_H = 1$. As solutions of the corresponding functional equations

$h(x^{t^a}) = h(x) - \log t, \quad x \in (0, 1)$

and

$l(x^{t^a}) = l(x) + \log t, \quad x \in (1, \infty)$

we obtain $h(x) = -\frac{1}{a}\log|\log x|$ and $l(x) = \frac{1}{a}\log(\log x)$ respectively. Put now $\alpha = \frac{1}{2a} > 0$ and observe that

$\tau_{12}(x) = -c_{12}e^{-h(x)/2} = -c_{12}|\log x|^{\alpha}$ for $x \in (0, 1),\ c_{12} > 0$,

and

$\tau_{12}(x) = c_{12}^{*}e^{l(x)/2} = c_{12}^{*}(\log x)^{\alpha}$ for $x \geq 1,\ c_{12}^{*} > 0$.

Finally we obtain the explicit form of the last limit distribution H_{12}, namely

$$H_{12}(x) = \begin{cases} \Phi(-c_{12}|\log x|^{\alpha}), & x \in (0, 1) \\ \Phi(c_{12}^{*}(\log x)^{\alpha}), & x \geq 1 \ . \end{cases}$$

At the end of this section let us gather all results concerning the model described above in the following

Theorem 4. *Let F be a continuous df. Assume that the limit relation*

$$F_{k_n,n}\left(G_n(x)\right) = P\left({G_n}^{-1}(X_{k_n,n}) < x\right) \xrightarrow[n]{w} H(x)$$

holds for a non-degenerate df H, where
i) $\{G_n\}$ is a regular norming sequence which corresponding function g_t in (4) satisfies the boundary conditions BC1 and BC2;
ii) $\{k_n\}$ is a sequence of integers satisfying the condition $(\frac{k_n}{n} - \theta)\sqrt{n} \to 0$ for a $\theta \in (0,1)$.
Then the limit df H belongs to one of the following 12 limit classes:

$$H_1(x) = \begin{cases} \Phi(-c_1|x|^\alpha), & x < 0 \\ 1, & x \geq 0 \end{cases} ;\ \alpha > 0, c_1 > 0.$$

$$H_2(x) = \begin{cases} 0, & x < 0 \\ \Phi(c_2 x^\alpha), & x \geq 0 \end{cases} ;\ \alpha > 0, c_2 > 0.$$

$$H_3(x) = \begin{cases} \Phi(-c_3|x|^\alpha), & x < 0 \\ \Phi(c_3^* x^\alpha), & x \geq 0 \end{cases} ;\ \alpha, c_3, c_3^* > 0.$$

$$H_4(x) = \begin{cases} \Phi(-c_4(\log|x|)^\alpha), & x < -1 \\ 1, & x \geq -1 \end{cases} ;\ \alpha > 0, c_4 > 0.$$

$$H_5(x) = \begin{cases} 0, & x < -1 \\ \Phi(c_5|\log|x||^\alpha), & -1 \leq x < 0 \\ 1, & x \geq 0 \end{cases} ;\ \alpha > 0, c_5 > 0.$$

$$H_6(x) = \begin{cases} 0, & x < -1 \\ \Phi(-c_6|\log|x||^{-\alpha}), & -1 \leq x < 0 \\ 1, & x \geq 0 \end{cases} ;\ \alpha > 0, c_6 > 0.$$

$$H_7(x) = \begin{cases} 0, & x < 0 \\ \Phi(c_7|\log x|^{-\alpha}), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} ;\ \alpha > 0, c_7 > 0.$$

$$H_8(x) = \begin{cases} 0, & x < 0 \\ \Phi(-c_8|\log x|^\alpha), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} ;\ \alpha > 0, c_8 > 0.$$

$$H_9(x) = \begin{cases} 0, & x < 1 \\ \Phi(c_9(\log x)^\alpha), & x \geq 1 \end{cases} \; ; \; \alpha > 0, c_9 > 0.$$

$$H_{10}(x) = \begin{cases} \Phi(-c_{10}(\log|x|)^\alpha), & x < -1 \\ \Phi(c^*_{10}|\log|x||^\alpha), & -1 \leq x < 0 \\ 1, & x \geq 0 \end{cases} \; ; \alpha, c_{10}, c^*_{10} > 0.$$

$$H_{11} = \begin{cases} 0, & x < -1 \\ \Phi(-c_{11}|\log|x||^{-\alpha}), & -1 \leq x < 0 \\ \Phi(c^*_{11}|\log x|^{-\alpha}), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases} \; ; \; \alpha, c_{11}, c^*_{11} > 0.$$

$$H_{12} = \begin{cases} 0, & x < 0 \\ \Phi(-c_{12}|\log x|^\alpha), & 0 \leq x < 1 \\ \Phi(c^*_{12}(\log x)^\alpha), & x \geq 1 \end{cases} \; ; \; \alpha, c_{12}, c^*_{12} > 0.$$

In their paper (2011) Barakat and Omar wrote: "... power normalization and linear normalization of central order statistics are leading to the same families of limit df's". The theorem above contains limit dfs for power normalization such as the dfs H_{10}, H_{11} or H_{12} which do not belong to the family of limit dfs for linear normalization. We leave to the reader to check that the limit dfs in Theorem 4 have non-empty domain of attraction.

4. Conclusions

We have seen that the class of a limit law depends on the solution of functional equation (5). Let us consider the possibilities for $\tau(x) = \Phi^{-1} \circ H(x)$:
for $x \in (l_H, m_H)$ $\quad -\infty < \tau(x) < 0$ or $\tau(x) \equiv -\infty$;
for $x \in (m_H, r_H)$ $\quad 0 < \tau(x) < \infty$ or $\tau(x) \equiv 0$ or $\tau(x) \equiv \infty$.

Formally, these possibilities result in 6 different combinations, namely
1. $-\infty < \tau(x) < 0$ for $x \in (l_H, m_H)$ and $0 < \tau(x) < \infty$ for $x \in (m_H, r_H)$,
2. $-\infty < \tau(x) < 0$ for $x \in (l_H, m_H)$ and $\tau(x) \equiv \infty$ for $x \in (m_H, r_H)$,
3. $\tau(x) \equiv -\infty$ for $x \in (l_H, m_H)$ and $0 < \tau(x) < \infty$ for $x \in (m_H, r_H)$,
4. $-\infty < \tau(x) < 0$ for $x \in (l_H, m_H)$ and $\tau(x) \equiv 0$ for $x \in (m_H, r_H)$,
5. $\tau(x) \equiv -\infty$ for $x \in (l_H, m_H)$ and $\tau(x) \equiv 0$ for $x \in (m_H, r_H)$,
6. $\tau(x) \equiv -\infty$ for $x \in (l_H, m_H)$ and $\tau(x) \equiv \infty$ for $x \in (m_H, r_H)$.

Let us go bottom-up. The last case 6 is of no interest for us since it corresponds to a degenerate df H. Case 5 can not appear if using power normalization since $\{g_t\}$ can not act as a translation group. In case 4 Supp$H$ is a disconnected set, namely Supp$H = (l_H, m_H) + \{r_H\}$ which case is impossible if using regular norming sequences. Case 3 results in a df H with jump 1/2 at $l_H = m_H$, continuous and strictly increasing on (l_H, r_H). In case 2 the limit df H has a jump 1/2 at $r_H = m_H$ being continuous and strictly increasing on (l_H, r_H). Case 1 leads to an everywhere continuous df H.

Note, with a power transformation $p(x) = b|x|^a sign(x)$, a and b positive, one can not transform e.g. Supp$H_5 = (-1, 0)$ to Supp$H_7 = (0, 1)$ although H_5 and H_7 both belong to case 3. On the other hand, there is always a mapping $g \in GMA$ which transforms into each other two distributions of the same form (case).

Consequently, the limit df's of Theorem 4 give rise to 12 different limit classes. They all belong to 3 different g-types described in cases 1, 2 and 3 above.

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E. I. Pancheva
Institute of Mathematics and Informatics, BAS,
Acad. G. Bonchev Str., Bl. 8, 113 Sofia, Bulgaria
pancheva@math.bas.bg

A. Gacovska-Barandovska
Institute of Mathematics,
Faculty of Natural Sciences and Mathematics,
Ss. Cyril and Methodius University,
Gazi Baba, bb. 1000 Skopje, Macedonia
aneta@pmf.ukim.mk