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Extremal Processes with One Jump

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Abstract. Convergence of a sequence of deterministic functions in the Skorohod topology $\mathscr{D}([0,\infty))$ implies convergence of the jumps. For processes with independent additive increments the fixed discontinuities converge. In this paper it will be shown that this is not true for processes with independent max-increments. The limit in $\mathscr{D}([0,\infty))$ of a sequence of stochastically continuous extremal processes may have fixed discontinuities. Our construction makes use of stochastically continuous extremal processes whose sample functions have only one jump.

Key words. extremal process, indecomposable, jump, Skorohod topology, total dependence

1. Introduction

The similarity between the theory of maxima and that of sums is considerable. Concepts like stability, decomposability, and infinite divisibility have been successfully transferred to the theory of maxima. Any max-id random variable (or vector) X with cdf F can be embedded as $Y(1)$ in a unique process Y with stationary independent max-increments. Simplifications occur in the theory of maxima: The cdf of $Y(t)$ for $t > 0$ is simply F^t; in the univariate case every rv is max-id. In the theory of maxima the cdf takes over the role of the characteristic function. As in the case of positive sums the max-id process is generated by a Poisson point process on the half plane $t > 0$ whose mean measure is invariant for translations along the time axis. This paper is concerned with extremal processes corresponding to processes with independent increments for sums. No stationarity of the increments is assumed. We restrict ourselves to non-negative variables (without loss of generality).

The apparent simplicity of extremal processes hides a number of deeper lying difficulties. For multivariate extremal processes the distribution of the max-increments need not be unique even if the process is stochastically continuous. This phenomenon of blotting has been treated in Balkema and Pancheva (1996). In the present paper we shall discuss a problem related to convergence. Continuous time processes may occur as limits for triangular arrays of increments X_{nk}. One assumes that the increments in each row are independent, and that the increments are asymptotically uniformly negligible. With the nth row one associates the random step function on $[0,1]$ with jumps X_{nk} in the time points k/n for $k = 1, \ldots, n$. Assume convergence of these random step functions in the space $\mathscr{D}([0,1])$ with respect to the Skorohod topology. Convergence in this topology implies that the jump in a fixed discontinuity can not arise as the limit of a superposition of smaller jumps. In the theory of processes with additive increments it is well known that the sample functions of the limit process are continuous with probability one if the size of the increments tends to zero uniformly. If one only assumes that the increment X_{nk} tends to zero in probability for $n + k \to \infty$ then the sample functions of the limit process may have jumps but the process is stochastically continuous.

In the case of max-increments this is no longer true. A sequence of stochastically continuous extremal processes may converge in $\mathscr{D}([0,\infty))$ to an extremal process with fixed discontinuities. In the univariate case the stochastically continuous extremal processes are dense in the space of all extremal processes. In the multivariate case we shall formulate conditions on the distribution of the max-increments which ensure convergence of the fixed jumps. Let Y_n be extremal processes for $n \geq 0$ and assume that $Y_n \Rightarrow Y_0$ in $\mathscr{D}([0,\infty))$. Convergence of the fixed jumps means: For each fixed discontinuity $t_0 > 0$ of the limit process Y_0 there exist time points $t_n \to t_0$ so that

$$(Y_n(t_n - 0), Y_n(t_n)) \Rightarrow (Y_0(t_0 - 0), Y_0(t_0)) \quad n \to \infty. \tag{1.1}$$

The construction of a sequence of stochastically continuous extremal processes converging to a given discontinuous limit process is based on a simple and unexpected observation: There exist stochastically continuous extremal processes with the property that each sample function of the process either is constant on $[0,1]$ or has exactly one jump in this time interval and is constant on the intervals on either side of the jump. Such a process will be called a one-jump process. In order to ensure stochastic continuity different sample functions have their jumps at different times. On the other hand one may not violate the condition of independence of the max-increments: The occurrence of a jump in an interval $[t_1, t_2)$ should not involve the occurrence or not of a jump in the intervals $[0, t_1)$ and $[t_2, 1]$. The condition of having only one jump thus is quite severe. Additive processes which satisfy these conditions are pathwise continuous.

For univariate extremal processes we have the following results:

Theorem 3.1: *For any random variable $X \geq 0$ there exist a one-jump process Y so that $Y(0) = 0$ and $Y(1)$ is distributed like X.*

Theorem 3.4: *Any univariate extremal process is the limit in $\mathscr{D}([0,\infty))$ of a sequence of stochastically continuous extremal processes.*

The multivariate situation is different. Complications in the multivariate case arise when the initial distribution vanishes on a neighborhood of the origin. This may happen even if the univariate marginals are strictly positive on $(0, \infty)$. If the initial vector of an extremal process has a distribution F which is not strictly positive on $(0, \infty)^d$ then $FG = FH$ does not imply $G = H$. The distribution of the max-increment is blotted out on the set where F vanishes. This phenomenon of blotting is peculiar to multivariate extreme value theory. It is not well understood.

One-jump processes exist in the multivariate case but only if the initial and final distributions are related: $F_1 = F_0\tilde{H}$, where the distribution $\tilde{H}$ is totally dependent. The concept of total dependence was introduced by Fréchet (1951) in his investigation of multivariate distributions with given marginals. If for each discontinuity point of an extremal process the distributions on either side of the discontinuity are related by the equation above then this process is the limit in $\mathscr{D}([0, \infty))$ of a sequence of stochastically continuous extremal processes.

These results raise the issue of formulating conditions which ensure that fixed discontinuities converge in the sense of (1.1). This question is studied in Sections 6 and 7. Relation (1.1) breaks down if the fixed discontinuity in the limit process corresponds to two fixed discontinuities in the approximating process which occur in quick succession and which between them only yield one jump in the limit. In order to study such behavior we introduce the discrete counterpart of the one-jump process. This is the one-jump triple:

$$(Z_0, Z_1, Z_2) = (Z_0, Z_0 \vee W_1, Z_1 \vee W_2) \tag{1.2}$$

where Z_0, W_1, W_2 are independent vectors with the property that the events $\{Z_1 \neq Z_0\}$ and $\{Z_2 \neq Z_1\}$ are exclusive and have positive probability. So there a.s. is only one jump. For any ω either $Z_1(\omega) = Z_0(\omega)$ or $Z_2(\omega) = Z_1(\omega)$. In addition for technical reasons it is assumed that Z_0 has lower endpoint in the origin. It is natural to ask what max-increments can not be broken down by a one-jump triple. A distribution H is called one-jump prime if there does not exist a one-jump triple (Z_0, Z_1, Z_2) with cdfs F_0, F_1, F_2 so that $F_2 = F_0H$. We shall characterize these distributions.

There is a different approach to the issue of convergence of fixed discontinuities. To explain that we need the concept of weight. The *weight* of an extremal process Y is a decreasing function y, which is defined as

$$y(t) = Ee^{-Y(t)} \quad t \geq 0. \tag{1.3}$$

The weight keeps track of the fixed discontinuities of the process. Thus (1.1) implies that the weights satisfy: $y_n(t_n - 0) \to y_0(t_0 - 0)$ and $y_n(t_n) \to y_0(t_0)$. For a multivariate process the weight may be taken as the sum of the weights of the component processes. Convergence of the weights then ensures convergence of the jumps.

Theorem 8.6: *Let $Y_n : [0, \infty) \to [0, \infty)^d$ be extremal processes for $n \geq 0$. Suppose $Y_n \Rightarrow Y_0$ in $\mathscr{D}([0, \infty))$. Then the fixed discontinuities converge in the sense of (1.1) if and only if the weights converge in $\mathscr{D}([0, \infty))$.*

The occurrence of fixed discontinuities in the limit process is a sign of degeneracy. For multivariate processes such discontinuities are irreparable; for univariate extremal processes it is possible to obtain a stochastically continuous limit process by a judicious choice of time changes. This will be shown in a forthcoming paper.

We now give a brief overview of the contents. Section 2 contains the necessary background on univariate extremal processes. Section 3 introduces one-jump extremal processes and shows how they can be used to approximate extremal processes with fixed discontinuities by stochastically continuous extremal processes. Sections 4 to 7 investigate the multivariate situation. They may be skipped by readers who are only interested in the univariate theory.

In the multivariate situation the concept of a one-jump triple plays the central role. For one-jump processes and one-jump triples starting in the origin the theory is simple. This theory is presented in Section 4 and applied to the convergence of fixed discontinuities in Section 6. Section 7 treats one-jump prime distributions. Section 5 describes the distribution of a multivariate one-jump process X. We prove:

Theorem 5.6: *The distribution of a multivariate one-jump process is determined by the distribution of the initial vector and the distributions of the univariate marginal processes.*

Section 8 treats the problem of break down of stochastic continuity in the more general framework of increasing processes and weak convergence. It is shown how convergence of the weight functions controls convergence of the processes: Stochastic continuity is preserved in the limit if one imposes the extra condition that the weight functions converge in $\mathscr{D}([0,\infty))$.

2. Extremal processes

An *extremal process* $Y:[0,\infty)\to[0,\infty)$ is a stochastic process with right-continuous increasing sample functions and with independent max-increments in the following sense: For each finite set of time points $0=t_0<t_1<\cdots<t_m$ there exist independent non-negative rv's $U_0,\ldots,U_m$ so that the following equality in distribution holds:

$$(Y(t_0),\ldots,Y(t_m))\stackrel{\mathrm{d}}{=}(V_0,\ldots,V_m)\quad V_i=U_0\vee\ldots\vee U_i,\quad i=0,\ldots,m.$$

The multivariate distributions of an extremal process are completely determined by the cdf's F_t of the rv's $Y(t)$ since these determine the cdf's of the max-increments U_i above: If $Y(t+s)=Y(t)\vee U$ with U and $Y(t)$ independent then the cdf H of U is just the quotient $H=F_{t+s}/F_t$. One has to be a little careful if $F_t(x)$ vanishes for certain $x>0$. Hence define $C(t)=\inf\{x\geq 0|F_t(x)>0\}$. This is the lower endpoint of the cdf F_t, and the curve $C:[0,\infty)\to[0,\infty)$ is the *lower curve* of the process Y. It is not difficult to see that C is increasing and right-continuous. The df H of the max-increment U above is unique if we impose the condition $U\geq C(t)$ a.s.

Since all rv's are max-id the extremal process Y can be described in terms of its lower curve and a Poisson point process N on the set above the lower curve:

$$Y(t) = C(t) \vee \sup\{X_k | T_k \leq t\}$$

where (T_k, X_k) are the points of N. The mean measure μ of the point process N is determined by the cdf's F_t: For $x > C(t)$

$$F_t(x - 0) = P\{Y(t) < x\} = P\{N([0, t] \times [x, \infty)) = 0\} = e^{-\mu([0,t] \times [x,\infty))}.$$

It is convenient to introduce the set $S \subset [0, \infty) \times [0, \infty]$ which consists of the points (t, x) above the lower curve, $x > C(t)$, including $x = \infty$. Then μ is a Radon measure on the locally compact space S and the measure μ lives on the set of points whose coordinates are finite. Any increasing right-continuous function $C : [0, \infty) \to [0, \infty)$ and any such Radon measure μ on S determine an extremal process and vice versa.

The *weight* of the extremal process Y is the function $t \mapsto y(t) = Ee^{-Y(t)}$. It plays the role which the variance plays for additive processes. It is monotone (decreasing) and shows up the fixed discontinuities of the process Y. Thus $t > 0$ is a fixed discontinuity of the extremal process Y if and only if it is a discontinuity of the weight function y. The weight is constant on an interval $[t_1, t_2)$ if and only if the extremal process is motionless on this interval.

For more details on extremal processes see Balkema and Pancheva (1996). That paper will from now on be referred to as BP (1996). The corresponding multivariate definitions may also be found there.

Definition 2.1: *A* one-jump process *is an extremal process* $X : [0, 1] : \to [0, \infty)$ *which is stochastically continuous, and whose sample functions have at most one jump and are constant on either side of the jump. (The process* X *may be extended to time points* $t > 1$ *by setting* $X(t) = X(t \wedge 1)$.)

For increasing functions convergence in $\mathscr{D}([0, \infty))$ may be formulated simply in terms of sequences: Let $\varphi_n : (0, \infty) \to [0, \infty)$ be increasing right continuous functions for $n \geq 0$. Then $\varphi_n \to \varphi_0$ in $\mathscr{D}([0, \infty))$ if $\varphi_n \to \varphi_0$ weakly, if $\varphi_n(0) \to \varphi_0(0)$, where $\varphi_n(0) = \lim_{t \downarrow 0} \varphi_n(t)$ for $n \geq 0$ by definition, and if for each discontinuity point $t_0 > 0$ of φ_0 there is a sequence $t_n \to t_0$ so that $\varphi_n(t_n) \to \varphi_0(t_0)$ and $\varphi_n(t_n - 0) \to \varphi_0(t_0 - 0)$.

3. Constructions

We start off with a very simple example of a one-jump process.

Example 3.1: Let N be the Poisson point process on $[0, \infty)$ with intensity 1. The corresponding additive process is the standard integer valued Poisson counting process with exponential holding times between jumps to the next integer. The corresponding extremal process is a process with one jump. Since the jump occurs at the first point of N, and this time point has an exponential distribution, the extremal process is stochastically continuous. It has stationary max-increments.

Theorem 3.2 (Existence): *For any rv $X \geq 0$ there exists a one-jump process Y on $[0, 1]$ so that $Y(0) = 0$ and $Y(1)$ is distributed like X.*

Proof: There exists a decreasing function $\psi : (0, 1) \to [0, \infty)$ so that X is distributed like $\psi(U)$ with U uniform on (0, 1). The function ψ is the generalised inverse function of the decreasing points. See Resnick (1987, Section 0.2) for details.

Let μ be the image of the measure $dt/(1-t)$ on the interval $(0, 1)$ under the map $t \mapsto (t; \psi(t))$ from $(0, 1)$ to $(0, 1) \times [0, \infty)$. Let N be the Poisson point process on the graph of ψ with mean measure μ and points (T_k, X_k). The points T_k form a Poisson point process on $[0, 1)$ with mean measure $dt/(1-t)$ and hence they can be numbered so that $T_1 < T_2 < \cdots$ Then

$$P\{T_1 > u\} = P\{N(\{0 \leq t \leq u\}) = 0\} = \exp - \int_0^u \frac{dt}{1-t} = 1 - u.$$

This means that the first point $X_1 = \psi(T_1)$ is distributed like X (by construction of ψ). Since the function ψ is decreasing and the sequence (T_n) increasing the points (T_k, X_k) with $k > 1$ have no influence on the value of the extremal process Y generated by N. Also $Y(0) = 0$ a.s.

For each ω the sample path $Y(\omega)$ has a jump $X_1(\omega) \geq 0$ in $Y(1) = Y(1-0)$. Then $Y(1) = Y(T_1) = X_1$ is distributed like X. □

Proposition 3.3 (Uniqueness): *Set $q = Ee^{-X}$. The process Y in Theorem 3.1 can be chosen so that $Ee^{-Y(t)} = q^t$ for $t \in [0, 1]$. This normalization makes the process Y unique.*

Proof: Assume that $p = P\{X > 0\} = 1 - H(0)$ is positive. Else $Y \equiv 0$ and the result is trivial. The weight $y : [0, 1] \to [0, 1]$ of the extremal process Y constructed in Proposition 3.1 is continuous and strictly decreasing on $[0, p]$ since Y is nowhere motionless. Hence there is a unique time change $\tau : [0, p] \to [0, 1]$ so that the extremal process $V = Y \circ \tau$ has weight $v(t) = y(\tau(t)) = q^t$ for $0 \leq t \leq 1$.

In the construction above the lower curve $C(t)$ of V vanishes on $[0, 1)$ since

$$P\{V(t) = 0\} = P\{T_1 > \tau^{-1}(t)\} > 0 \qquad t < 1.$$

Now let Y be a one-jump process which satisfies the conditions of Theorem 1 with lower curve $C \equiv 0$ on $[0, 1)$. Stochastic continuity implies that the extremal process Y is generated by a Poisson point process N whose mean measure μ does not charge vertical lines. The condition of not more than one jump implies that the points of N lie on a decreasing curve

$y = \varphi(t)$, and X is distributed like $\varphi(T_1)$. If we choose the time scale so that T_1 is uniformly distributed on $(0, 1)$ then $\varphi = \psi$ in all continuity points of ψ on $(0, 1)$. □

A one-jump process $Y : [0, 1] \to [0, \infty)$ may be decomposed as

$$Y(t) = Y(0) \vee U(t) \qquad t \in [0, 1]$$

where U is a one-jump process starting in the origin. We assume here and henceforth that the underlying probability space is sufficiently rich, see BP (1996, Section 7).

So the distribution of Y may be described by the boundary distributions $F_0 = F$ and $F_1 = FH$ of $Y(0)$ and $Y(1)$ respectively and the *rate function*

$$r(t) = P\{U(t) \neq 0\} \qquad t \in [0, 1] \tag{3.1}$$

which describes the rate at which the jump occurs. Then $Y(t)$ has cdf $F_t = FH_t$ where H_t is the cdf of $U(t)$, which has the form

$$H_t(u) = H(u) \vee (1 - r(t)) \qquad u \geq 0, t \in [0, 1], \tag{3.2}$$

since $\{U(t) \leq u\} = \{T_1 \geq t\} \cup \{U(1) \leq u\}$ and $U(1) = \varphi(T_1)$. A univariate one-jump process is completely characterized by the triple (F, H, r). Conversely such a triple determines a one-jump process. A one-jump process may be described in terms of two independent variables.

Theorem 3.4: *Let F and H be cdfs on $[0, \infty)$, and let $r : [0, 1] \to [0, 1]$ be continuous and increasing. Suppose $r(0) = 0, r(1) + H(0) = 1$ and $F(x) > 0$ for all $x > 0$. Let $\psi : (0, 1) \to [0, \infty)$ be the right-continuous generalized inverse function of $1 - H$, and let X, with cdf H, and T, uniformly distributed on $(0, 1)$, be independent. Then*

$$Y(t) = \begin{cases} Y(0) & \text{if } 0 \leq r(t) < T \\ Y(0) \vee \psi(T) & \text{if } T \leq r(t) \end{cases}$$

is a one-jump process with rate function r.

Proof: If $r(t) = t \wedge (1 - H(0))$ then Y is the process constructed in Theorem 3.2. This process is unique up to time scale. The rate function r determines the time scale. □

Any extremal process $Y_0 : [0, \infty) \to [0, \infty)$ has a decomposition $Y_0 = W_0 \vee Z_0$ where W_0 is stochastically continuous and Z_0 has only fixed discontinuities:

$$Z_0(t) = C(t) \vee \sup\{U_r | r \in R, r \leq t\}.$$

Here R is the set of fixed discontinuities of Y_0 and U_r is the max-increment in r. The variables U_r are independent and the family (U_r) is independent of Y_0. See BP (1996, Section 7). By expanding the max-increment in each fixed discontinuity point into a one-jump process we obtain an approximation to Y_0 which is stochastically continuous.

Theorem 3.5 (Convergence): *Any univariate extremal process is the limit in* $\mathscr{D}([0,\infty))$ *of a sequence of stochastically continuous extremal processes.*

Proof: Let $Y_0 : [0,\infty) \to [0,\infty)$ be an extremal process with lower curve C_0. Let R be the set of fixed discontinuities of the process Y_0. With each $r \in R$ associate a closed interval J_r of length $\delta_r > 0$ with $\sum \delta_r < \infty$. For $t > 0$ set

$$\sigma(t) = t + \sum \{\delta_r | t_r \le t\}.$$

Take $J_r = [a_r, b_r] = [\sigma(r-0), \sigma(r)]$ and let U_r be the max-increment of Y_0 in the point r, i.e. $Y_0(r) = Y_0(r-0) \vee U_r$. Using Theorem 3.1 we construct for each $r \in R$ a stochastically continuous one-jump extremal process $X^r : J_r \to [0,\infty)$ such that

$$X^r(a_r) = 0 \qquad X^r(b_r) = U_r.$$

The sample functions of X^r are motionless on either side of the jump. Take the processes $X^r, r \in R$, independent and independent of the stochastically continuous part of Y_0.

Define the extremal process Y by setting

$$Y(\sigma(t)) = Y_0(t) \qquad t \ge 0. \tag{3.3}$$

For $t \in J_r$ set $Y(t) = Y_0(a_r) \vee X^r(t)$. This definition agrees with (3.3) for $t = a_r$ and for $t = b_r$. The new process Y is stochastically continuous in each time point $t > 0$ since $P\{Y(t-0) = Y(t)\} = 1$ for all $t > 0$.

Now define time changes τ_n with slope $\tau_n' = 1/n$ on the intervals J_r and $\tau_n' = 1$ elsewhere on $[0, \infty)$ and let $Y_n = Y \circ \sigma_n$ where $\sigma_n = \tau_n^{-1}$. Then Y_n is a stochastically continuous extremal process since the functions τ_n are strictly increasing and continuous and $Y_n \Rightarrow Y_0$ in $\mathscr{D}([0,\infty))$ since this convergence holds for all sample functions:

Let $\varphi = Y(\omega)$ be a sample function of Y. Then $\varphi_0(t) = \varphi(\sigma(t)) = \varphi_n(\tau_n(\sigma(t)))$ is the corresponding sample function of Y_0. Convergence of $\sum \delta_r$ ensures that $\tau_n \circ \sigma \to \mathrm{id}$ uniformly on bounded intervals $[0, t]$. For $r \in R$ there is a unique time point $r' \in J_r$ so that $\varphi(r' \pm 0) = \varphi_0(r \pm 0)$. Take $r_n = \tau_n(r')$ and we find that $r_n \to r$ and $\varphi_n(r_n \pm 0) = \varphi_0(r \pm 0)$. □

In BP (1996, Theorem 6.1) it was shown that convergence $Y_n \Rightarrow Y_0$ with respect to the topology of weak convergence for increasing functions implies $Y_n(t) \Rightarrow Y_0(t)$ in a dense set of time points t, and hence implies weak convergence of the weights. The corresponding statement for convergence in $\mathscr{D}([0,\infty))$ is false:

Proposition 3.6: *There exist extremal processes which converge in* $\mathscr{D}([0,\infty))$ *but whose weights do not converge in* $\mathscr{D}([0,\infty))$.

Proof: Take a sequence of stochastically continuous processes which converge to a process with a fixed discontinuity. □

4. Total dependence

Recall that the lower endpoint of a random vector $X = (X_1, \ldots, X_d) \in \mathbf{R}^d$ is the point $q = (q_1, \ldots, q_d) \in [-\infty, \infty)^d$ where q_i is the lower endpoint of the cdf of X_i. Thus q is the largest point so that $P\{X \geq q\} = 1$. (Inequalities in $\mathbf{R}^d$ are interpreted coordinatewise, in particular the inequality $a < b$ means that strict inequality $a_i < b_i$ holds for each of the d coordinates.) Our vectors all have non-negative components. This may be achieved by a coordinatewise exponential transformation.

A multivariate extremal process $Y : [0, \infty) \to [0, \infty)^d$ then is defined as in the univariate case. The lower curve $C(t) \in [0, \infty)^d, t \geq 0$, is defined coordinatewise: $C_i(t)$ is the lower endpoint of the univariate marginal cdf $F_i(t)$ of $Y_i(t)$. The weight $y(t)$ is the sum of the weights of the d marginals. The max-increment U_t of the extremal process Y in the fixed discontinuity t need not be unique even if $C(t) \equiv 0$ since the multivariate cdf of $Y(0)$ may vanish on a neighborhood of the origin.

The concept of decomposition in the theory of sums of independent rv's is well known and has been discussed in Zolotarev (1998). Stability of decomposition is also valid for maxima provided the lower endpoint is kept constant for all components. See Pancheva (1994). In this paper we are interested in one-jump decomposability.

Definition 4.1: *A vector $U \geq 0$ is* one-jump decomposable *if there exist independent non-negative vectors U_1 and U_2 so that*

1. $U_1 \vee U_2$ *is distributed like* U;
2. *the events* $\{U_1 \neq 0\}$ *and* $\{U_1 \vee U_2 \neq U_1\}$ *are disjoint and have positive probability.*

If U is one-jump decomposable then $(0, U_1, U_1 \vee U_2)$ is a one-jump triple starting in the origin, see (1.2).

Lemma 4.2: *Let X and Y be independent vectors such that $X \leq Y$* a.s. *Then the upper endpoint of X lies below the lower endpoint of Y.*

Proof: This holds for the univariate marginals. □

Assume $U = U_1 \vee U_2$ is a one-jump decomposition. Then $U_2 \leq U_1$ on the event $E = \{U_1 \neq 0\}$. By Lemma 4.2 this implies that $U_1 \geq q$ on E where q is the upper endpoint of U_2. So there is a vector $q \geq 0$ so that

$$P(\{U \leq q\} \cup \{U \geq q\}) = 1, \qquad P\{U \leq q\} P\{U \geq q\} > 0. \tag{4.1}$$

Such a point will be called a *pivot* for U.

Now suppose U has a pivot $q \neq 0$. Choose an event E so that $\{U \leq q\} \subset E \subset \{U \geq q\}$

and $0 < PE < 1$. Some variation is possible if q is an atom. Let $U_1 = U1_E$ and let $U_2 = U$ on E^c. This determines the conditional distribution of U_2 given E^c. Now take U_2 to be independent of the σ-field generated by U_1 and E. Then (U_1, U_2) is a one-jump decomposition of U.

These observations yield an analytic criterium for one-jump decomposability:

Proposition 4.3: *The vector $U \geq 0$ is one-jump decomposable if and only if there is a pivot $a \neq 0$ for U, see (4.1).*

Example 4.4: The vector $a + U$ may be one-jump decomposable while U is not. Let U have a strictly positive density on $(0, \infty)^d$ except for an atom of size 1/2 in the origin, and let $a \in [0, \infty)^d$. Then a is a pivot for $a + U$, and $a + U$ is one-jump decomposable only if $a \neq 0$. □

One-jump decomposability implies that the support S of U can be covered by two blocks $[0, a]$ and $[a, \infty)$ with $a \neq 0$. An extreme form occurs if the support is contained in the image of an increasing continuous function $t \mapsto a(t) \in [0, \infty)^d$. The vector U then is totally dependent.

Definition 4.5: *Let X be a random vector with cdf H with univariate marginals H_i. The relation*

$$\tilde{H}(x_1, \ldots, x_d) = H_1(x_1) \wedge \ldots \wedge H_d(x_d) \tag{4.2}$$

defines a multivariate distribution which is called totally dependent.

Totally dependent vectors turn up in various parts of probability theory. They have a simple representation:

$$\tilde{X} = (\psi_1(U), \ldots, \psi_d(U)) \tag{4.3}$$

where U is a random variable which is uniformly distributed on $(0, 1)$ and ψ_i is the generalised inverse function of $1 - H_i$. The cdf $\tilde{H}$ is the maximal multivariate cdf with the given univariate marginals. See Fréchet (1951) and Gutman et al. (1991). Relation (4.2) implies that totally dependent vectors are max-id (since univariate cdf's are max-id). Zempléni (1987) characterises the totally dependent distributions for $d = 2$ as the max-antiireducible elements in the semigroup (under the max operation) of probability distributions on $[0, \infty)^2$ with lower endpoint in the origin. For max-stable totally dependent vectors there is a simple characterization due to Takahashi (1988). See Huesler (1989) for the non-max-stable case.

The proof of the next two theorems is similar to the univariate case, and omitted.

Theorem 4.6: *Let X be a random vector in $[0, \infty)^d$. There exists a stochastically continuous extremal process Y: $[0, 1] \to [0, \infty)^d$ with the boundary conditions*

$Y(0) = 0, Y(1) = X$ and lower curve C which vanishes on $[0,1)$ so that every sample path has at most one jump if and only if X is totally dependent.

The construction is obvious from the representation (4.3). The process Y becomes unique under the additional assumption that the weight has the special form $y(t) = q^t, 0 \le t \le 1$ where $q = E(e^{-X_1} + \cdots + e^{-X_d})$.

Theorem 4.7: *Let Y: $[0,\infty) \to [0,\infty)^d$ be a multivariate extremal process. Suppose for each fixed discontinuity point $t > 0$ one can choose the max-increment to be totally dependent. Then there exists a sequence of stochastically continuous extremal processes which converge to Y in $\mathscr{D}([0,\infty))$.*

5. One-jump processes

So far our one-jump processes started in the origin. In the univariate setting this is an innocuous assumption. The multivariate case is different.

Example 5.1: Let X and U be independent random vectors in $[0,\infty)^3$, U uniformly distributed on the cube $[0,1]^3$ and X uniformly distributed on the half of the cube $[0,4]^3$ above the plane $H: x_1 + x_2 + x_3 = 6$. Set $X' = X \vee U$. Although the unit cube lies below the plane H the event $\{X' \neq X\}$ has positive probability. The vector U is not totally dependent, not even one-jump-decomposable. Yet there exists a one-jump process with initial value $Y(0) = X$ and final value $Y(1) = X'$.

Proof: Write $U = U_1 \vee U_2 \vee U_3$ where U_i is uniformly distributed along the unit interval on the ith axis and the U_i are independent. Since each vector U_i is totally dependent there is a three-jump process $Z: [0,3] \to [0,\infty)^3$ so that $Z(0) = 0$ and $Z(3) = U$. Now set $Y(t) = X \vee Z(3t)$. Since any value x of the vector X has at most one component in $[0,1)$ only one of the three jumps of Z will be visible as a jump for Y. (Let x lie in the support of X. Suppose $x_1 < 1$ and $x_2 < 1$. Then the inequality $x_1 + x_2 + x_3 \ge 6$ implies $x_3 > 4$. Contradiction.) □

Example 5.1 shows that one-jump processes $Y: [0,1] \to [0,\infty)^d$ with a non-constant initial vector $Y(0) = X$ may have an unexpected structure. In particular they need not be unique up to a time change. Clearly one may alter the distribution of the initial vector X in the example as long as it remains concentrated on the upper half of the large cube. Less trivial is the fact that one may replace the uniform distribution of the max-increment U by any probability measure which lives on the unit cube.

Say that two vectors X and X' can be *connected* if there exists a one-jump process Y so that $Y(0) = X$ and $Y(1) = X'$. Given the initial vector X what conditions should the cdf of U satisfy, with U independent of X, in order that X and $X \vee U$ can be connected?

Theorem 5.2: *Let X and U be independent vectors in $[0,\infty)^d$ with cdf's F and H. Then X and $X \vee U$ can be connected if and only if $FH = F\tilde{H}$, with $\tilde{H}$ totally dependent, see (4.2).*

Necessity of the condition is obvious. Sufficiency follows from the technical Proposition 5.3 below. We shall use the notation of Theorem 5.2. In addition let D denote the closure in $[0,\infty)^d$ of $\{F>0\}$. The set D is increasing in the sense that it is the union of sets $[x,\infty), x\in D$. Let the vector $\tilde{U}$ with df $\tilde{H}$ be independent of X.

Proposition 5.3: *The following are equivalent*:

1. *X and $X\vee U$ can be connected*;
2. *x and $x\vee U$ can be connected for each $x\in D$*;
3. *$x\vee U$ is totally dependent for each $x\in D$*;
4. *$x\vee U$ is distributed like $x\vee \tilde{U}$ for each $x\in D$*;
5. *$X\vee U \overset{d}{=} X\vee \tilde{U}$*.

Proof: First note that 2, 3 and 4 are equivalent for any $x\geq 0$. The equivalence of 2 and 3 follows from Proposition 4.3; that of 3 and 4 holds since $H1_{[x,\infty)}$ and $\tilde{H}1_{[x,\infty)}$ have the same univariate marginals.

If 3 holds for x it holds for any $y\geq x$. If it holds for all x in a set it holds for all x in the closure of that set since by (4.2) the class of totally dependent distributions is closed under weak convergence.

By conditioning 1 implies 2 for X-almost every x, and hence by the arguments above for all $x\in D$. Conversely 2 implies 1 since D contains the support of X. A similar argument shows that 4 and 5 are equivalent. □

The condition $FH = F\tilde{H}$ may be hard to check for $d>2$. So it is convenient to know that it suffices to check it for the bivariate marginals. First note:

Lemma 5.4: *The point a is a pivot for the random vector U in $\mathbf{R}^d$ if*

$$P\{U_i<a_i, U_j>a_j\}=0 \qquad i\neq j, 1\leq i,j\leq d.$$

Theorem 5.5: *If the bivariate marginals of H are totally dependent then H is totally dependent*.

Proof: Each point in the support is a pivot. □

We need a stronger form of this result.

Theorem 5.6: *Let X and U be independent non-negative random vectors with cdf's F and H. The vectors X and $X'=X\vee U$ can be connected if and only if the bivariate marginals of H satisfy*: $H_{ij}(x_i,x_j)=H_i(x_i)\wedge H_j(x_j)$ *whenever* $F_{ij}(x_i,x_j)>0$.

Proof: We have to show that $H=\tilde{H}$ on $W=\text{int}\{F>0\}$ when this relation holds for the two dimensional marginals: $H_{ij}=\tilde{H}_{ij}$ on $W_{ij}=\text{int}\{F_{\text{ij}}>0\}$. Here $\tilde{H}$ is defined in (4.2). It suffices to prove that $dH_I=d\tilde{H}_I$ on W_I for all $I\subset D=\{1,\ldots,d\}$. Here for any cdf G

on $\mathbf{R}^d$ we let G_I denote the marginal distribution on $\mathbf{R}^I$ and W_I the image of W under the projection from $\mathbf{R}^d$ to $\mathbf{R}^I$. For simplicity of notation we shall prove this only for $I = D$.

The support $\tilde{S}$ of $\tilde{H}$ is linearly ordered. Set $a = \sup(\tilde{S} \backslash W)$ and $b = (\tilde{S} \cap W)$. Then $a \leq b$ and $\tilde{S} \subset [0, a] \cup [b, \infty)$ and $\tilde{S} \subset [0, a] \cup W$. We claim that these inclusions also hold for S.

Consider the bivariate marginals. Claim: $dH_{ij} = d\tilde{H}_{ij}$ off the rectangle $R = [0, a_i] \times [0, a_j]$. Let $d\mu$ be the restriction of $d\tilde{H}_{ij}$ to the complement of R. Then μ lives on W_{ij} since $\tilde{S}_{ij} \subset R \cup W_{ij}$. By assumption $dH_{ij} = d\tilde{H}_{ij} = d\mu$ on $W_{ij} \backslash R$. Since H and $\tilde{H}$ have the same univariate marginals the measures dH_{ij} and $d\mu$ agree on the half planes $\{x_i > a_i\}$ and $\{x_j > a_j\}$. So $dH_{ij} = d\mu = d\tilde{H}_{ij}$ off R. In particular dH_{ij} lives on the union of R and $[b_i, \infty) \times \{b_j, \infty)$.

This holds for all bivariate marginals. So a is a pivot for H by Lemma 5.4 and so is b. The vectors $a \vee U$ and $a \vee \tilde{U}$ have the same distribution since $a \vee U$ is totally dependent by Theorem 5.5. Since dH and $d\tilde{H}$ both live on $[0, a] \cup [b, \infty)$ it follows that $dH = d\tilde{H}$ off $[0, a]$, hence on W. □

Corollary 5.7: *Suppose the cdf's H_i of U_i have upper endpoint $q_i < \infty$. If the bivariate cdf's F_{ij} vanish on the rectangles $[0, q_i) \times [0, q_j)$ then X and $X \vee U$ can be connected.*

In Example 5.1 the bivariate cdf's F_{ij} vanish on $\{x_i + x_j < 2\}$ and hence on the square $[0, 1)^2$ on which (U_i, U_j) lives. The Corollary applies.

We now come to the main result of this section.

Theorem 5.8: *The distribution of a multivariate one-jump process is determined by the distribution of the initial vector and the distributions of the univariate marginal processes.*

Proof: This follows from the more specific result in Theorem 5.9 below. □

In the univariate case a one-jump process is characterized by (F, H, r) Here F is the initial distribution, FH the final distribution, and r the rate function of the jump. See (3.1) and Theorem 3.4. Since the marginals of one-jump processes are one-jump processes we may associate with the multivariate one-jump process X the distribution F of $X(0)$ and the rate functions r_i and distributions H_i of the univariate marginal processes X^i. Then $(F, r_1, \ldots, H_d)$ are characteristics of the process X. We shall now show that these characteristics determine the distribution of the one-jump process X.

Theorem 5.9: *Let $X : [0, 1] \to [0, \infty)^d$ be a one-jump process. Let F be the cdf of $X(0)$ and $F_i\, H_i$ the cdf of $X^i(1)$ for $i = 1, \ldots, d$. Let r_i be the rate function of the marginal process $X^i : [0, 1] \to [0, \infty)$. Then the cdf of $X(t)$ is*

$$F_t = F\tilde{H}_t \qquad H_{it}(u) = H_i(u) \vee (1 - r_i(t)) \qquad i = 1, \ldots, d, t \in [0, 1], u \geq 0, \quad (5.1)$$

where $\tilde{H}_t$ is the totally dependent cdf with marginals H_{it}, see (4.2).

Proof: The relation for $\tilde{H}^i_t(u)$ holds for univariate one-jump processes by (3.2). Then $F_t = F\tilde{H}_t$ by Proposition 5.3 since $X(0)$ and $X(t)$ can be connected. □

The structure of a multivariate one-jump process in terms of point processes may be quite complicated. We shall not pursue this topic here.

6. Convergence of fixed discontinuities

Total dependence is an extreme form of one-jump decomposability. Total dependence means that each point of the support of the random vector is a pivot.

Convergence of the fixed discontinuities in a sequence of extremal processes as in (1.1) may already fail if the max-increment U in the time point t_0 of the limit process is one-jump decomposable. Choose Y_n to have fixed discontinuities in $t_n' < t_n''$ both converging to t_0. Then (1.1) fails. However if each sample function of Y_n has a jump in at most one of these two time points then convergence in $\mathscr{D}([0,\infty))$ is not violated.

Can one impose conditions on the limit process Y_0 which ensure that the fixed discontinuities converge in the sense of (1.1)?

We introduce some notation which will be used throughout this section.

1. $Y_0, Y_1, \ldots$ are extremal processes in $[0,\infty)^d$ and $Y_n \Rightarrow Y_0$ in $\mathscr{D}([0,\infty))$;
2. t_0 is a fixed discontinuity point of Y_0, U a max-increment of Y_0 at time t_0;
3. F_0 is the cdf of $Y_0(t_0)$, F the cdf of $Y_0(t_0-0)$ and H the cdf of U;
4. C_0 is the lower curve of Y_0, c_0 the lower endpoint of F_0, c the lower endpoint of F:

$$C_0(t_0-0) \le c \le c_0 = C_0(t_0) \qquad H = H1_{[c_0,\infty)} \qquad F_0 = FH. \tag{6.1}$$

We start with a simple result.

Proposition 6.1: *The notation above is used. Suppose $d > 1$ and $C_0 \equiv 0$. The conditions*

a. F is strictly positive on $(0,\infty)^d$ and U is not one-jump decomposable;
b. U has a density which is strictly positive on $(0,\infty)^d$;

each imply (1.1) for a sequence of time points $t_n \to t_0$. If (a) holds then $U_n \Rightarrow U$ where U_n is a max-increment of Y_n at time t_n.

Proof: The proof will follow from more general results established below. □

We now first give an example to illustrate the inequalities in (6.1).

Example 6.2: Let the bivariate process Y_0 be generated by a Poisson point process on the diagonal in $[0,\infty)^3$ with mean measure μ, and a max-increment U at time $t=1$. Suppose μ projected on the time axis has density $1/(1-t)$ on $[0,1)$ and vanishes on $[1,\infty)$. Let U have density e^{-x-y} on $[r,\infty)^2$ and an atom of mass $1-e^{-2r}$ in the point (r, r). We assume

$r \geq 1$. Then $C_0(1-0) = (0,0), c = (1,1)$ since $Y_0(1-0) \equiv (1,1)$ and $c_0 = (r,r)$. Only $r = 1$ ensures convergence of fixed discontinuities.

Theorem 6.3: *The notation above (6.1) is used. Let F be strictly positive on* $(c_0, \infty) \subset \mathbf{R}^d$. *If* $U - c$ *is not one-jump decomposable then (1.1) holds for a sequence* $t_n \to t_0$. *Moreover* $U_n \vee c_0 \Rightarrow U$ *where* U_n *is a max-increment of* Y_n *in* t_n.

Proof: By assumption the weight functions converge weakly. Suppose (1.1) does not hold. Then there is a sequence (or a subsequence if necessary) $t_n \to t_0$ so that the weight functions y_n satisfy

$$y_n(t_n) \to \theta \in (\beta, \alpha) \qquad \alpha = y_0(t_0 - 0), \beta = y_0(t_0).$$

We claim that this makes the vector $U - c$ one-jump-decomposable. For simplicity assume $c = 0$. This may be achieved by replacing Y_n by $(Y_n - c) \vee 0$.

Choose $t'_n < t_n < t''_n$ continuity points of y_n converging to t_0 so that $y_n(t'_n) \to \alpha$ and $y_n(t''_n) \to \beta$. Let U_n be a max-increment of Y_n over the interval $[t'_n, t''_n]$. Then $U_n = V_n \vee W_n$ where V_n is a max-increment of Y_n over $(t'_n, t_n]$ and W_n a max-increment over $(t_n, t''_n]$, and V_n and W_n are independent, and independent of max-increments over intervals disjoint from these two.

Since $Y_n(t''_n)$ converges in distribution by tightness one can find a subsequence (k_n) so that

$$(X_{k_n}, V_{k_n}, W_{k_n}) \Rightarrow (X, V, W)$$

for independent vectors *V, W*. It follows that

$$(X, X \vee V \vee W), \stackrel{\mathrm{d}}{=} (Y_0(t_0 - 0), Y_0(t_0)).$$

This implies that U is distributed like $V \vee W \geq c_0$.

The sequence $0, V, V \vee W$ is a one-jump decomposition of the max-increment U. It is non-trivial since the weights α, θ, β of the three vectors $X, X \vee V$ and $X \vee V \vee W$ are unequal.

Since F_0 is positive on (c_0, ∞) convergence $V_n \vee W_n \Rightarrow U$ holds for the full sequence. This implies $U_n \vee c_0 \Rightarrow U$. □

In Pancheva (1994) a vector with lower endpoint q is called max-decomposable if it is the maximum of two independent vectors with the same lower endpoint q.

Corollary 6.4: *If* $c = c_0$ *and* $U - c_0$ *is max-indecomposable then (1.1) holds and* $U_n \vee c_0 \Rightarrow U$.

Proof: One-jump decomposability implies max-decomposability. □

Now drop the condition that F is strictly positive on (c_0, ∞). Blotting occurs. One can no longer speak of "the" max-increment U and hence cannot expect convergence $U_n \Rightarrow U$.

Example 6.5: *Let Y_0 have standard bivariate normal max-increments at times 1, 2, ... and suppose $Y_0(0)$ has a normal distribution conditioned to lie above the line $x + y = 0$. This is a process with values in $\mathbf{R}^2$. The lower endpoint of $Y_0(t)$ is $(-\infty, -\infty)$ for all $t \geq 0$, but the cdf F_t of $Y_0(t)$ vanishes for $x + y \leq 0$. On this part of the plane the max-increments are not uniquely determined by the distribution of the extremal process. Suppose $Y_n \Rightarrow Y_0$ in $\mathscr{D}([0, \infty))$. Does this imply convergence of the discontinuities in the sense of (1.1)?*

In order to understand the breakdown of (1.1) insight in the structure of one-jump triples is indispensible. An important question is: Which max-increments can not give rise to a one-jump triple?

Definition 6.6: *Let Z_0, W_1, W_2 be independent vectors in $[0, \infty)^d$ with cdf's F_0, H_1, H_2. The triple*

$$(Z_0, Z_1, Z_2) = (Z_0, Z_0 \vee W_1, Z_1 \vee W_2)$$

see (1.2) is called the one-jump triple generated by F_0, H_1, H_2 *if*

1. *F_0 has lower endpoint in the origin and*
2. *the events $\{Z_1 \neq Z_0\}$ and $\{Z_2 \neq Z_1\}$ are disjoint and have positive probability.*

If $Z_0 = 0$ in (1.2) then $Z_2 = W_1 \vee W_2$ is one-jump decomposable, the upper endpoint q of W_2 is a pivot, and W_1 lives on $\{0\} \cup [q, \infty)$. The general case is slightly different.

Proposition 6.7: *The vector W_2 in the one-jump triple (1.2) is bounded. Let $q \in [0, \infty)^d$ denote the upper endpoint of this vector W_2. Then the support S of the vector W_1 satisfies*

$$S \subset (\{0\} \cup [q_1, \infty)) \times \cdots \times (\{0\} \cup [q_d, \infty)).$$

If q_i is positive then the set $S^i = S \cap \{x_i = 0\}$ is bounded.

Proof: Suppose W_2 is unbounded. The two independent events

$$\{\|W_2\|_\infty > m\} \qquad \{Z_1 \neq Z_0, \|Z_1\|_\infty \leq m\}$$

have positive probability for sufficiently large m on the intersection $Z_0 \neq Z_1 \neq Z_2$. So with positive probability there are two jumps

Suppose $\{0 < W_1^i < q_i\}$ has positive probability. Then this is also true for the event

$\{W_1^i \in [\alpha, \beta]\}$ for some $\alpha < \beta$ in $(0, q_i)$. The events $\{Z_0^i < \alpha\}$ and $\{W_2^i > \beta\}$ have positive probability. By independence there is a positive probability of two jumps.

Suppose $q_i > 0$. Choose $\theta \in (0, q_i)$. If S^i is unbounded then the independent events

$$\{Z_0^i < \theta, \|Z_0\|_\infty \leq m\} \qquad \{W_1^i = 0, \|W_1\|_\infty > m\} \qquad \{W_2^i \geq \theta\}$$

have positive probability for large m, so two jumps are possible. □

Definition 6.8: *A non-constant random vector $W \geq 0$ with cdf* H *is* prime (*more specifically* one-jump prime) *if there is no one-jump triple (1.2) so that $F_2 = F_0 H$ where F_i is the cdf of Z_i.*

We can now formulate a condition for convergence of the fixed discontinuities which does not presuppose that the cdfs are strictly positive above the lower curve.

Theorem 6.9: *The notation introduced above (6.1) is used. If $U - c$ is prime then (1.1) holds for some sequence $t_n \to t_0$.*

Proof: The proof is as for Proposition 6.1. Assume $c = 0$. If the result were not true we could find a subsequence k_n and time points $t'_{k_n} < t_{k_n} < t''_{k_n}$ so that

$$(Y_{k_n}(t'_{k_n}), Y_{k_n}(t_{k_n}), Y_{k_n}(t''_{k_n})) \Rightarrow (X_0, X_1, X_2)$$

for a one-jump triple with $X_0 = Y_0(t_0 - 0), X_2 = Y_0(t_0)$. This contradicts the primality of the max-increment U. □

It remains to describe the prime distributions.

7. One-jump prime distributions

This section is devoted to a characterization of primality, see Definition 6.8. Independence of the max-increments in a one-jump triple implies that we are dealing with products of (at most three) distribution functions. This does not make the theory trivial.

Example 7.1: Let U_1 and U_2 be uniformly distributed on $(0, 1)$ and let X_1 and X_2 have a standard exponential distribution on $(0, \infty)$. Assume that these four random variables are independent. The vectors (X_1, X_2), (U_1, X_1, X_2) and (U_1, U_2, X_1, X_2) are prime, the vectors (U_1, U_2), (U_1, X_1), (U_1, U_2, X_1) are not. See Corollary 7.12 below.

Definition 7.2: *Two distributions H and G on $[0, \infty)^d$ are said to be* equivalent *(in this section) and we write $G \sim H$ if there is a distribution* F *with lower endpoint in the origin so that $FG = FH$.*

Remark 7.3: The relation $\sim$ is reflexive, symmetric and transitive, so it indeed is an equivalence relation. (If F_1 and F_2 have positive univariate marginals then so has the product F_1F_2). If two distributions are equivalent then so are their marginals (both univariate and multivariate). So equivalent distributions have the same univariate marginals.

Lemma 7.4: *Equal mixtures of equivalent cdf's are equivalent: If $G_i \sim H_i$ for $i = 1, \ldots, m$ then $G = p_1G_1 + \cdots + p_mG_m \sim H = p_1H_1 + \cdots + p_mH_m$ for any probability vector $p_1, \ldots, p_m$.*

Proof: Suppose $G_iF_i = H_iF_i$. Then $GF = HF$ for $F = F_1 \ldots F_m$. □

Proposition 7.5: *If H is prime then so is G for any cdf $G \sim H$.*

Proof: Suppose $FG = FH$. If F_0, H_1, H_2 generates a one-jump triple, then so does FF_0, H_1, H_2. So if $GF_0 = H_1H_2F_0$ then $HFF_0 = H_1H_2FF_0$. □

Corollary 7.6: *Prime vectors are unbounded.*

Proof: A bounded vector is equivalent to a totally dependent vector, see Section 5. □

The next example is basic. We use the notation

$$x = (x', x'') \in [0, \infty)^{d-1} \times [0, \infty) \tag{7.1}$$

for the decomposition of a vector in a horizontal part and a vertical component.

Example 7.7: *Let $\theta > 0$ and let $R = [a', b'] \times \{0\}$ be a rectangle in the horizontal plane with $0 \leq a' \leq b'$, see (7.1). Suppose Z_0, W_1 and W_2 are independent vectors: Z_0 lives on the union of the halfspace $\{x'' \geq \theta\}$ and the orthant $[b' \infty) \times [0, \infty)$, W_1 lives on the union of $[a', \infty) \times [\theta, \infty)$ and the rectangle R, and W_2 lives on the vertical line segment $\{a'\} \times [0, \theta]$. Then (1.2) has at most one jump: If $Z_1 \neq Z_0$ then $Z_1 \geq (a', \theta) \geq W_2$. (A figure may be helpful here.)*

If Z_0 has lower endpoint in the origin, $P\{W_1 \in R\} \in (0, 1)$, and $P\{W_1'' > 0\} > 0$ we have a one-jump triple.

Proposition 7.8: *Let $\theta > 0$. Suppose $W \geq 0$ lives on the union of the halfspace $\{x'' \geq \theta\}$ and a compact set B. If $P\{W \in B, 0 < W'' < \theta\} > 0$ then W is not prime.*

Proof: Let a be the lower endpoint of the cdf H of W. We may assume that B is a box $[a, b]$ with $b \geq a$ and $b'' = \theta$. We may also assume that W is unbounded, so $P\{W \in B\} = p \in (0, 1)$.

Write $H = pG + (1 - p)F$ where G lives on the box B and F on the halfspace $\{x'' \geq \theta\}$.

Let (V', V'') have cdf G. Choose independent vectors $(U', 0)$ and (a', U'') with cdf G_1 and G_2 respectively so that U' is distributed like V' and U'' like V''. Then $G \sim G_1G_2$, (bounded support and the same marginals), and $FG_2 = F$ since $G_2 = 1$ in the lower endpoint of F. Now let W_1 and W_2 be independent vectors with cdf $H_1 = pG_1 + (1-p)F$ and $H_2 = G_2$. Then

$$H_1H_2 = pG_1G_2 + (1-p)FG_2 = pG_1G_2 + (1-p)F \sim pG + (1-p)F = H.$$

By Example 7.7 the cdf H_1H_2 is not prime. So neither is H. □

Let W have cdf H and support $S \subset [0, \infty)^d$. Define the *essential* lower endpoint of H as

$$c(H) = c = \sup c(n) \quad c(n) = (S \backslash [0, n]^d). \tag{7.2}$$

Recall that the inf and sup of a set of vectors is defined coordinatewise. The sequence $c(n)$ is increasing. One may think of the limit $c \in [0, \infty]^d$ as the maximal lower endpoint of H when one is allowed to alter H on compact sets. Equivalent distributions (obviously) have the same lower endpoint. They also have the same essential lower endpoint.

Proposition 7.9: *If $H \sim G$ then $c(H) = c(G)$ with c defined in (7.2).*

Proof: Suppose $\theta < c_i(H)$. For simplicity of notation take $i = 1$. Then there is an integer n so that $H(\theta, x_2, \ldots, x_d) = H(\theta, n \ldots, n)$ for $x_i \geq n$. Suppose $FG = FH$ for a cdf F with lower endpoint in the origin. Then $F(\theta, n, \ldots, n) > 0$ eventually. So G has the same property as H and $\theta \leq c_1(G)$. □

Given the essential lower endpoint primality can be expressed in terms of the one-dimensional marginals:

Theorem 7.10: *Let H be the cdf of a non-constant vector $W \geq 0$ with essential lower endpoint $c \in [0, \infty]^d$, see (7.2). The cdf H is prime if and only if*

$$H^i(c_i) = H^i(0) \quad i = 1, \ldots, d. \tag{7.3}$$

Proof: Let T denote the support of W and $a = \inf T$ the lower endpoint.

First assume the condition does not hold. There is an index j so that $H^j(c_j) > H^j(0)$. There are two cases. (1) If $H^j(c_j - 0) = H^j(0)$ then Proposition 7.8 applies with $\theta = c_j$ since $T \cap \{x_j = 0\}$ is bounded by definition of c. (2) If $H^j(c_j - 0) > H^j(0)$ then there is a $\theta \in (0, c_j)$ so that $H^j(\theta) > H^j(0)$ and Proposition 7.8 applies.

Now suppose H is not prime. Then $H \sim H_1H_2$ where F_0, H_1, H_2 generates a one-jump triple. We may and shall assume that $H = H_1H_2$ by Proposition 7.5 and 7.9. Let q denote the upper endpoint of H_2. Recall Proposition 6.7. Note that $c \geq q$ since the set S^i is bounded when q_i is positive. Now assume (7.3). We shall derive a contradiction.

There is a coordinate j so that $q_j > 0$ and $H_1^j(0) > 0$. (Else $W_1 \geq q$ and $Z_2 = Z_1$ almost surely.) Then $H^j(q_j) > 0$ and $H^j(0) = H^j(c_j) = H^j(q_j)$ implies $H_2^j(0) = H_2^j(q_j) = 1$. So $W_2^j \equiv 0$ and $q_j = 0$. Contradiction. □

Corollary 7.11: *Primality depends only on the bivariate marginal distributions.*

Proof: These determine the essential lower endpoint. □

Corollary 7.12: *Let the components W^i be pairwise independent with lower endpoint a_i. Then H is prime if and only if $H^i(a_i) = H(0)$ for all i and if at least two components are unbounded.*

Proof: If two components or more are unbounded then $c(H) = a$, else $c(H) = \infty$. □

8. The weight

Realizations of extremal processes are increasing right-continuous functions from $[0, \infty)$ to $[0, \infty)^d$. In this section we drop the assumption of independence of the max-increments. This means that we consider the class of all processes $Y : [0, \infty) \to [0, \infty)^d$ with right-continuous increasing sample functions. For such functions the topology of weak convergence is applicable. The space J_d of all increasing right-continuous functions $\varphi : (0, \infty) \to [0, \infty)^d$ with the topology of weak convergence is a Polish space. Hence so is the space $\mathscr{M}_1(J_d)$ of all probability measures on J_d.

Definition 8.1: *An* increasing *process is a process $Y : (0, \infty) \to [0, \infty)^d$ whose sample functions are right-continuous and increasing.*

For increasing processes Y_n we say that

$$Y_n \Rightarrow Y_0 \text{ weakly} \quad n \to \infty \tag{8.1}$$

if the probability distributions converge in the topology on $\mathscr{M}_1(J_d)$ of weak convergence in J_d. The theory of weak convergence of increasing processes has been treated in BP (1996). On the set $\mathscr{M}_1(J_d)$ we also have the Skorohod topology. We say that

$$Y_n \Rightarrow Y_0 \text{ in } \mathscr{D}([0, \infty)) \quad n \to \infty \tag{8.2}$$

if the probability distributions converge in the Skorohod topology. See Billingsley (1968) or Resnick (1987). Note that (8.2) implies (8.1). For weak convergence it is convenient to have functions defined on an open interval. For convergence in the Skorohod topology the sample functions are extended to $[0, \infty)$ by right continuity: $\varphi(0) := \varphi(0+0) = \lim_{t \downarrow 0} \varphi(t)$.

For many operations one prefers to work in the Skorohod topology. This will be the case if one wants to keep track of the largest jump, or when one integrates non-continuous processes with respect to the increasing process.

We begin with a more general definition of weight for increasing processes.

Let $\psi : [0,\infty)^d \to [0,\infty)$ be continuous and strictly decreasing in the sense that $0 \le x \le y$ and $x \ne y$ implies $\psi(x) > \psi(y)$. Then ψ is bounded by $\psi(0)$ and ψ has the ability to enforce convergence: Let $x_0 \in [0,\infty)^d$, and let the points $x_n \ge 0$ satisfy one of the inequalities $x_n \le x_0$ or $x_n \ge x_0$ for each $n \ge 1$. Then

$$\psi(x_n) \to \psi(x_0) \Rightarrow x_n \to x_0. \tag{8.3}$$

We call $t \mapsto y(t) = E\psi(Y(t))$ the *weight* of the increasing process Y. One regains the original definition on taking

$$\psi(x_1,\dots,x_d) = e^{-x_1} + \cdots + e^{-x_d}. \tag{8.4}$$

Recall from Proposition 3.6 that a.s. convergence of a sequence of extremal processes in the Skorohod topology does not imply convergence of the weights in this topology.

Proposition 8.2: *Suppose $Y_n, n \ge 0$, are increasing processes in $[0,\infty)^d$ and $Y_n \Rightarrow Y_0$ in the topology of weak convergence on $(0,\infty)$. Suppose $Y_n(0) \Rightarrow Y_0(0)$. For each fixed discontinuity $t_0 > 0$ of Y_0 let there exists a sequence $t_n \to t_0$ so that (1.1) holds. Then the weights converge in $\mathscr{D}([0,\infty))$.*

Proof: Convergence $Y_n \Rightarrow Y_0$ in the topology of weak convergence implies weak convergence $y_n \to y_0$ on $(0,\infty)$ by BP (1996, Theorem 6.1). Since ψ is continuous and bounded, $y_n(0) \to y_0(0)$. Similarly (1.1) implies $y_n(t_n \pm 0) \to y_0(t_0 \pm 0)$. □

Lemma 8.3: *Let $X_n \ge 0$ be random vectors for $n \ge 0$ such that for each $n \ge 1$ either $X_n \le X_0$ a.s. or $X_n \ge X_0$ a.s. Then $E\psi(X_n) \to E\psi(X_0)$ implies $X_n \to X_0$ in probability.*

Proof: Let $U_n = \psi(X_n)$. Then $U_n \ge U_0$ a.s. or $U_n \le U_0$ a.s. In either case one has that $E|U_n - U_0| = |EU_n - EU_0| \to 0$. Hence $U_n \to U_0$ in $\mathbf{L}^1$ and then also in probability. This implies that $X_n \to X_0$ in probability. (There are events E_n with $PE_n \to 1$ so that $(U_n - U_0)1_{E_n} \to 0$ a.s. By (8.3) also $(X_n - X_0)1_{E_n} \to 0$ a.s.) □

Proposition 8.4: *Let $Y_n : (0,\infty) \to [0,\infty)^d$ be increasing processes for $n \ge 0$ with weights y_n. Assume that $Y_n \to Y_0$ weakly holds a.s. Let $t_n \to t_0$ in $[0,\infty)$.*

1. *If $y_n(t_n + 0) \to y_0(t_0 + 0)$ then $Y_n(t_n + 0) \to Y_0(t_0 + 0)$ in probability;*
2. *If $t_0 > 0$ and $y_n(t_n - 0) \to y_0(t_0 - 0)$ then $Y_n(t_n - 0) \to Y_0(t_0 - 0)$ in probability.*

Proof: Write $U_n = Y_n(t_n + 0)$. Let $s > t_0$ and $j \in \{1, \ldots, d\}$. Take an a.s. continuity point t of Y_0 with $t_0 < t < s$. Then

$$\limsup U_n^j \leq \lim Y_n^j(t) = Y_0^j(t) \leq Y_0^j(s) \text{ a.s.}$$

Since $s > t_0$ is arbitrary we conclude that $\limsup U_n \leq U_0$ a.s. Set $X_n = U_n \wedge U_0$. Then we have just proved $U_n - X_n \to 0$ a.s., and the inequalities

$$E\psi(X_0) \geq E\psi(X_n) \geq y_n(t_n) \to y_0(t_0) = E\psi(X_0)$$

allow us to conclude by Lemma 8.2 that $X_n \to X_0$ in probability. Together with $U_n - X_n \to 0$ a.s. this implies $U_n \to U_0$ in probability.

The proof of 2 is similar. Use $U_n = Y_n(t_n - 0)$ and $X_n = U_n \vee U_0$. □

Theorem 8.5: *Let* $Y_n : (0, \infty) \to [0, \infty)^d$ *be increasing processes for* $n \geq 0$. *Suppose* $Y_n \Rightarrow Y_0$ *weakly and the weights converge in* $\mathscr{D}([0, \infty))$. *Then* $Y_n(0 + 0) \Rightarrow Y_0(0 + 0)$ *and the fixed discontinuities converge in the sense of (1.1).*

Proof: One may assume almost sure convergence by using the Skorohod representation theorem, see Resnick (1987, p. 151). Now apply Proposition 8.4. □

Let us now return to convergence in the Skorohod topology. By combining Proposition 8.2 and Theorem 8.5 we obtain

Theorem 8.6: *Let* $Y_n : [0, \infty) \to [0, \infty)^d$ *be extremal processes for* $n \geq 0$. *Suppose* $Y_n \Rightarrow Y_0$ *in* $\mathscr{D}([0, \infty))$. *Then the fixed discontinuities converge in the sense of (1.1) if and only if the weights converge in* $\mathscr{D}([0, \infty))$.

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