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A FUNCTIONAL EXTREMAL CRITERION*

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1. Introduction

In this paper, we are concerned with the following model:

(A) Let $\mathcal{N} = \{(t_k, X_k): k \geq 1\}$ be a point process with time space $[0, \infty)$ and state space $[0, \infty)^d$, where $\{t_k\}$ are distinct nonrandom time points. We assume them ordered and increasing to ∞, i.e., $t_1 < t_2 < \dots$. So, the point process $\mathcal{N}$ is simple in time; $\{X_k\}$ are independent and identically distributed (i.i.d.) random vectors (r.v.'s) on a given probability space with values in $[0, \infty)^d$ and with common distribution function (d.f.) F nondefective at $+\infty$.

Assume that almost all realizations of $\mathcal{N}$ are Radon measures on $\mathcal{S} := [0, \infty) \times E$, where $E := [0, \infty]^d \backslash \{0\}$, i.e.,

$$\mathcal{N}(A) < \infty \quad \text{a.s.} \quad \forall \text{ compact sets } A \in \mathcal{B}(\mathcal{S}). \tag{1}$$

We consider two random processes associated with $\mathcal{N}$, namely, the extremal process

$$X(t) = \{\vee X_k\colon\ t_k \leq t\}$$

and the process Z with additive increments

$$Z(t) = \left\{\sum X_k\colon\ t_k \leq t\right\}.$$

Because of (1), the sum and the maximum are a.s. finite for every fixed $t > 0$. Both processes are right continuous with increasing sample paths. Here and further on we use the notion increasing in the sense of nondecreasing.

We denote by $\mathcal{M}$ the set of all increasing, right-continuous functions $y\colon (0, \infty) \to [0, \infty)^d$. Then the set $\mathcal{P}$ of all probability measures on $\mathcal{M}$ is compact. Let $\{P_n\}$ be a sequence of probability measures on $\mathcal{M}$. We say $\{P_n\}$ is weakly convergent to $P \in \mathcal{P}$, briefly $P_n \Rightarrow P$, if $\int \varphi\, dP_n \to \int \varphi\, dP$ for bounded $\varphi\colon \mathcal{M} \to R$ which are continuous in the weak topology of $\mathcal{M}$. Now denote by $\mathcal{P}_e$ and $\mathcal{P}_s$ the subsets of $\mathcal{P}$ corresponding to an extremal process (with independent max-increments) and to a sum process (with independent additive increments), respectively. In [1, Theorem 6.4] it is shown that the space $\mathcal{P}_e$ with the topology of weak convergence is closed in $\mathcal{P}$. The same is also true for $\mathcal{P}_s$. So, the weak convergence of extremal processes $Y_n \Rightarrow Y$ and of sum processes $S_n \Rightarrow S$ is equivalent to the convergences in distribution $Y_n(t) \xrightarrow{\text{d}} Y(t)$ and $S_n(t) \xrightarrow{\text{d}} S(t)$ for each continuity point t of the limit process.

Further, for normalizing we use an unboundedly increasing in n sequence of mappings

$$\zeta_n(t, x) = (\tau_n(t), u_n(x)),$$

continuous and strictly increasing in each coordinate. We call them time-space changes. Suppose $\{\zeta_n\}$ is regular in the sense that there exists a pointwise limit of $\zeta_n^{-1} \circ \zeta_{[ns]}$ for $n \to \infty$ and $s > 0$ which is again continuous and strictly increasing (cf. [6]). We assume the weak convergence

$$Y_n(t) = \{\vee u_n^{-1}(X_k)\colon t_k \leq \tau_n(t)\} \Longrightarrow Y(t), \quad n \to \infty, \tag{2}$$

to a nondegenerate extremal process Y with initial value $Y(0) \overset{\text{a.s.}}{=} 0$.

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We also form the associated processes with additive increments

$$S_n(t) := \left\{ \sum u_n^{-1}(X_k) \colon t_k \le \tau_n(t) \right\}.$$

Note that the space changes $\{u_n\}$ preserve the max-operation, i.e., $u_n^{-1}(\vee X_k) = \vee u_n^{-1}(X_k)$, but do not preserve (in general) the summing operation. Hence, $Y_n(t) = u_n^{-1} \circ X \circ \tau_n(t)$ but $S_n(t) \ne u_n^{-1} \circ Z \circ \tau_n(t)$ in general. If u_n preserves both operations $\vee$ and $\sum$, then u_n is just a scale change and the convergence $S_n = u_n^{-1} \circ Z \circ \tau_n \Rightarrow S$ implies that S is a self-similar process (cf. [4]).

Our main result, proved in Sec. 2, concerns the convergence $S_n \Rightarrow S$, if given (2). We call it a functional extremal criterion (for the convergence $S_n \Rightarrow S$), having in mind the extremal criterion in [5, §22.4.c].

Theorem 1. *Let $\mathcal{N} = \{(t_k, X_k),\ k \ge 1\}$ be the point process described in* (A) *and let $\zeta_n(t,x) = (\tau_n(t), u_n(x))$ be a regular norming sequence of time-space changes of $(0,\infty)^{d+1}$ such that the sequence of the associated extremal processes $Y_n(t) = \{\vee u_n^{-1}(X_k) \colon t_k \le \tau_n(t)\}$ is weakly convergent to a nondegenerate extremal process $Y(t)$. Assume that the d.f. G of $Y(1)$ satisfies the condition.*

$$I_G = \int\limits_{A_{q,v}} \|x\| \, d(\log G(x)) < \infty.$$

Then there exists a time-space change $\zeta(t,x) = (\tau(t), h_\alpha(x))$ such that the sequence of the associated sum processes $S_n(t) := \{\sum u_n^{-1}(X_k) \colon t_k \le \tau_n(t)\}$ is weakly convergent to an infinitely divisible process $S(t)$ whose characteristic function is given by

$$\mathbf{E} e^{i\langle \theta, S(t) \rangle} = \exp\left\{ \tau(t) \int\limits_E \{ e^{i\langle \theta, h_\alpha^{-1}(x) \rangle} - 1 \} \, d\Pi(x) \right\},$$

where $\Pi(dx)$ is the Lévy measure of Proposition 2, (iii).

2. Stepwise Proof of the Functional Extremal Criterion

We put $t_{nk} = \tau_n^{-1}(t_k)$, $X_{nk} = u_n^{-1}(X_k)$, and consider the point process $\mathcal{N}_n := \{(t_{nk}, X_{nk} \colon k \ge 1)\}$ associated with S_n and Y_n.

Step 1. Denote by $k_n(t)$ the nonrandom counting function of $\mathcal{N}_n$, i.e.,

$$k_n(t) = \max\{k \colon t_k \le \tau_n(t)\} = \sum_k I_{[0,t]}(t_{nk}).$$

Here $I_A(\bullet)$ is the indicator of the set A. By (2), we have the weak convergence

$$\mathbf{P}(Y_n(t) < x) = F^{k_n(t)}(u_n(x)) \xrightarrow{w} g(t,x), \quad n \to \infty, \tag{3}$$

where g is the d.f. of the limit process Y. Thus, for fixed $t > 0$ F belongs to the partial max-DA of $g_t(x) := g(t,x)$. Hence Y (respectively, g) is max-ID.

Moreover, by Propositions 2.1 and 2.3 in [6], Y (respectively, g) is self-similar (briefly $Y \in \mathrm{SS}$) and stochastically continuous. The condition $Y(0) \overset{\text{a.s.}}{=} 0$ guarantees that $G(x) := g_1(x)$ does not have a defect at $+\infty$. In our case, where $\{X_k\}$ are i.i.d., one can determine more precisely the subclass of SS which g belongs to.

Lemma 1. *The regularity of the time-space changes ζ_n implies the regularity of the sequence $k_n := k_n(1)$.*

Proof. For $t = 1$, (3) reads as

$$F^{k_n}(u_n(x)) \xrightarrow{w} G(x), \quad n \to \infty. \tag{4}$$

Now we take $s > 0$ and observe the convergence in distribution for $n \to \infty$

$$u_n^{-1} \circ X \circ \tau_{[ns]} = u_n^{-1} \circ u_{[ns]} \circ u_{[ns]}^{-1} \circ X \circ \tau_{[ns]} \xrightarrow{\mathrm{d}} U_s \circ Y,$$

where $U_s(x) := \lim_{n\to\infty} u_n^{-1} \circ u_{[ns]}(x)$, $\forall x \in \{G > 0\}$. Hence,

$$\mathbf{P}(u_n^{-1} \circ X \circ \tau_{[ns]}(1) < x) = F^{k_{[ns]}}(u_n(x)) \underset{n\to\infty}{\xrightarrow{w}} \mathbf{P}(Y(1) < U_s^{-1}(x)) = G(U_s^{-1}(x)).$$

On the other hand, $F^{k_{[ns]}}(u_n(x)) = [F^{k_n}(u_n(x))]^{k_{[ns]}/k_n}$. So there exists $\lim_{n\to\infty}(k_{[ns]}/k_n) = k(s) \in (0,\infty)$. As is known, the last convergence is uniformly in s and $k(s)$ is a power function of s. Say s^β.

The next examples show the consequences of the regularity of k_n for the time process $\{t_k\}$.

Example 1. Let $t_k = e^k$, $k \in \{1,2,\dots\}$, and take a time change $\tau_n(t) = (tn)^2$; then $t_{nk} = \tau_n^{-1}(t_k) = (1/n)e^{k/2}$ and $k_n = \sum_k I_{[0,1]}(t_{nk}) = 2\log n$ is not a regular sequence (but is slowly varying).

Example 2. Let $t_n = (n(n+1))/2$ and $\tau_n(t)$ be the same as in Example 1. Now the sequence $\{t_n\}$ is regular, $t_{nk} = (1/n)\sqrt{k(k+1)/2}$, and $k_n \sim 2n$ is regular, too.

Let us come back to (4) and observe that the limit d.f. $G(x)$ satisfies $G^{k(s)}(x) = G(U_s^{-1}(x))$ for all $s > 0$, where $k(s) = s^\beta$.

Denote $L_t(\cdot) := U_{\sqrt[\beta]{t}}(\cdot)$. Now from $G^t(x) = G(L_t^{-1}(x))$ $\forall t > 0$ one can conclude that G is max-stable with respect to the continuous one-parameter group $\mathcal{L} = \{L_t\colon\ t > 0\}$. Note that $\mathcal{L}$ bears the regularity of both sequences $\{k_n\}$ and $\{u_n\}$.

Analogously one can see that there exists $\lim_{n\to\infty} k_n(t)/k_n =: \tau(t)$ and, finally, we get

$$g(t,x) = G^{\tau(t)}(x).$$

The mapping $t \to \tau(t)$ is continuous and increasing (since $Y \in \mathrm{SS}$), hence it is a time change and we can write

$$g(\tau^{-1}(t),x) = G^t(x) = G(L_t^{-1}(x)) = g(1, L_t^{-1}(x)).$$

This means that $Y \circ \tau^{-1}(t) \overset{\mathrm{d}}{=} L_t \circ Y(1)$, $\forall t > 0$. Thus the process $Y \circ \tau^{-1}$ has homogeneous max-increments. Note that $L_1 = \mathrm{id}$, $\tau(1) = 1$, $\tau(t) \to 0$, $t \to 0$, $\tau(t) \to \infty$, $t \to \infty$.

Step 2. Choose $\alpha = (\alpha_1,\dots,\alpha_d)$ with $0 < \alpha_i < 1$ (later below we discuss this choice). Let $\Phi_\alpha^*(x_1,\dots,x_d)$ be the d.f. on $[0,\infty)^d$ with Frechet univariate marginals

$$\Phi_{\alpha_i}^*(x_i) = e^{-x_i^{-\alpha_i}}, \quad i = 1,\dots,d,$$

whose dependence (copula) function is the same as that of the limit d.f. $G(x)$. We determine the mapping

$$h_\alpha(x) = (h_{\alpha_1}(x_1),\dots,h_{\alpha_d}(x_d))$$

for all x in the support of G (briefly $\operatorname{Supp} G$) by $G(h_\alpha^{-1}(x)) = \Phi_\alpha^*(x)$ and use it to define the r.v. $Y^*(1) := h_\alpha \circ Y(1)$. It is distributed by Φ_α^*. Note that the mapping $h_\alpha\colon \operatorname{Supp} G \to (0,\infty)^d$ is continuous and strictly increasing in each component.

Now the limit relation (4) implies that the d.f. $F'(x) := F \circ h_\alpha^{-1}(x)$ belongs to the max-DA(Φ_α^*), i.e.,

$$[F'(T_n(x))]^{k_n} \xrightarrow{w} \Phi_\alpha^*(x), \quad n \to \infty, \tag{5}$$

with norming sequence $T_n(\cdot) = h_\alpha \circ u_n \circ h_\alpha^{-1}(\cdot)$, which is regular at ∞.

Step 3. Combine Step 1 and Step 2. So, we started with the limit extremal process $Y(t)$ distributed by $g(t,x)$ and came to the time-space changed extremal process $Y^*(t) := h_\alpha \circ Y \circ \tau^{-1}(t)$, $t > 0$, with d.f. $(\Phi_\alpha^*)^t$. It is self-similar with respect to the continuous one-parameter group

$$\mathcal{U}_\alpha = \{U_t(x) := (\sqrt[\alpha_1]{t}\,x_1,\dots,\sqrt[\alpha_d]{t}\,x_d)\colon t > 0\}. \tag{6}$$

More precisely, the extremal process $Y^*(t) = U_t \circ Y^*(1)$ is stochastically continuous, has homogeneous max-increments, and $Y^*(0) \overset{\text{a.s.}}{=} 0$. Thus it is a Lévy process in the max-framework.

In what follows, we denote the vector $(\sqrt[\alpha_1]{s}\,x_1,\dots,\sqrt[\alpha_d]{s}\,x_d)$ simply by $\sqrt[\alpha]{s}\,x$. Then we can write

$$Y^*(t) = \sqrt[\alpha]{t}\,Y^*(1).$$

Proposition 1. *Let (5) hold, i.e., $F' \in \text{max-DA}(\Phi_\alpha^*)$ with respect to a regular norming sequence $\{T_n\}$ of space changes and a regular subsequence $\{k_n\}$ of $\{n\}$. Then there exists a space change T such that*

$$[F' \circ T(\sqrt[\alpha]{k_n}\,x)]^{k_n} \xrightarrow{w} \Phi_\alpha^*(x), \quad n \to \infty. \tag{7a}$$

Proof. For $T_n(x) = (T_{1n}(x_1), \dots, T_{dn}(x_d))$ and $\alpha = (\alpha_1, \dots, \alpha_d)$

$$1 - F_i'(T_{in}(x_i)) \sim \frac{x_i^{-\alpha_i}}{k_n}, \quad n \to \infty,$$

or equivalently,

$$T_{in}(x_i) \sim \left(\frac{1}{1 - F_i'}\right)^{\leftarrow} (k_n x_i^{\alpha_i})$$

($F^{\leftarrow}$ means the left inverse of F). In fact, the assumption that $\{T_n\}$ is regularly varying in $n \to \infty$ is the same as $[1 - F']$ is regularly varying in $x \to \infty$. Put

$$\hat{T}_i(x_i) := \left(\frac{1}{1 - F_i'}\right)^{\leftarrow} (x_i^{\alpha_i}), \quad i = 1, \dots, d.$$

These mappings are positive, increasing, and asymptotically continuous. The latter means

$$\frac{\hat{T}_i(x+0) - \hat{T}_i(x-0)}{\hat{T}_i(x)} \longrightarrow 0, \quad x \to \infty, \quad i = 1, \dots, d.$$

Thus, there exists (cf. [2, Lemma 2]) a continuous and strictly increasing mapping T (space change) such that

$$T(x) \sim (\hat{T}_1(x_1), \dots, \hat{T}_d(x_d)), \quad x \to \infty.$$

Now we can see that $T_n(x) \sim T(\sqrt[\alpha]{k_n}\, x)$.

Statement (7a) of Proposition 1 is equivalent to (cf. [8, Proposition 5.17]): $1 - F' \circ T$ is regularly varying at ∞, i.e., for $A_x^c := [0, \infty]^d \backslash [0, x)$ and $\mathbf{e} = (1, \dots, 1) \in \mathbf{R}^d$

$$\frac{1 - F' \circ T(sx)}{1 - F' \circ T(s\mathbf{e})} \longrightarrow \frac{\nu_\alpha(A_x^c)}{\nu_\alpha(A_{\mathbf{e}}^c)} := \lambda(x), \quad s \to \infty, \tag{7b}$$

where ν_α is the exponent measure of Φ_α^* satisfying

$$s^{-1}\nu_\alpha(A_x^c) = \nu_\alpha(A^c_{\sqrt[\alpha]{s}\, x}). \tag{8}$$

Let us summarize what we have achieved within the three steps: we have transformed continuously our initial model (A) to a model (B), where the sequence of extremal processes needs scale normalization to converge. And scale normalizations preserve both the $\vee$ and $\sum$ operations. In the next step, we pursue convergence of the associated processes with additive increments.

Step 4. Model (B): Denote $t_k^* = \tau(t_k)$, $\sigma_n(t) = \tau \circ \tau_n \circ \tau^{-1}(t)$, and

$$X_k^* = T^{-1} \circ h_\alpha(X_k), \quad k = 1, 2, \dots,$$

with common d.f. $F^* := F' \circ T$. In this model, we use the point process $\mathcal{N}^* = \{(t_k^*, X_k^*)\colon k \in \{1, 2, \dots\}\}$ and the norming sequence $\eta_n(t, x) = (\sigma_n(t), \sqrt[\alpha]{k_n}\, x)$ to generate the asymptotically homogeneous point process

$$\mathcal{N}_n^* = \left\{(t_{nk}^* = \sigma_n^{-1}(t_k), X_{nk}^* = \frac{1}{\sqrt[\alpha]{k_n}} X_k^*\colon\ k \geq 1\right\}.$$

Consider now the process with additive increments

$$S_n^*(t) = \left\{\sum X_{nk}^*\colon\ t_{nk}^* \leq t\right\} = \frac{1}{\sqrt[\alpha]{k_n}} \sum_{k=1}^{k_n^*(t)} X_k^*$$

and the extremal process

$$Y_n^*(t) = \{\vee X_{nk}^*\colon\ t_{nk}^* \leq t\} = \frac{1}{\sqrt[\alpha]{k_n}} \bigvee_{k=1}^{k_n^*(t)} X_k^*$$

associated with the same point process $\mathcal{N}_n^*$ and with the same counting function

$$k_n^*(t) = \sum_k I_{[0,t]}(t_{nk}^*) \sim k_n t.$$

The last asymptotic relation is a consequence of $k_n^*(t) = k_n(\tau^{-1}(t))$ and $k_n(t) \sim k_n\tau(t)$ established in the first step. Further, by (7a),

$$Y_n^* \Longrightarrow Y^*.$$

In model (B), a lot of results are well known. We gather them in the next proposition. Let $\mathcal{B}(E)$ denote the Borel σ-algebra of subsets of E.

Proposition 2. *The following statements are equivalent*:
(i) $Y_n^* \Rightarrow Y^*$ *and the limit process is max-stable with respect to the multiplicative group* $\mathcal{U}_\alpha$ *defined in* (6);
(ii) $\mathcal{N}_n^* \Rightarrow \pi$ *and the limit point process* π *is a homogeneous Poisson point process whose structural measure* μ *does not charge instant spaces and* $\mu([0,t] \times A) = t\nu_\alpha(A)$ *for* $A \in \mathcal{B}(E)$;
(iii) $S_n^* \Rightarrow S^*$ *and the limit process is* α*-stable. Its Lévy measure* Π *satisfies*

$$\Pi(A) = \nu_\alpha(A), \qquad A \in \mathcal{B}(E), \qquad \Pi(\{0\}) = 0.$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is a special case of Proposition 3.21 in [8]. Recall that every max-ID extremal process (with d.f. g) is associated with a Poisson point process (with structural measure μ) and the connection between them is given by

$$g(t, x) = e^{-\mu([0,t]\times A_x^c)}$$

(cf. [1]). Let (T_k, Y_k^*), $k = 1, 2, \ldots$, be the points of π. Then
- $Y^*(t) = \{\vee Y_k^*\colon\ T_k \le t\}$ is max-ID $\Leftrightarrow$ π is Poisson;
- $Y^*(t)$ is stochastically continuous $\Leftrightarrow$ μ does not charge instant spaces, i.e., $\mu(\{t\} \times A) = 0$, $A \in \mathcal{B}(E)$;
- $Y^*(t)$ has homogeneous max-increments $\Leftrightarrow$

$$\mu([s,t] \times A) = (t - s)\nu_\alpha(A), \quad 0 \le s < t < \infty.$$

Note that ν_α is a Radon measure on E, i.e., finite on compact subsets far away from zero.

On (i) $\Rightarrow$ (iii). The process

$$S^*(t) = \left\{\sum Y_k^*\colon\ T_k \le t\right\}$$

is associated with the time-homogeneous Poisson point process π on $\mathcal{S}$, whose structural measure does not charge instants. Hence it is stochastically continuous. Further it has nonnegative independent increments. Thus for $\theta \in (0,\infty)^d$ its characteristic function $\varphi_t(\theta) := \mathbf{E}e^{i\langle\theta, S^*(t)\rangle}$ has the form

$$\varphi_t(\theta) = \exp\left\{t \int_E \{e^{i\langle\theta,x\rangle} - 1\}\,\Pi(dx)\right\}, \tag{9}$$

where the σ-finite Lévy measure Π has the properties

$$\int_{A_\mathbf{e}} \|x\|\,\Pi(dx) < \infty, \quad \Pi(\{0\}) = 0.$$

Further, Π is determined by the limit relation

$$k_n[1 - F^*(\sqrt[\alpha]{k_n}\,x)] \longrightarrow \Pi(A_x^c), \quad \forall x > 0, \quad n \to \infty.$$

By (7a) and since $\mathcal{B}(E)$ is generated by sets of the form A_x^c, $x > 0$,

$$\Pi(A) = \nu_\alpha(A), \quad \forall A \in \mathcal{B}(E).$$

The last limit relation together with the regularity of the tail $(1-F^*)$, expressed in (7b), is equivalent to the weak convergence

$$S_n^*(1) \sim \frac{1}{\sqrt[\alpha]{k_n}} \sum_1^{k_n} X_k^* \xrightarrow{\mathrm{d}} S^*(1), \quad n \to \infty, \tag{10}$$

where $S^*(1)$ is a one-sided α-stable r.v. (see, e.g., [9]) with $\alpha_i \in (0,1)$, $i = 1, \ldots, d$. From here and the asymptotic $k_n^*(t) \sim k_n t$ we get

$$S_n^*(t) \xrightarrow{\mathrm{d}} S^*(t), \quad \forall t > 0, \quad n \to \infty. \tag{11}$$

So $S^*(t)$, $t > 0$, is one-sided α-stable process with $\alpha_i \in (0,1)$ and $S^*(0) = 0$ a.s. In fact,

$$S^*(t) \overset{\mathrm{d}}{=} \sqrt[\alpha]{t}\, S^*(1).$$

The inverse implication (iii) ⇒ (i) is obvious.

Remarks. 1. Now the choice of α with $\alpha_i \in (0,1)$ is plausible: in this case $\vee X_k^*$ and $\sum X_k^*$ need the same scale normalization $\sqrt[\alpha]{k_n}$.

2. It is no surprise that the spectral measure Π of $S^*(1)$ and the exponent measure ν_α of $Y^*(1)$ coincide on $\mathcal{B}(E)$. By construction, $Y^*(1)$ is the largest jump of S^* in $[0,1]$ and $\Pi(A_x^c)$ is just the expected value of the number of jumps in $[0,1]$ larger than x (cf. [5, XI]). More interesting is that the dependence structure of the process $S^*(t)$ for all $t > 0$ is determined by the dependence structure of the maximal jump of S^* in $[0,1]$. Indeed, in the integral expression of the exponent measure

$$\nu_\alpha(A_x^c) = \int_{S_d^+} \max_{1 \le i \le d} \left(\frac{s_i}{x_i}\right)^{\alpha_i} Q(ds),$$

the dependence structure of the r.v. $Y^*(1) = (Y_1^*, \ldots, Y_d^*)$ is borne by Q. Here S_d^+ is the intersection of E and the unit sphere in $\mathbf{R}^d$, and Q is a finite Borel measure on S_d^+ (cf., e.g., [8]). In the case of full dependence, i.e., if $\mathbf{P}(Y_1^* = \cdots = Y_d^*) = 1$, ν_α is concentrated on the orbit $\{\sqrt[\alpha]{s}\,\mathbf{e}\colon s > 0\}$, respectively, Q is concentrated at the point $\mathbf{e}/\|\mathbf{e}\|$. Hence $\Pi(A_x^c) = \nu_\alpha(A_x^c) = \bigvee_{i=1}^d x_i^{-\alpha_i}$. In the case of independent marginals,

$$\mathbf{P}(Y_1^* < x_1, \ldots, Y_d^* < x_d) = \exp\left\{-\sum_{i=1}^d x_i^{-\alpha_i}\right\}.$$

So $\Pi(A_x^c) = \nu_\alpha(A_x^c) = \sum_{i=1}^d x_i^{-\alpha_i}$. Consequently, Q is discrete and concentrated on $\mathbf{e}_i$, $i = 1, \ldots, d$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith coordinate.

How Q reflects the dependence structure of the associated vector in $\mathbf{R}^d$ is in general a hard problem (cf. [3]).

Now we come back to our starting problem: to determine the limit behavior of the sum process in the initial model (A)

$$S_n(t) = \sum_{k=1}^{k_n(t)} u_n^{-1}(X_k), \quad n \to \infty.$$

By the multivariate Central Criterion of Convergence (CCC), this sum is convergent if and only if
(C_1) $k_n(t)[1 - \mathbf{P}(u_n^{-1}(X_k) < x)]$ converges weakly to a nonnegative, nonincreasing, and right-continuous function;
(C_2) $k_n(t)\mathbf{E}(u_n^{-1}(X_k) I\{u_n^{-1}(X_k) < v\})$ converges to a finite vector, say $a(t,v)$.

Below, we shall check these conditions in our case.

The limit relation (iii) of Proposition 2 implies

$$\mathbf{P}(S_n^*(t) < x) = \mathbf{P}\left\{\sum_{k=1}^{k_n^*(t)} \frac{1}{\sqrt[\alpha]{k_n}} T^{-1} \circ h_\alpha(X_k) < x\right\} \sim \mathbf{P}\left\{\sum_{k=1}^{k_n(\tau^{-1}(t))} h_\alpha \circ u_n^{-1}(X_k) < x\right\} \xrightarrow{w} \mathbf{P}(S^*(t) < x), \quad n \to \infty.$$

Here we have used the relation $T(\sqrt[\alpha]{k_n}\,x) \sim T_n(x) = h_\alpha \circ u_n \circ h_\alpha^{-1}(x)$. Hence necessarily we have for $t > 0$ and $x > q := \inf \operatorname{Supp} G$

$$k_n(t)[1 - \mathbf{P}(u_n^{-1}(X_k) < x)] \longrightarrow \tau(t)\Pi(A_{h_\alpha(x)}^c), \quad n \to \infty. \tag{12a}$$

Furthermore, the sequence of the truncated by $v > q$ mean of $u_n^{-1}(X_k)$ is formally convergent, i.e., for $A_v = \{y \in E: y < v\}$

$$k_n(t)\mathbf{E}\{u_n^{-1}(X_k)I(u_n^{-1}(X_k) < v)\} = k_n(t)\int\limits_{A_v} x\,d\mathbf{P}(u_n^{-1}(X_k) < x)$$

$$= -\int\limits_{A_v} x\,d\{k_n(t)(1 - \mathbf{P}(u_n^{-1}(X_k) < x))\} \xrightarrow{w} \tau(t)\int\limits_{A_{q,v}} x\,d\Pi(h_\alpha(x)), \qquad A_{q,v} := A_v \cap \{x > q\}. \tag{12b}$$

Consequently,

$$a(t,v) = \tau(t)\int\limits_{A_{q,v}} x\,d\Pi(h_\alpha(x)). \tag{13}$$

Here we have used that $\mathbf{P}(u_n^{-1}(X_k) < x) \to 0$ for $x < q$ and $k_n(t) \sim \tau(t)k_n$. Observe that $a(1,v) =: a(v)$ is zero if $v = q$.

Caution: With abuse of notation we denote the Lévy measure and its d.f. by the same letter Π. So, $\Pi(A_x^c) = \Pi(\infty) - \Pi(x) = -\Pi(x)$.

At this stage we have to clarify both of the following questions:
(a) Is the measure $\mathbf{\Psi}(A) := \Pi(h_\alpha(A))$ a spectral measure (here $h_\alpha(A) =: \{h_\alpha(x): x \in A\}$)?
(b) Is $a(v)$ well defined, i.e., $a(v) < \infty$?

Note that $\Pi(A^c_{h_\alpha(x)}) = -\log G(x)$. The d.f. $\Psi(x)$ of the measure $\mathbf{\Psi} = \Pi \circ h_\alpha$ is defined by $\mathbf{\Psi}(A_x^c) = \Psi(\infty) - \Psi(x)$, i.e., $\Psi(x) = \log G(x)$. Thus, it possesses the following properties:
(1) it is nondecreasing in each component;
(2) $\Psi(\infty) = 0$;
if, additionally,
(3) $\int\limits_{A_{q,v}} \|x\|\,d\Psi(x)$ is finite,
then $\mathbf{\Psi}(A)$ is the Lévy measure of an infinitely divisible random vector whose characteristic function has the form (9). Thus, questions (a) and (b) are positively answered if

$$I_G = \int\limits_{A_{q,v}} \|x\|\,d(\log G(x)) < \infty. \tag{14}$$

Obviously, no max-stable d.f. $G(x)$ satisfies condition (14). (Recall in $\mathbf{R}^1$ each continuous and strictly increasing d.f. is max-stable with respect to a certain one-parameter norming group (cf. [7]).)

Example 3. $G(x) = e^{-x^{-\alpha}}$ for $x > 0$ and $\alpha > 0$ is a univariate max-stable d.f. with respect to the group $\mathcal{L} = \{\mathbf{L}_t(x) = x\sqrt[\alpha]{t}: t > 0\}$, since $G^t(x) = G(\mathbf{L}_t^{-1}(x))$, $\forall t > 0$.
(a) Let $0 < \alpha < 1$. In this case, $I_G = a(v) = (v^{1-\alpha})/(1-\alpha) < \infty$.
(b) Let $\alpha \geq 1$. Here I_G is infinite, so the corresponding measure $\mathbf{\Psi}(A)$ is not a Lévy measure of a distribution of the kind (9).

Now let us come back to conditions (12) with Lévy measure $\mathbf{\Psi} := \Pi \circ h_\alpha$ and finite $a(v)$. They are equivalent to the convergence $S_n(t) \xrightarrow{\text{d}} S(t)$, $t > 0$, $n \to \infty$. The limit process $\{S(t): t > 0\}$ has nonnegative and independent increments and is nonhomogeneous in the general case. The characteristic function of $S(t)$ is expressed by

$$\mathbf{E}e^{i\langle\theta,S(t)\rangle} = \exp\left\{\tau(t)\int\limits_{[q,\infty]\setminus\{q\}} (e^{i\langle\theta,x\rangle} - 1)\,d\Pi(h_\alpha(x))\right\} = \exp\left\{\tau(t)\int\limits_{E} (e^{i\langle\theta,h_\alpha^{-1}(x)\rangle} - 1)\,d\Pi(x)\right\}. \tag{15}$$

Note the shift parameter here is zero, because the limit of the truncated means in C_2 is just $\tau(t)\int_{A_{q,v}} x\,d\Pi(h_\alpha(x))$. Recall, in the general case, that the shift parameter $\gamma(t)$ is $a(t,v) - \int_{A_{q,v}} x\,d\Psi_t(x)$ and does not depend on v.

From (15) one can see that the Lévy measure $\mathbf{\Psi}_t$ of $S(t)$ admits the factorization $d\mathbf{\Psi}_t(h_\alpha^{-1}(x)) = \tau(t)\,d\Pi(x)$.

In this way, we complete the proof of our main theorem, formulated in Sec. 1.

Example 4. Consider the point process $\mathcal{N} = \{(t_k, X_k): k \in \{1, 2, \ldots\}\}$, where $t_k = k(k+1)/2$ and X_k are i.i.d. r.v's with d.f.

$$F(x) = e^{-(\log x)^{-\alpha}}, \quad x \in [1,\infty), \quad 0 < \alpha < 1.$$

The distribution function F is max-stable with respect to the norming group $\{\mathbf{L}_t(x) = x^{1/\sqrt[\alpha]{t}}: t > 0\}$. Indeed,

$$F^t(x) = \exp\{-t(\log x)^{-\alpha}\} = \exp\{-(\log x^{1/\sqrt[\alpha]{t}})^{-\alpha}\} = F(x^{1/\sqrt[\alpha]{t}}).$$

The sequence of the following time-space changes

$$\zeta_n(t,x) = (\tau_n(t), u_n(x)) = (n^2 t, (x+1)^{\sqrt[\alpha]{n}})$$

satisfies the conditions of the Theorem 1, namely,
(i) it is regular:

$$u_{[ns]}^{-1} \circ u_n(x) \longrightarrow (x+1)^{1/\sqrt[\alpha]{s}} - 1 =: \mathbf{L}_s(x), \quad \forall s > 0, \quad n \to \infty$$
$$\tau_{[sn]}^{-1} \circ \tau_n(t) \longrightarrow ts^{-2} = \tau_s(t), \quad \forall s > 0, \quad n \to \infty;$$

(ii) the random variables $X_{nk} = u_n^{-1}(X_k) = X_k^{1/\sqrt[\alpha]{n}} - 1$ are asymptotically negligible, i.e.,

$$\mathbf{P}(X_k^{1/\sqrt[\alpha]{n}} - 1 > x) = 1 - \sqrt[n]{F(x+1)} \longrightarrow 0, \quad \forall x > 0, \quad n \to \infty;$$

(iii) the sequence of extremal processes $Y_n(t) = \bigvee_{n=1}^{k_n(t)} u_n^{-1}(X_k)$ is weakly convergent for $n \to \infty$.
Indeed, since

$$k_n(t) = \sum_k I\left\{\frac{k(k+1)}{2n^2} \in [0,t]\right\} \sim n\sqrt{t} \quad \text{for } n \to \infty,$$

we have

$$\mathbf{P}(Y_n(t) < x) \sim F^{n\sqrt{t}}(u_n(x)) = F^{\sqrt{t}}(x+1) := g(t,x),$$

$F^n(u_n(x)) = F(x)$ since $F \in \mathrm{MS}$ with respect to $\{u_n\}$. Put $G(x) := g(1,x)$. Then $\mathbf{P}(Y_n(t) < x) \to G^{\sqrt{t}}(x)$, $n \to \infty$, and the limit d.f. F is MS with respect to the one-parameter group $\{\mathbf{L}_t\colon t > 0\}$ defined in (i), i.e., $G^t(x) = G(\mathbf{L}_t^{-1}(x))$, $\forall t > 0$, $\forall x > 0$.

Furthermore, $G(x)$ satisfies (14), since $\Psi(x) = \log G(x) = -(\log(x+1))^{-\alpha}$ and

$$\int_0^v x\, d(\log G(x)) = \int_0^v x\, d(\log(x+1))^{-\alpha} < \infty.$$

One can observe that the process $Y^*(t)$ has d.f. Φ_α^t, where

$$Y^*(t) := h_\alpha \circ Y \circ \tau^{-1}(t) = \log(Y(t^2) + 1)$$

with $\tau(t) = \sqrt{t}$ and $h_\alpha(x) = \log(x+1)$, $\forall x > 0$. Indeed,

$$\mathbf{P}(Y^*(t) < x) = \mathbf{P}(Y(t^2) < h_\alpha^{-1}(x)) = F^t(e^x) = e^{-x^{-\alpha}}.$$

Now, by Theorem 1,

$$\sum_{k=1}^{k_n(t)} ((X_k)^{1/\sqrt[\alpha]{n}}) \xrightarrow{\mathrm{d}} S(t), \quad n \to \infty.$$

The characteristic function of $S(t)$ is

$$\exp\left\{\int_0^\infty (e^{i\theta x} - 1)\, d\mathbf{\Psi}_t(x)\right\},$$

where

$$d\mathbf{\Psi}_t(x) = \sqrt{t}\, d\Pi(h_\alpha(x)) = \frac{\sqrt{t}(-\alpha)\, dx}{(x+1)(\log(x+1))^{1+\alpha}}, \quad x > 0,$$

as $\Pi(A^c_{\log(x+1)}) = [\log(x+1)]^{-\alpha}$.

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