Functional Transfer Theorems for Maxima of iid Random Variables *

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Abstract

This note discusses limit theorems for a sequence of extremal processes associated with a Bernoulli point process with random time and space points, and a regular norming sequence of time-space changes.

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The framework of our study is set by a given Bernoulli point process (Bpp) $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ on the time-state space $\mathcal{S} = (0, \infty) \times (0, \infty)$. By definition (cf. [1]) $\mathcal{N}$ is simple in time ($T_k \neq T_j$ a.s. for $k \neq j$), its mean measure is finite on compact subsets of $\mathcal{S}$ and all restrictions of $\mathcal{N}$ to slices over disjoint time intervals are independent. We assume that:
a) the sequences $\{T_k\}$ and $\{X_k\}$ are independent and defined on the same probability space;

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b) the state points $\{X_k\}$ are independent and identically distributed random variables (iid rv's), on $(0,\infty)$ with common distribution function (df) F which is asymptotically continuous at infinity;
c) the time points $\{T_k\}$ are increasing to infinity, i.e. $0 < T_1 < T_2 < ..., T_k \to \infty$ a.s.

The main problem in the Extreme Value Theory is the asymptotic of the extremal process $\{\bigvee_k X_k : T_k \leq t\} = \bigvee_{k=1}^{N(t)} X_k$, associated with $\mathcal{N}$, for $t \to \infty$. Here the maximum operation is denoted by " $\vee$ " and $N(t) := max\{k : T_k \leq t\}$ is the counting process of $\mathcal{N}$. The method usually used is to choose proper time-space changes $\zeta_n = (\tau_n(t), u_n(x))$ of $\mathcal{S}$, strictly increasing and continuous in both components, such that for $n \to \infty$ and $t > 0$ the weak convergence

$$\tilde{Y}_n(t) := \{\bigvee_k u_n^{-1}(X_k) : \tau_n^{-1}(T_k) \leq t\} \Longrightarrow \tilde{Y}(t) \tag{1}$$

to a non-degenerate extremal process holds. (For weak convergence of extremal processes consult e.g. [1], th 7.)

In fact, the classical Extreme Value Theory deals with Bpp's $\{(t_k, X_k) : k \geq 1\}$ with deterministic time points t_k, $0 < t_1 < t_2 < ...$, $t_k \to \infty$. One investigates the weak convergence to a non-degenerate extremal process

$$Y_n(t) := \{\bigvee_k u_n^{-1}(X_k) : t_k \leq \tau_n(t)\} \Longrightarrow Y(t) \tag{2}$$

under the assumption that the norming sequence $\{\zeta_n\}$ is regular. The later means that for all $s > 0$ and for $n \to \infty$ there exist pointwise

$$\lim_{n\to\infty} u_n^{-1} \circ u_{[ns]}(x) = \mathbf{U}_s(x)$$

$$\lim_{n\to\infty} \tau_n^{-1} \circ \tau_{[ns]}(t) = \sigma_s(t)$$

and the time-space change $(\sigma_s(t), \mathbf{U}_s(x))$ is strictly increasing and continuous in t, x and s. As usual " $\circ$ " means the composition and $[s]$ the integer part of s.

Let us denote the (deterministic) counting function $k(t) = max\{k : t_k \leq t\}$, and put $k_n(t) := k(\tau_n(t))$, $k_n := k_n(1)$. The df of the limit extremal process in (2) we denote by $g(t,x) := \mathbf{P}(Y(t) < x)$, and set $G(x) := g(1,x)$. Then necessary and sufficient conditions for convergence (2) are the following

1. $F^{k_n}(u_n(x)) \xrightarrow{w} G(x), \quad n \to \infty$

2. $\frac{k_n(t)}{k_n} \longrightarrow \lambda(t), \quad n \to \infty, \quad t > 0.$

The regularity of the norming sequence $\{\zeta_n\}$ has some important consequences (cf. [2]):

0. $\frac{k_{[ns]}}{k_n} \longrightarrow s^a, n \to \infty$, for some $a > 0$ and all $s > 0$;

1'. the limit df G is max-stable in the sense that

$$G^s(x) = G(L_s^{-1}(x)) \quad \forall s > 0, \quad L_s := \mathbf{U}_{\sqrt[a]{s}}; \tag{3}$$

2'. the intensity function $\lambda(t)$ is continuous.

Thus, under conditions 1. and 2. and the regularity of the norming sequence, the limit extremal process $Y(t)$ is stochastically continuous with df $g(t,x) = G^{\lambda(t)}(x)$ and the process $Y \circ \lambda^{-1}(t)$ is max-stable in the sense of (3).

Let us come back to the point process $\mathcal{N}$ with the random time points T_k. The Functional Transfer Theorem (FTT) in this framework gives conditions on $\mathcal{N}$ for the weak convergence (1) and determines the explicit form of the limit df $f(t,x) := \mathbf{P}(\tilde{Y}(t) < x)$. In other words, the weak convergence (2) in the framework with non-random time points can be transfer to the framework of $\mathcal{N}$ if some additional conditions on the point process $\mathcal{N}$ are met. In our case these are conditions d) and 3. below.

Denote by $\mathcal{M}([0,\infty))$ the space of all strictly increasing, cadlac functions $y : [0,\infty) \to [0,\infty)$, $y(0) = 0$, $y(t) \to \infty$ as $t \to \infty$.

We assume additionally to a) - c) the following condition
d) there exists a stochastically continuous time process $\theta(t)$ with sample paths in $\mathcal{M}([0,\infty))$ such that $N(t) = k(\theta(t))$ a.s., where $k(t)$ is a deterministic counting function with $k(0) = 0$ and $k(t) \uparrow \infty$ for $t \uparrow \infty$.

In the most cases the random counting function N is determined by the model we have to work in and the deterministic counting function k is known from the classical limit theory for extremes.

Proposition: *Let $\{(T_k, X_k) : k \geq 1\}$ and $\{(t_k, X_k) : k \geq 1\}$ be Bpp's with counting functions $N(t)$ and $k(t)$, respectively. Then there always exists a time process θ which satisfies d).*

Proof: Denote by $Q_t(s) := \mathbf{P}(\theta(t) < s)$. Then for $n = 0, 1, 2, ...$ and $t_0 = 0$

$$\begin{aligned} \mathbf{P}(N(t) = n) &= \sum_{k=0}^{\infty} \mathbf{P}(k(\theta(t)) = n, \quad t_k \leq \theta(t) < t_{k+1}) = \\ &= \mathbf{P}(t_n \leq \theta(t) < t_{n+1}) = Q_t(t_{n+1}) - Q_t(t_n). \end{aligned}$$

By this iteration formula we obtain the values of $Q_t(s)$ for $s \in \Gamma = \{t_0, t_1, t_2, ...\}$. For $s \notin \Gamma$ we can interpolate $Q_t(s)$ by preserving the properties required in d), e.g. linearly. ■

Now we are ready to state a general FTT for maxima of iid rv's on $(0, \infty)$. We set $N_n(t) := N(\tau_n(t))$.

Theorem 1 (FTT): *Let $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ be a Bpp described by conditions a) - d). Assume further that there is a regular norming sequence $\zeta_n(t, x) = (\tau_n(t), u_n(x))$ of time-space changes of $\mathcal{S}$ such that for $n \to \infty$ and $t > 0$ conditions 1., 2. and*
3. $\theta_n := \tau_n^{-1} \circ \theta \circ \tau_n \Longrightarrow \Lambda$, in $\mathcal{M}([0,\infty))$
hold. Then
i) $\mathbf{P}(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x) \xrightarrow{w} \mathbf{E}[G(x)]^{\lambda \circ \Lambda(t)}$
ii) $\frac{N_n(t)}{k_n} \xrightarrow{d} \lambda \circ \Lambda(t)$

Proof: i) One can express the partial maxima $\tilde{Y}_n(t)$ as

$$\tilde{Y}_n(t) = \bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) = \bigvee_{k=1}^{k_n(\theta_n(t))} u_n^{-1}(X_k) = Y_n \circ \theta_n(t).$$

Since Y_n converges weakly to a stochastically continuous extremal process Y and since θ_n and Λ are time processes with sample paths in $\mathcal{M}([0,\infty))$, the composition $Y_n \circ \theta_n$ is continuous under the weak convergence (cf. [3] th 3). Thus $Y_n \circ \theta_n \Longrightarrow Y \circ \Lambda$ and

$$\mathbf{P}(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x) \longrightarrow f(t,x) = \mathbf{P}(Y \circ \Lambda(t) < x) =$$

$$\int_0^\infty [g(1,x)]^{\lambda(s)} dQ_t(s) = \mathbf{E}[G(x)]^{\lambda \circ \Lambda(t)}$$

Note, (3) implies the stochastic continuity of Λ.
ii) $\mathbf{P}(\theta_n(t) < s) = \mathbf{P}(\frac{k_n(\theta_n(t))}{k_n} \leq \frac{k_n(s)}{k_n}) =$

$$= \mathbf{P}(\frac{N_n(t)}{k_n} \leq \lambda(s) + [\frac{k_n(s)}{k_n} - \lambda(s)]) \xrightarrow{w} \mathbf{P}(\Lambda(t) < s).$$

Since λ is continuous, convergence 2. is uniform and we conclude for $n \longrightarrow \infty$ and $\lambda(s) = u$ that

$$\mathbf{P}(\frac{N_n(t)}{k_n} \leq u) \xrightarrow{w} \mathbf{P}(\lambda \circ \Lambda(t) \leq u).$$

■

As a by-product one notices the following

Corollary: *The convergences 2. and 3. imply ii).*

Below we give some special cases of FTT. For $\alpha > 0$ we denote the Frechet df by $\Phi_\alpha(x) = e^{-x^{-\alpha}}$ for $x > 0$, and $= 0$ otherwise. Set $X_{nk} := \frac{x_k}{B_n}$ with $B_n \sim \sqrt[\alpha]{n} L(n)$. Here $L(n)$ is a slowly varying function. We note that the norming sequences $\tau_n(t) = nt$ and $u_n(x) = B_n x$, proper in the case when $F \in max - DA(\Phi_\alpha)$, are regular. Just analogously to Theorem 1 one obtain the next two statements.

Theorem 2: (cf.[4], th 1) *Let* $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ *be a Bpp described by conditions a) - c). We assume*

1. $\bigvee_{k=1}^{n} X_{nk} \xrightarrow{d} Y_\alpha(1)$ *with df* Φ_α*;*

2. $k_n(t) \sim nt$ *(i.e.* $\lambda(t) = t$*);*

3. $\frac{N_n(t)}{n} \Longrightarrow \lambda t, \quad \lambda > 0$ *(i.e.* $\Lambda(t) = \lambda t$*).*

Then $\tilde{Y}_n(t) = \bigvee_{k=1}^{N_n(t)} X_{nk} \Longrightarrow \tilde{Y}(t) = Y_\alpha(\lambda t)$ *and* $Y_\alpha(\lambda t) \stackrel{d}{=} \lambda^{1\backslash\alpha} Y_\alpha(t)$.

Theorem 3: *Let* $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ *be a Bpp described by conditions a) - d). Assume further that*

1. $\bigvee_{k=1}^{n} X_{nk} \xrightarrow{d} Y_\alpha(1)$ *with df* Φ_α*;*

2. $k_n(t) \sim nt^\beta$ *(i.e.* $\lambda(t) = t^\beta$*);*

3. $\theta_n \Longrightarrow \Lambda$*, in* $\mathcal{M}([0, \infty))$*.*

Then: i) $\frac{N_n(t)}{n} \Longrightarrow \Lambda^\beta(t)$

ii) $\bigvee_{k=1}^{N_n(t)} X_{nk} \Longrightarrow \tilde{Y}(t) = Y_\alpha \circ \Lambda^\beta(t)$ *and* $\tilde{Y}(st) \stackrel{d}{=} s^{\beta\backslash\alpha} \tilde{Y}(t) \quad \forall s > 0$.

Let us below denote the one-sided α-stable Levy motion by $S_\alpha(t)$. We recall from the classical fluctuations theory for sums and maxima of iid rv's on $[0, \infty)$ that the following statements are equivalent (cf. e.g. [5]):

"$1 - F$ is regularly varying with index $-\alpha$, $\alpha > 0$ (i.e. $\bar{F} \in RV_{-\alpha}$)";

"$\bigvee_{k=1}^{[nt]} X_{nk} \Longrightarrow Y_\alpha(t)$ with df Φ_α^t (i.e. $F \in max - DA(\Phi_\alpha)$)";

"$\sum_{k=1}^{[nt]} X_{nk} \Longrightarrow S_\alpha(t)$ (i.e. $F \in DA(\mathcal{L}(S_\alpha))$), where $\mathcal{L}(S_\alpha)$ denotes the law of $S_\alpha(1)$".

The following result is due to P. Jordanova (2003). Assume that $B_n \sim \sqrt[\alpha]{n}L_1(n)$, $b_n = b(n) \sim \sqrt[\beta]{n}L_2(n)$, $\alpha, \beta \in (0,1)$ and determine $\tilde{b}_n = \tilde{b}(n)$ by the asymptotic relation $b(\tilde{b}(n)) \sim n$.

Theorem 4: *Let*

1. $\bar{F} \in RV_{-\alpha}$, *i.e.* $Y_n(t) := \bigvee_{k=1}^{[nt]} \frac{X_k}{B_n} \Longrightarrow Y_\alpha(t)$;

2. $T_n = \sum_{k=1}^{n} J_k$ where $\{J_k\}$ are iid rv's on $(0,\infty)$ which df $G \in DA(\mathcal{L}(S_\beta))$, i.e. for $n \to \infty$ $\sum_{k=1}^{[nt]} \frac{J_k}{b_n} \Longrightarrow S_\beta(t)$.

Then

i) $\frac{N(nt)}{\tilde{b}(n)} \longrightarrow E(t) := inf\{s : S_\beta(s) > t\}$ the hitting time process of $S_\beta(t)$;

ii) $\bigvee_{k=1}^{N(nt)} \frac{X_k}{C_n} \Longrightarrow \tilde{Y}(t) = Y_\alpha(E(t))$ where $C_n \sim B(\tilde{b}_n)$ and

$$\tilde{Y}(st) \stackrel{d}{=} s^{\beta\backslash\alpha}\tilde{Y}(t) \quad s > 0. \tag{4}$$

Proof: Statement i) is proved by Meerschaert M.M. and Scheffler P.H. (2002) in Limit Theorems For Continuous Time Random Walks (submitted). For ii) we observe that

$$\bigvee_{k=1}^{N(nt)} \frac{X_k}{C_n} = \bigvee_{k=1}^{\frac{N(nt)}{\tilde{b}(n)}\tilde{b}(n)} \frac{X_k}{C_n} = Y_{\tilde{b}(n)}(\frac{N(nt)}{\tilde{b}(n)})$$

Since the sample paths of neither $E(t)$ nor $\frac{N(nt)}{\tilde{b}(n)}$ are in $\mathcal{M}([0,\infty))$ one can not at once use the continuity property of the composition. But the processes $S_\beta(t)$ and $Y_\alpha(t)$ have a.s. no simultaneous jumps (because of their stochastic continuity and independence). In this case one is allowed to apply again the continuity of composition (cf. e.g. [3], th. 3) and thus

$$Y_{\tilde{b}(n)}(\frac{N(nt)}{\tilde{b}(n)}) \Longrightarrow Y_\alpha \circ E(t).$$

The $\frac{\beta}{\alpha}$ - selfsimilarity property (4) is a direct consequence of $\frac{1}{\alpha}$ - selfsimilarity of Y_α and β - selfsimilarity of the hitting time process $E(t)$. ∎

Limit theorems for maxima of random number of rv's appear in many theoretical and applied works. The first Transfer Theorem for extremes seems to be the one in Gnedenko B.V. and Gnedenko D.B. [6]. The recent one we like to mention is the paper [7] studying the infinite divisibility of a random time-changed process.

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