Sum and Extremal Processes over Explosion Area

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Abstract

Let $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}), k \geq 1\}$ be a Bernoulli p.p. on $Z = (0, \infty) \times [0, \infty)^d$. We discuss weak limit theorems for $\mathcal{N}_n$ as well as for the associated sum and extremal processes $\mathbf{S}_n(t) = \{\sum \mathbf{X}_{nk} : T_{nk} \leq t\}$ and $\mathbf{Y}_n(t) = \{\vee \mathbf{X}_{nk} : T_{nk} \leq t\}$ on an open subset of Z.

Key Words and Phrases: point processes; increasing processes; weak convergence; random sample size

1 Insurance Model Interpretation

An insurance model can be interpreted as point process $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ on particular time-state space: the time components T_k mark the customers' claim arrival times and the state components $\mathbf{X}_k$ represent the claim sizes. We suppose that
a) the claim arrival times $\{T_k\}$ are distinct positive, increasing to infinity random variables;
b) the claim sizes $\{\mathbf{X}_k\}$ are independent random vectors in $[0,\infty)^d$;
c) both sequences $\{T_k\}$ and $\{\mathbf{X}_k\}$ are independent.

For risk management purposes one is interested in the behaviour of the total claim amount process $\mathbf{S}(t) = \sum_{\{k:T_k \leq t\}} \mathbf{X}_k$ for large t. In fact, it is related with the behaviour of the extremal claim process

$$\mathbf{Y}(t) = \mathbf{C}(t) \vee \{\vee_k \mathbf{X}_k : T_k \leq t\} \tag{1}$$

Here $\mathbf{C} : [0,\infty) \to [0,\infty)^d$ is the lower curve of $\mathbf{Y}$. It is defined coordinatewise $C^{(i)}(t) = \inf\{x : \mathbf{P}(Y^{(i)}(t) < x) > 0\}$ as the left endpoint of the support of $Y^{(i)}(t), i = 1,\ldots,d$. The lower curve is right continuous and increasing. The last notion we use here and below in the sense of non-decreasing. We understand the maximum operation $(\vee)$ coordinatewise. All notions on multivariate extremal processes we use here are stated in Balkema and Pancheva (1996) and in the unpublished paper "Convergence of multivariate extremal processes" (2000) by the same authors. The later we refer to as BP(2000).

The above insurance model interpretation gives us a reason for a pure theoretical investigation of the relationship $\mathcal{N} \leftrightarrow \mathbf{Y}$ between the point process and the associated extremal process and also the relationship $\mathbf{S} \leftrightarrow \mathbf{Y}$ between the extremal and the sum process. The method we use is to transform properly the time-state space $Z = (0,\infty) \times [0,\infty)^d$ so that the number of claims in any interval $[0,t]$ gets larger and the claim sizes get smaller. In this way we achieve a sequence of point processes (p.p.'s) on Z, $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}) : k \geq 1\}, n \geq 1$ satisfying the above assumptions a) - c). Denote by $\mu_n(\cdot)$ the mean measure $\mathbb{E}\mathcal{N}_n$ of $\mathcal{N}_n$.

Remark 1. In Balkema and Pancheva (1996) the authors introduce Bernoulli point process $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ as time-space point process defined on a locally compact metric space $\mathcal{S}$ and satisfying the conditions:
1) its mean measure μ is Radon measure on $\mathcal{S}$, i.e. it is finite on compact subsets of $\mathcal{S}$;
2) it is simple in time: $T_i \neq T_j$ a.s. for $i \neq j$;
3) restrictions of $\mathcal{N}$ to slices over disjoint time intervals are independent random variables.
One can see that our p.p.'s $\mathcal{N}_n$, satisfying a) - c) are in fact Bernoulli p.p.'s.

Our paper is organized as follows:
In Section 2 we discuss the convergence notions of the three stochastic objects: point process, extremal process and sum process with positive independent increments. In Section 3 we introduce the accompanying processes and use them in the proof of our main results - Theorem 3, 4 and 5. In the last Section 4 we explain how we understand a sum process over explosion area.

2 Vague convergence of point processes and weak convergence of increasing processes

Let $\mathcal{S}$ be an open subset of Z, hence it is locally compact. We say a sequence of p.p.'s $\mathcal{N}_n$, defined on Z, is vaguely convergent to a p.p. $\mathcal{N}$ on $\mathcal{S}$, briefly $\mathcal{N}_n \stackrel{v}{\to} \mathcal{N}$, if for arbitrary relatively compact subset $K \subset \mathcal{S}$ with $\mathbb{P}(\mathcal{N}(\partial K) = 0) = 1$ the convergence $\mathcal{N}_n(K) \stackrel{d}{\to} \mathcal{N}(K)$ holds. If $\mathcal{N}$ is simple in time then $\mathcal{N}_n \stackrel{v}{\to} \mathcal{N}$ is equivalent to the weak convergence $\mathcal{N}_n \Rightarrow \mathcal{N}$ (see Th.14.16 in Kallenberg (1997)). Below we denote by $K(t)$ the instant section of K at time t. The following statement from BP(2000) gives conditions for the vague convergence on $\mathcal{S}$ of a sequence $\mathcal{N}_n$. Its proof is included in our forthcoming paper "Relationship between extremal and sum processes generated by the same point process", briefly PVM(2006).

Theorem 1. *Suppose $\mathcal{N}_n$ is a Bernoulli point process on an open subset $\mathcal{S} \subset Z$ with mean measure μ_n, for $n \geq 1$. Let μ be a Radon measure on $\mathcal{S}$. If*

$$\mu_n \stackrel{v}{\to} \mu \text{ on } \mathcal{S} \tag{i}$$

and

$$\sup\{\mu_n(K(s)) : 0 \leq s \leq t\} \to 0 \text{ as } n \to \infty \tag{ii}$$

for every $t > 0$ and every relatively compact subset $K \subset \mathcal{S}$, then the sequence $\mathcal{N}_n$ converges vaguely to a Poisson p.p. $\mathcal{N}$ on $\mathcal{S}$ with mean measure μ.

An extremal process $\mathbf{Y}$ with lower curve $\mathbf{C}$ is infinitely divisible w.r.t. the operation maximum, or briefly max-id, if $\forall n > 1$ there exist n i.i.d. extremal processes $\mathbf{Y}_{nk}$, with the same lower curve $\mathbf{C}, k = 1, \ldots, n$ such that $\mathbf{Y} = \mathbf{Y}_{n1} \vee \ldots \vee \mathbf{Y}_{nn}$. Moreover necessary and sufficient condition for max-infinitely divisibility of $\mathbf{Y}$ is the generating point process $\mathcal{N}$ to be Poisson on the set $\mathcal{S} = [0, \mathbf{C}]^c$, see Balkema and Pancheva (1996). In this case there is a close relation between the distribution function (d.f.) $f(t, \mathbf{x}) = \mathbb{P}(\mathbf{Y}(t) < \mathbf{x})$ of the extremal process $\mathbf{Y}$ generated by $\mathcal{N}$ via (1) and the mean measure μ of $\mathcal{N}$, which is given by

$$f(t, \mathbf{x}) = \begin{cases} \exp\{-\mu([0, t] \times [0, \mathbf{x})^c)\}, & (t, x) \in \mathcal{S} \\ 0, & \textit{otherwise}. \end{cases} \tag{2}$$

Obviously $\mu < \infty$ if $f > 0$, i.e. if $\mathbf{x} > \mathbf{C}(t)$. The closed set $[0, \mathbf{C}]$ below the lower curve $\mathbf{C}$ is an "explosion area" for the mean measure μ, i.e. μ is not finite there.

For example, every stochastically continuous extremal process is max-id. The mean measure μ of the associated Poisson p.p. does not charge instant spaces, i.e. $\mu(\mathcal{S}(t)) = 0, \forall t \geq 0$. Furthermore, $\mu([0, t] \times [0, \mathbf{x})^c) = \infty$ whenever $\mathbf{x}$ lies between the left and the right value of $\mathbf{C}$ at a discontinuity point t, i.e. $\mathbf{x} \in [\mathbf{C}(t - 0), \mathbf{C}(t)]$.

Let $\mathcal{M}([0, \infty))$ be the space of all increasing right-continuous functions $y : [0, \infty) \to [0, \infty)^d$, equipped with the topology of the weak convergence: $y_n \stackrel{w}{\to} y$ if $y_n(t) \to y(t)$ for all continuity points of the limit function. The last set we denote by $C(y)$. The sample paths of extremal processes $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ and sum processes $\mathbf{S} : [0, \infty) \to [0, \infty)^d$ with positive independent increments belong to $\mathcal{M}([0, \infty))$. Below we call them "increasing processes" and denote by $\mathbf{V}$. The following result is well known.

Theorem 2. *Assume $\mathbf{V}, \mathbf{V}_1, \mathbf{V}_2, \ldots : [0, \infty) \to [0, \infty)^d$ are increasing processes, such that $\mathbf{V}_n(t) \stackrel{d}{\to} \mathbf{V}(t)$ for all $t \in C(\mathbf{V})$. Then $\mathbf{V}_n \Rightarrow \mathbf{V}$.*

Indeed, as known, $\mathcal{M}([0,\infty))$ is closed w.r.t. the weak convergence and the set $\mathcal{P}$ of all probability measures on $\mathcal{M}([0,\infty))$ is relatively compact (c.f. Corollary 12.5.1., Whitt (2002)). Denote by $\mathcal{P}_I$ the subset of $\mathcal{P}$ corresponding to the increasing processes $\mathbf{V} : [0,\infty) \to [0,\infty)^d$. Using the fact that the set of all components of a distribution on $[0,\infty)^d$ is relatively compact (see §6, Zolotarev(1997)) one can proof (c.f. Th.4 in Balkema and Pancheva (1996)) that $\mathcal{P}_I$ is closed w.r.t. the weak topology, hence it is relatively compact too, and any sequence $\mathbf{V}_n$ of increasing processes is tight. Thus $\mathbf{V}_n \stackrel{f.d.d.}{\longrightarrow} \mathbf{V}$ implies $\mathbf{V}_n \Rightarrow \mathbf{V}$. Moreover, as we are dealing with processes with independent increments the convergence $\mathbf{V}_n(t) \stackrel{d}{\to} \mathbf{V}(t)$, for each $t \in C(\mathbf{V})$ implies $\mathbf{V}_n \stackrel{f.d.d.}{\longrightarrow} \mathbf{V}$, hence $\mathbf{V}_n \Rightarrow \mathbf{V}$.

3 Accompanying processes

Let us come back to our sequence of p.p.'s $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}), k \geq 1\}, n \geq 1$ on Z. The corresponding counting process $N_n(t) = \max\{k : T_{nk} \leq t\}$ is a.s. finite in view of condition a). The generated by $\mathcal{N}_n$ sum and extremal processes we denote by $\mathbf{S}_n(t) = \sum_{j=1}^{N_n(t)} \mathbf{X}_{nj}$ and $\mathbf{Y}_n(t) = \bigvee_{j=1}^{N_n(t)} \mathbf{X}_{nj}$. For a seek of simplicity we assume that the lower curve of $\mathbf{Y}_n$ is $\mathbf{C}_n \equiv \mathbf{0}, \forall n \geq 1$. For studying the asymptotic behaviour of both processes we make the following

Basic Assumption: *For every $n \geq 1$ there exists a deterministic counting function k_n and a random time change θ_n such that*

$$N_n(t) = k_n(\theta_n(t)) \ a.s. \ \forall t \geq 0 \tag{BA}$$

Recall, a random time change is stochastically continuous and strictly increasing process $\theta : (0,\infty) \to (0,\infty)$ with $\theta(0) = 0$ and $\theta(t) \to \infty$ for $t \to \infty$. A given counting function k_n determines uniquely an associated sequence of deterministic distinct time points $0 < t_{n1} < t_{n2} < \ldots$ such that $k_n(t) = \max\{k : t_{nk} \leq t\}$ is finite for $t \geq 0$. Furthermore, given both counting processes N_n and k_n, the random time change θ_n can be determined uniquely at $t_{n1}, t_{n2}, \ldots$ and defined piecewise linear between them (e.g. Pancheva, Yordanova (2004)). Now the basic assumption reads

$$\sum_k \delta_{T_{nk}}([0,t]) = \sum_k \delta_{t_{nk}}([0,\theta_n(t)]) \quad a.s.$$

In our model described by a) - c) the deterministic counting function k_n is not arbitrary but such that guarantees the weak convergence

$$\sum_{j=1}^{k_n(t)} \{1 - \mathbf{P}(\mathbf{X}_{nj} < \mathbf{x})\} \stackrel{w}{\longrightarrow} \mu([0,t] \times [\mathbf{0},\mathbf{x})^c), \quad (t,\mathbf{x}) \in \mathcal{S}$$

where μ is Radon measure on $\mathcal{S}$.

Definition 1. *A simple in time point process $\mathcal{N}_n^{(a)} = \{(t_{nk}, \mathbf{X}_{nk}), k \geq 1\}$ which state components are the same as these of $\mathcal{N}_n$ and which time components are related to the time components of $\mathcal{N}_n$ by the basic assumption is called below "accompanying point process." Analogously, the generated by $\mathcal{N}_n^{(a)}$ sum and extremal processes $\mathbf{S}_n^{(a)}(t) = \sum_{j=1}^{k_n(t)} \mathbf{X}_{nj}$ and $\mathbf{Y}_n^{(a)}(t) = \bigvee_{j=1}^{k_n(t)} \mathbf{X}_{nj}$ we call "accompanying sum" and "accompanying extremal" processes.*

We observe that $\mathbf{S}_n(t) = \sum_{j=1}^{N_n(t)} \mathbf{X}_{nj} = \sum_{j=1}^{k_n(\theta_n(t))} \mathbf{X}_{nj} = \mathbf{S}_n^{(a)}(\theta_n(t))$ and analogously $\mathbf{Y}_n = \mathbf{Y}_n^{(a)} \circ \theta_n$. Denote $A_{t,\mathbf{x}} = [0,t] \times [\mathbf{0},\mathbf{x})^c$. In this model our main result claims the following.

Theorem 3. *Let $\mathcal{N}_n = \{(T_{nk}, \mathbf{X}_{nk}), k \geq 1\}$ be a Bernoulli p.p. on Z which counting process satisfies the basic assumption, i.e. $N_n(t) = k_n(\theta_n(t))$ a.s.$\forall t \geq 0$. Suppose the random time changes θ_n are weakly convergent to a random time change Λ. If the sequence of the accompanying p.p.'s $\mathcal{N}_n^{(a)}$ is vaguely convergent to a simple in time Poisson p.p. $\mathcal{N}$ on an open subset $\mathcal{S} \subset Z$ with mean measure μ then the sequence $\mathcal{N}_n$ is weakly convergent to a Cox p.p. $\tilde{\mathcal{N}}$ with mean measure*

$$\tilde{\mu}(A_{t,\mathbf{x}}) = \mathbb{E}\mu\left([0,\Lambda(t)] \times [\mathbf{0},\mathbf{x})^c\right) \tag{3}$$

Proof. Denote the time and space coordinates of the limiting Poisson p.p. $\mathcal{N}$ by T_k and $\mathbf{X}_k$, respectively. The $\sigma-$algebra of the compact subsets of Z is generated by the class of all sets of the form $A_{t,\mathbf{x}} = [0,t] \times [\mathbf{0},\mathbf{x})^c, \quad (t,\mathbf{x}) \in Z$. Thus, it suffices to show that $\mathcal{N}_n(A_{t,\mathbf{x}}) \xrightarrow{d} \tilde{\mathcal{N}}(A_{t,\mathbf{x}})$. Indeed, for $A_{\mathbf{x}}^c = [\mathbf{0},\mathbf{x})^c$

$$\mathcal{N}_n(A_{t,\mathbf{x}}) = \sum_{k=1}^{N_n(t)} \delta_{\mathbf{X}_{nk}}(A_{\mathbf{x}}^c) = \sum_{k=1}^{k_n(\theta_n(t))} \delta_{\mathbf{X}_{nk}}(A_{\mathbf{x}}^c) = \mathcal{N}_n^{(a)}([0,\theta_n(t)] \times A_{\mathbf{x}}^c)$$

By assumption $\mathcal{N}_n^{(a)}(A_{t,\mathbf{x}}) \xrightarrow{d} \mathcal{N}(A_{t,\mathbf{x}}) = \sum_{k=1}^{N(t)} \delta_{\mathbf{X}_k}(A_{\mathbf{x}}^c), \quad \forall (t,\mathbf{x}) \in \mathcal{S}$ and $\theta_n \Rightarrow \Lambda$. The counting process is increasing and the random time-change is strictly increasing and stochastically continuous, hence N and Λ have a.s. no simultaneous jumps and by the continuity of composition theorem (c.f. Th.13.2.3, Whitt (2002))

$$\mathcal{N}_n(A_{t,\mathbf{x}}) = \mathcal{N}_n^{(a)}([0,\theta_n(t)] \times A_{\mathbf{x}}^c) \xrightarrow{d} \sum_{k=1}^{N(\Lambda(t))} \delta_{\mathbf{X}_k}(A_{\mathbf{x}}^c) =: \tilde{\mathcal{N}}(A_{t,\mathbf{x}})$$

for all $(t,\mathbf{x}) \in \tilde{\mathcal{S}}$. Here $\tilde{\mathcal{S}}$ is the definition domain of the mean measure $\tilde{\mu}(\cdot) = \mathbb{E}\tilde{\mathcal{N}}(\cdot)$. Furthermore, formally we can express

$$\tilde{\mu}(A_{t,\mathbf{x}}) = \mathbb{E} \sum_{k=1}^{N(\Lambda(t))} \delta_{\mathbf{X}_k}(A_{\mathbf{x}}^c) = \mathbb{E}\left(\mathbb{E}\sum_{k=1}^{N(s)} \delta_{\mathbf{X}_k}(A_{\mathbf{x}}^c)\middle|\Lambda(t) = s\right)$$

$$= \mathbb{E}\mu\left([0,\Lambda(t)] \times [\mathbf{0},\mathbf{x})^c\right)$$

Let us denote by M the random measure on Z defined by

$$M(A_{t,\mathbf{x}}) := \mu([0,\Lambda(t)] \times A_{\mathbf{x}}^c).$$

Then $\tilde{\mathcal{N}}$ is a Cox process directed by M since, conditional on M, $\tilde{\mathcal{N}}$ is Poisson p.p. □

Note, (3) is a formal expression. In fact

$$\tilde{\mu}(A_{t,\mathbf{x}}) = \int \mu(A_{s,\mathbf{x}}) d\mathbf{P}(\Lambda(t) < s) = \int_0^{t_{\mathbf{x}}} \mu(A_{s,\mathbf{x}}) d\mathbf{P}(\Lambda(t) < s)$$

where $t_{\mathbf{x}} = \sup\{s : \mu(A_{s,\mathbf{x}}) < \infty\}$. Hence the definition domain $\tilde{\mathcal{S}}$ of $\tilde{\mu}$ depends on the definition domain $\mathcal{S}$ of μ and the distribution of Λ, and $\tilde{\mathcal{S}} \subseteq \mathcal{S}$. By (2), the definition domain $\mathcal{S}$ is just the complement of the set $[\mathbf{0},\mathbf{C}]$ below the lower curve $\mathbf{C}$ of the extremal process $\mathbf{Y}$ generated by the Poisson p.p. $\mathcal{N}$.

Theorem 4. *Let $\mathbf{Y}_n, n \geq 1$ be extremal processes on Z generated by the point processes $\mathcal{N}_n$ from Theorem 3. Suppose that*

$$\sum_{j=1}^{k_n(t)} \{1 - \mathbf{P}(\mathbf{X}_{nj} < \mathbf{x})\} \xrightarrow{w} \mu([0,t] \times [\mathbf{0}, \mathbf{x})^c), \quad (t, \mathbf{x}) \in \mathcal{S} \tag{i'}$$

where μ is Radon measure on $\mathcal{S}$;

$$\sup_{0 \leq t_{nj} \leq t} \{1 - \mathbf{P}(\mathbf{X}_{nj} < \mathbf{x})\} \to 0, \quad n \to \infty \qquad \textit{for } \mathbf{x} > \mathbf{C}(t), t \geq 0; \tag{ii'}$$

$$f_n(t, \mathbf{x}) = \mathbf{P}(\mathbf{Y}_n^{(a)}(t) < \mathbf{x}) \to 0, \quad n \to \infty \quad \forall (t, \mathbf{x}) : \mathbf{x} < \mathbf{C}(t-0); \tag{iii}$$

$$\theta_n \Rightarrow \Lambda \text{ - random time change} \tag{iv}$$

Then the sequence $\mathbf{Y}_n$ is weakly convergent to the composition $\tilde{\mathbf{Y}} := \mathbf{Y} \circ \Lambda$ with d.f. $\tilde{f}$ where $\mathbf{Y}$ is a max-id extremal process and

$$\tilde{f}(t, \mathbf{x}) = \mathbb{E}e^{-\mu([0,\Lambda(t)] \times A_{\mathbf{x}}^c)} = \mathbb{E}e^{-M(A_{t,\mathbf{x}})}$$

Proof. Note, conditions (i') and (ii') are equivalent to conditions (i) and (ii) in Theorem 1. Condition (ii') is an analogue to the asymptotic negligibility condition in the classical limit theory. It means an asymptotic closeness of $\mathbf{X}_{nk}$ to the lower curve $\mathbf{C}$, the boundary of the explosion area of μ. By Theorem 1 $\mathcal{N}_n^{(a)} \xrightarrow{v} \mathcal{N}$, where $\mathcal{N}$ is Poisson p.p. with mean measure μ on $\mathcal{S}$. This convergence (in case $\mathcal{N}$ is simple in time) together with (iii) implies $\mathbf{Y}_n^{(a)} \Rightarrow \mathbf{Y}$. The limit extremal process is max-id with d.f. f of the form (2). Now the weak convergence

$$\mathbf{Y}_n = \mathbf{Y}_n^{(a)} \circ \theta_n \Rightarrow \mathbf{Y} \circ \Lambda = \tilde{\mathbf{Y}}$$

is a consequence of the continuity of composition theorem and

$$\tilde{f}(t, \mathbf{x}) = \mathbf{P}(\mathbf{Y} \circ \Lambda(t) < \mathbf{x}) = \int \mathbf{P}(\mathbf{Y}(s) < \mathbf{x}) d\mathbf{P}(\Lambda(t) < s) = \mathbb{E}f(\Lambda(t), \mathbf{x}) =$$

$$= \mathbb{E}\exp(-\mu([0, \Lambda(t)] \times A_{\mathbf{x}}^c)) = \mathbb{E}\exp(-M(A_{t,\mathbf{x}}))$$

□

One can see that $\tilde{\mathbf{Y}} = \mathbf{Y} \circ \Lambda$ does not need to be a max-id extremal process. As known (e.g. Pancheva,Kolkovska, Yordanova (2003)), $\tilde{\mathbf{Y}}$ has independent max-increments if and only if Λ has independent (additive) increments and $\mathbf{Y}$ has homogeneous max-increments. If $\tilde{\mathbf{Y}}$ were a max-id extremal process then the generating p.p. $\tilde{\mathcal{N}}$ should be Poisson, hence Λ should be deterministic.

Before stating the analogous to Theorem 3 result for the associated sum process $\mathbf{S}_n$ we have to explain how we understand a sum process over an explosion area.

4 Sum process over explosion area

Let $\mathbf{C} : [0, \infty) \to [0, \infty)^d$ be an increasing right-continuous curve and $T = \sup\{t : |\mathbf{C}(t)| = 0\}$. Given a simple in time Poisson p.p. $\mathcal{N} = \{(T_k, \mathbf{X}_k) : k \geq 1\}$ on the open set $\mathcal{S} = [\mathbf{0}, \mathbf{C}]^c$ with $\sigma-$finite mean measure μ, assume μ satisfies the conditions

$$\int_0^T \int_{[0,\infty)^d \setminus \{0\}} (|\mathbf{x}| \wedge 1)\mu(ds, \mathbf{dx}) < \infty \tag{a1}$$

$$\int_T^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \mu(ds, \mathbf{dx}) < \infty, \quad \forall t > T \tag{a2}$$

$$\mu\{\mathcal{S}(t)\} = 0, \forall t \in (0, \infty) \tag{a3}$$

$$\mu([0,t] \times [0,\mathbf{x})^c) = \infty, \quad x \in [\mathbf{C}(t-0), \mathbf{C}(t)] \tag{a4}$$

The stochastic process $\mathbf{S}$ defined by $\mathbf{S}(t) = \sum_{\{k:T_k \leq t\}} \mathbf{X}_k$ is:
• a.s. finite, because of conditions (a1) and (a2);
• stochastically continuous, as a consequence of conditions (a3) and (a4), (c.f. Balkema and Pancheva (1996));
• and has independent increments because the random vectors $\mathbf{X}_k$ are independent. It can be decomposed as sum of two independent sum processes $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$. The process $\mathbf{S}^{(2)}$ is compound Poisson while $\mathbf{S}^{(1)}$ admits the decomposition $\mathbf{S}^{(1)}(t) = \mathbf{a}(t) + \lim_{c \downarrow 0} \mathbf{S}_c^{(1)}(t)$. Here $\mathbf{a}(t)$ is continuous increasing function with $\mathbf{a}(0) = 0$ and $\mathbf{S}_c^1(t) = \sum_{\{k:T_k \leq t\}} \mathbf{X}_k \mathbb{I}_{\{|\mathbf{X}_k| > c\}}, \forall t \in [0,T], c > 0$. More precisely

$$\mathbf{S}(t) = \begin{cases} \mathbf{S}^{(1)}(t), & 0 \leq t \leq T \\ \mathbf{S}^{(1)}(T) + \mathbf{S}^{(2)}(t), & t > T. \end{cases} \tag{4}$$

Thus, the characteristic function of the sum process $\mathbf{S}$ over the explosion area $[\mathbf{0}, \mathbf{C}]^c$ is given by

$$\psi_t(z) = \mathbb{E}e^{i\mathbf{z}.\mathbf{S}(t)} = \exp\{i\mathbf{z}.\mathbf{a}_T(t) + \int_0^t \int_{[\mathbf{0},\mathbf{C}(s)]^c} \left(e^{i\mathbf{z}.\mathbf{x}} - 1\right) \mu(ds, \mathbf{dx})\} \tag{5}$$

where $\mathbf{a}_T(t)$ is such that,

$$\mathbf{a}_T(t) = \begin{cases} \mathbf{a}(t), & 0 \leq t \leq T \\ 0, & t > T. \end{cases}$$

Note that when $\mathbf{C} \equiv 0$ the set $[\mathbf{0}, \mathbf{C}(s)]^c$ means $[0,\infty]^d \setminus \{0\}$.
The above results (4) and (5) are due to I. Mitov (2006) and their proof is included in PVM(2006). Finally, we can state

Theorem 5. *Let $\mathbf{S}_n, n \geq 1$ be sum process on Z generated by the p.p. $\mathcal{N}_n$ from Theorem 3. Suppose*

$$\mathbf{Y}_n^{(a)} \Rightarrow \mathbf{Y} \text{ stochastically continuous with d.f. } f(t,\mathbf{x}) = e^{-\mu(A_{t,\mathbf{x}})}, (t,\mathbf{x}) \in \mathcal{S}; \tag{C1}$$

$$\sum_{j=1}^{k_n(t)} \mathbb{E}\left(\mathbf{X}_{nj}\mathbb{I}\{|\mathbf{X}_{nj}| \leq c\}\right) \to \mathbf{a}(t) - \int_0^t \int_{|\mathbf{x}| \leq c} |\mathbf{x}|\mu(ds, \mathbf{dx}) < \infty \text{ for } t \leq T \text{ and } c > 0; \tag{C2}$$

$$\theta_n \Rightarrow \Lambda \text{ - random time change.} \tag{C3}$$

Then $\mathbf{S}_n \Rightarrow \tilde{\mathbf{S}} = \mathbf{S} \circ \Lambda$ *where* $\mathbf{S}$ *is a stochastically continuous sum process with characteristic function* $\psi \sim (\mathbf{a}, \mu)$ *and*

$$\tilde{\psi}_t(\mathbf{z}) = \mathbb{E} e^{i\mathbf{z}.\tilde{\mathbf{S}}(t)} = \mathbb{E}\psi_{\Lambda(t)}(\mathbf{z})$$

Proof. In fact, condition (C1) implies that

$$\sum_{j=1}^{k_n(t)} \{1 - \mathbf{P}(\mathbf{X}_{nj} < \mathbf{x})\} \xrightarrow{w} \mu(A_{t,\mathbf{x}}), \quad (t, \mathbf{x}) \in \mathcal{S}$$

and together with (C2) ensure that:

• the exponent measure $\mu = -\log f$ of the limit extremal process is a Levy measure satisfying the conditions (a1)-(a4);

• $\mathbf{S}_n^{(a)}(t) \xrightarrow{d} \mathbf{S}(t)$ where $\mathbf{S}$ is stochastically continuous with characteristic function ψ of the form (5). The latter convergence combined with Theorem 2 and condition (C3) results in $\mathbf{S}_n \Rightarrow \mathbf{S} \circ \Lambda$. Furthermore,

$$\tilde{\psi}_t(\mathbf{z}) = \mathbb{E} e^{i\mathbf{z}.\tilde{\mathbf{S}}(t)} = \mathbb{E}\left(\mathbb{E} e^{i\mathbf{z}.\mathbf{S}(s)} \middle| \Lambda(t) = s\right) \mathbb{E}\psi_{\Lambda(t)}(\mathbf{z})$$

□

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