Upper and lower bounds for ruin probability

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Abstract

In this note we discuss upper and lower bound for the ruin probability in an insurance model with very heavy-tailed claims and interarrival times.

Key Words and Phrases: compound extremal processes; α-stable approximation; ruin probability

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1 Backgrounds

The framework of our study is set by a given Bernoulli point process (Bpp) $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ on the time-state space $\mathcal{S} = (0,\infty) \times (0,\infty)$. By definition (cf. Balkema and Pancheva 1996) $\mathcal{N}$ is simple in time ($T_k \neq T_j$ a.s. for $k \neq j$), its mean measure is finite on compact subsets of $\mathcal{S}$ and all restrictions of $\mathcal{N}$ to slices over disjoint time intervals are independent. We assume that:
a) the sequences $\{T_k\}$ and $\{X_k\}$ are independent and defined on the same probability space;
b) the state points $\{X_k\}$ are independent and identically distributed random variables (iid rv's) on $(0,\infty)$ with common distribution function (df) F which is asymptotically continuous at infinity;
c) the time points $\{T_k\}$ are increasing to infinity, i.e. $0 < T_1 < T_2 < ..., T_k \to \infty$ a.s.

The main problem in the Extreme Value Theory is the asymptotic of the extremal process $\{\bigvee_k X_k : T_k \leq t\} = \bigvee_{k=1}^{N(t)} X_k$, associated with $\mathcal{N}$, for $t \to \infty$. Here the maximum operation between rv's is denoted by " $\vee$ " and $N(t) := max\{k : T_k \leq t\}$ is the counting process of $\mathcal{N}$. The method usually used is to choose proper time-space changes $\zeta_n = (\tau_n(t), u_n(x))$ of $\mathcal{S}$ (i.e. strictly increasing and continuous in both components) such that for $n \to \infty$ and $t > 0$ the weak convergence

$$\tilde{Y}_n(t) := \{\bigvee_k u_n^{-1}(X_k) : \tau_n^{-1}(T_k) \leq t\} \Longrightarrow \tilde{Y}(t) \qquad (1)$$

to a non-degenerate extremal process holds. (For weak convergence of extremal processes consult e.g. Balkema and Pancheva 1996.)

In fact, the classical Extreme Value Theory deals with Bpp's $\{(t_k, X_k) : k \geq 1\}$ with deterministic time points t_k, $0 < t_1 < t_2 < ...,\ t_k \to \infty$. One investigates the weak convergence to a non-degenerate extremal process

$$Y_n(t) := \{\bigvee_k u_n^{-1}(X_k) : t_k \leq \tau_n(t)\} \Longrightarrow Y(t) \qquad (2)$$

under the assumption that the norming sequence $\{\zeta_n\}$ is regular. The later means that for all $s > 0$ and for $n \to \infty$ there exist point-wise

$$\lim_{n\to\infty} u_n^{-1} \circ u_{[ns]}(x) = U_s(x)$$

$$\lim_{n\to\infty} \tau_n^{-1} \circ \tau_{[ns]}(t) = \sigma_s(t)$$

and $(\sigma_s(t),\ U_s(x))$ is a time-space change. As usual "$\circ$" means the composition and $[s]$ the integer part of s. The family $\mathcal{L} = \{(\sigma_s(t),\ U_s(x)) : s > 0\}$ forms a continuous one-parameter group w.r.t. composition.

Let us denote the (deterministic) counting function $k(t) = max\{k : t_k \leq t\}$, and put $k_n(t) := k(\tau_n(t))$, $k_n := k_n(1)$. The df of the limit extremal process in (2) we denote by $g(t,x) := \mathbf{P}(Y(t) < x)$, and set $G(x) := g(1,x)$. Then necessary and sufficient conditions for convergence (2) are the following

1. $F^{k_n}(u_n(x)) \xrightarrow{w} G(x), \quad n \to \infty$

2. $\frac{k_n(t)}{k_n} \longrightarrow \lambda(t), \quad n \to \infty, \quad t > 0.$

The regularity of the norming sequence $\{\zeta_n\}$ has some important consequences (cf. Pancheva 1998). First of all, the limit extremal process $Y(t)$ is self-similar w.r.t. $\mathcal{L}$, i.e.

$$U_s \circ Y(t) \stackrel{d}{=} Y \circ \sigma_s(t), \quad \forall s > 0 \ .$$

Furthermore:
0. $\frac{k_{[ns]}}{k_n} \longrightarrow s^a, n \to \infty$, for some $a > 0$ and all $s > 0$;
1'. the limit df G is max-stable in the sense that

$$G^s(x) = G(L_s^{-1}(x)) \quad \forall s > 0, \quad L_s := \mathbf{U}_{\sqrt[a]{s}}; \tag{3}$$

2'. the intensity function $\lambda(t)$ is continuous.

Thus, under conditions 1. and 2. and the regularity of the norming sequence, the limit extremal process $Y(t)$ is stochastically continuous with df $g(t,x) = G^{\lambda(t)}(x)$ and the process $Y \circ \lambda^{-1}(t)$ is max-stable in the sense of (3).

Let us come back to the point process $\mathcal{N}$ with the random time points T_k. The Functional Transfer Theorem (FTT) in this framework gives conditions on $\mathcal{N}$ for the weak convergence (1) and determines the explicit form of the limit df $f(t,x) := \mathbf{P}(\tilde{Y}(t) < x)$. In other words, the weak convergence (2) in the framework with non-random time points can be transfer to the framework of $\mathcal{N}$ if some additional condition on the point process $\mathcal{N}$ is met. In our case this is condition d) below.

Denote by $\mathcal{M}([0,\infty))$ the space of all strictly increasing, cadlac functions $y : [0,\infty) \to [0,\infty)$, $y(0) = 0$, $y(t) \to \infty$ as $t \to \infty$. We assume additionally to a) - c) the following condition

d) $\theta_n(s) := \tau_n^{-1}(T_{[sk_n]}) \Longrightarrow T(s)$

where $T : [0,\infty) \to [0,\infty)$ is a random time change, i.e. stochastically continuous process with sample paths in $\mathcal{M}([0,\infty))$. Let us set $N_n(t) := N(\tau_n(t))$. In view of condition d) the sequence

$$\begin{aligned} \Lambda_n(t) := \frac{N_n(t)}{k_n} &= \frac{1}{k_n} \max\{k : T_k \le \tau_n(t)\} \\ &= \sup\{s > 0 : \tau_n^{-1}(T_{[sk_n]}) \le t\} \\ &= \sup\{s > 0 : \theta_n(s) \le t\} \end{aligned}$$

is weakly convergent to the inverse process of $T(s)$. Let us denote it by Λ and let $Q_t(s) = P(\Lambda(t) < s)$.

Now we are ready to state a general FTT for maxima of iid rv's on $(0, \infty)$.

Theorem (FTT): Let $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ be a Bpp described by conditions a) - c). Assume further that there is a regular norming sequence $\zeta_n(t, x) = (\tau_n(t), u_n(x))$ of time-space changes of $\mathcal{S}$ such that for $n \to \infty$ and $t > 0$ conditions 1., 2. and d) hold. Then

i) $\frac{N_n(t)}{k_n} \xrightarrow{d} \Lambda(t)$

ii) $\mathbf{P}(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x) \xrightarrow{w} \mathbf{E}[G(x)]^{\Lambda(t)}$

Indeed, we have to show only ii). Observe that for $n \to \infty$

$$N_n(t) = k_n . \frac{N_n(t)}{k_n} \sim k_n . \Lambda(t) \sim k_n(\lambda^{-1} \circ \Lambda(t))$$

In the last asymptotic relation we have used condition 2). Then by convergence (2)

$$\tilde{Y}_n(t) = \bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) \Longrightarrow Y(\lambda^{-1} \circ \Lambda(t))$$

and

$$\mathbf{P}(\bigvee_{k=1}^{N_n(t)} u_n^{-1}(X_k) < x) \longrightarrow f(t, x) =$$

$$\int_0^\infty G^s(x) dQ_t(s) = \mathbf{E}[G(x)]^{\Lambda(t)}$$

Let us apply these results to a particular insurance risk model.

2 Application to ruin probability

The insurance model, we are dealing with here, can be described by a particular Bpp $\mathcal{N} = \{(T_k, X_k) : k \geq 1\}$ where

a) the claim sizes $\{X_k\}$ are positive iid random variables which df F has a regularly varying tail, i.e. $1 - F \in RV_{-\alpha}$. We consider the "very heavy tail case" $0 < \alpha < 1$ when EX does not exist, briefly $EX = \infty$;

b) the claims occur at times $\{T_k\}$ where $0 < T_1 < T_2 < ... < T_k \to \infty$ a.s. We denote the inter-arrival times by $J_k = T_k - T_{k-1},\ k \geq 1,\ T_0 = 0$ and assume the random variables $\{J_k\}$ positive iid with df H. Suppose $1 - H \in RV_{-\beta},\ 0 < \beta < 1$;

c) both sequences $\{X_k\}$ and $\{T_k\}$ are independent and defined on the same probability space.

The point process $\mathcal{N}$ generates the following random processes we are interested in.

i) The counting process $N(t) = \max\{k : T_k \le t\}$. It is a renewal process with $\frac{N(t)}{t} \to 0$ as $t \to 0$ for $EJ = \infty$. By the Stable CLT there exists a normalizing sequence $\{b(n)\}$, $b(n) > 0$, such that $\sum_{k=1}^{[nt]} \frac{J_k}{b(n)}$ converges weakly to a β- stable Levy process $S_\beta(t)$. One can choose $b(n) \sim n^{1/\beta} L_J(n)$, where L_J denotes a slowly varying function. Let us determine $\tilde{b}(n)$ by the asymptotic relation $b(\tilde{b}(n)) \sim n$ as $n \to \infty$. Now the normalized counting process $\frac{N(nt)}{\tilde{b}(n)}$ is weakly convergent to the hitting time process $E(t) = \inf\{s : S_\beta(s) > t\}$ of S_β, see Meerschaert and Scheffler (2002). As inverse of S_β, $E(t)$ is β-selfsimilar.

ii) The extremal claim process $Y(t) = \{\vee X_k : T_k \le t\} = \bigvee_{k=1}^{N(t)} X_k$. In view of assumption a) there exist norming constants $B(n) \sim n^{1/\alpha} L_X(n)$ such that $\bigvee_{k=1}^{[nt]} \frac{X_k}{B(n)}$ converges weakly to an extremal process $Y_\alpha(t)$ with Frechet marginal df, i.e. $P(Y_\alpha(t) < x) = \Phi_\alpha^t(x) = \exp -tx^{-\alpha}$. Consequently,

$$Y_n(t) := \bigvee_{k=1}^{N(nt)} \frac{X_k}{B(\tilde{b}(n))} \Longrightarrow Y_\alpha(E(t)).$$

Below we use the $\frac{\beta}{\alpha}$ - selfsimilarity of the compound extremal process $Y_\alpha(E(t))$ (see e.g. Pancheva et al. 2003).

iii) The accumulated claim process $S(t) = \sum_{k=1}^{N(t)} X_k$. Using the same norming sequence as above we observe that

$$S_n(t) := \sum_{k=1}^{N(nt)} \frac{X_k}{B(\tilde{b}(n))} \Longrightarrow Z_\alpha(E(t)).$$

Here Z_α is an α-stable Levy process and the composition $Z_\alpha(E(t))$ is $\frac{\beta}{\alpha}$-selfsimilar.

iv) The risk process $R(t) = c(t) - S(t)$. Here $u := c(0)$ is the initial capital and $c(t)$ denotes the premium income up to time t, hence it is an increasing curve. We assume $c(t)$ right-continuous.

Note, the extremal claim process $Y(t)$ and the accumulated claim process $S(t)$ need the same time-space changes $\zeta_n(t,x) = (nt, \frac{x}{B(\tilde{b}(n))})$ to achieve weak convergence to a proper limiting process. In fact, $\{\zeta_n\}$ makes the claim sizes smaller and compensates this by increasing their number in the interval $[0,t]$. Both processes $Y_n(t)$ and $S_n(t)$ are generated by the point process $\mathcal{N}_n = \{(\frac{T_k}{n}, \frac{X_k}{B(\tilde{b}(n))}) : k \ge 1\}$. With the latter we also associate the sequence of risk processes $R_n(t) = \frac{c(nt)}{B(\tilde{b}(n))} - S_n(t)$. Let us assume additionally to a) - c) the condition

d) $\frac{c(nt)}{B(\tilde{b}(n))} \xrightarrow{w} c_0(t)$, c_0 increasing curve with $c_0(0) > 0$.

Under conditions a) - d) the sequence R_n converges weakly to the risk process (cf Furrer et al. 1997) $R_{\alpha,\beta}(t) = c_0(t) - Z_\alpha(E(t))$ with initial capital $u_0 = c_0(0)$. Using the $R_{\alpha,\beta}$ - approximation of the initial risk process $R(t)$, when time and initial capital increase with n, we next obtain upper ($\bar{\psi}$) and lower($\underline{\psi}$) bound for the ruin probability $\Psi(c,t) := P(\inf_{0\le s\le t} R(s) < 0)$. Let $Z_\alpha(1)$ and $E(1)$ have df's G_α and Q, resp. Then we have :

$$\begin{aligned}
\psi(c_0,t) &:= P(\inf_{0\le s\le t} R_{\alpha,\beta}(s) < 0) \\
&\le P(\sup_{0\le s\le t} Z_\alpha(E(s)) > u_0) \\
&\le P(Z_\alpha(E(t)) > u_0) \\
&= \int_0^\infty \bar{Q}(\left(\frac{u_0}{xt^{\frac{\beta}{\alpha}}}\right)^\alpha) dG_\alpha(x) =: \bar{\psi}(c_0,t)
\end{aligned}$$

Here $\bar{Q} = 1 - Q$. On the other hand

$$\begin{aligned}
\psi(c_0,t) &\ge P(Y_\alpha(E(t)) > c_0(t)) \\
&= \int_0^\infty \bar{Q}(\left(\frac{c_0(t)}{xt^{\frac{\beta}{\alpha}}}\right)^\alpha) d\Phi_\alpha(x) =: \underline{\psi}(c_0,t)
\end{aligned}$$

Here we have used the self-similarity of the processes Z_α, Y_α and E. Thus, finally we get

$$\underline{\psi}(c_0,t) \le \psi(c_0,t) \le \bar{\psi}(c_0,t)$$

Remember, our initial insurance model was described by the point process $\mathcal{N}$ with the associated risk process $R(t)$. We have denoted the corresponding ruin probability by $\Psi(c,t)$ with $u = c(0)$. Then

$$\Psi(c,t) = P(\inf_{0\le s\le t}\{c(s) - \sum_{k=1}^{N(s)} X_k\} < 0)$$

$$= \quad P(\inf_{0\le s\le \frac{t}{n}}\{\frac{c(ns)}{B(\tilde{b}(n))} - \sum_{k=1}^{N(ns)} \frac{X_k}{B(\tilde{b}(n))}\} < 0)$$

Now let initial capital u and time t increase with $n \to \infty$ in such a way that $\frac{u}{B(\tilde{b}(n))} = u_0$, $\frac{t}{n} = t_0$. We observe that under conditions a) - d) we may approximate

$$\Psi(c,t) \approx \psi(c_0,t_0)$$

and consequently for u and t "large enough"

$$\underline{\psi}(c_0,t_0) \le \Psi(c,t) \le \bar{\psi}(c_0,t_0) \tag{4}$$

3 Examples

Assume that our model is characterized by $\alpha = 0.5$, i.e. the df of $Z_\alpha(1)$ is the Levy df $G_\alpha(x) = 2(1 - \Phi(\sqrt{\frac{1}{x}}))$. Here Φ is the standard normal df. We suppose also that the random variable $E(1)$ is Exp(1)-distributed, namely $Q(s) = 1 - e^{-s}$, $s \geq 0$. Further, let us take the income curve c_0 to be of the special form $c_0(t) = u_0 + t^{\frac{\beta}{\alpha}} c$, c positive constant, that agrees with the self-similarity of the process $Z_\alpha(E(t))$. Now the upper bound depends on (u_0, t_0, β) and the lower bound depends on (u_0, t_0, β, c). We calculate the bounds $\underline{\psi}$ and $\bar{\psi}$ in two cases $\alpha > \beta = 0.25$ and $\alpha < \beta = 0.75$ by using MATLAB7. The results of the calculations show clearly that in case $\beta > \alpha$, when "large" claims arrive "often", the bounds of the ruin probability are larger than in the case $\beta < \alpha$, even in small time interval.

Note, if we choose the income curve in the above special form, we may calculate the ruin probability $\psi(c_0, t_0)$ in the approximating model exactly, namely

$$
\begin{aligned}
\psi(c_0, t_0) &= P(\inf_{0 \leq s \leq t_0} \{u_0 + s^{\frac{\beta}{\alpha}} c - Z_\alpha(E(s))\} < 0) \\
&= P(\inf_{0 \leq s \leq t_0} \{s^{\frac{\beta}{\alpha}} (c - Z_\alpha(E(1)))\} < -u_0) \\
&= P(\inf_{0 \leq s \leq t_0} \{s^{\frac{\beta}{\alpha}} (c - Z_\alpha(E(1)))\} < -u_0, \quad c - Z_\alpha(E(1)) < 0) \\
&= P(t_0^{\frac{\beta}{\alpha}} (c - Z_\alpha(E(1))) < -u_0, \quad c - Z_\alpha(E(1)) < 0) \\
&= P(Z_\alpha(E(1)) > c + \frac{u_0}{t_0^{\frac{\beta}{\alpha}}}) \\
&= \int_0^\infty \bar{Q}(\left(\frac{c_0(t_0)}{x t_0^{\frac{\beta}{\alpha}}}\right)^\alpha) dG_\alpha(x)
\end{aligned}
$$

Below we give graphical results related to the computation of $\underline{\psi}(c_0, t_0)$, $\psi(c_0, t_0)$ and $\bar{\psi}(c_0, t_0)$ in the 6 cases: c=0.1, c=1, c=10 when $\alpha = 0.5$ and $\beta = 0.25 \quad \beta = 0.75$.

4 Graphics of $\underline{\psi}(c_0, t_0), \psi(c_0, t_0)$ *and* $\bar{\psi}(c_0, t_0)$

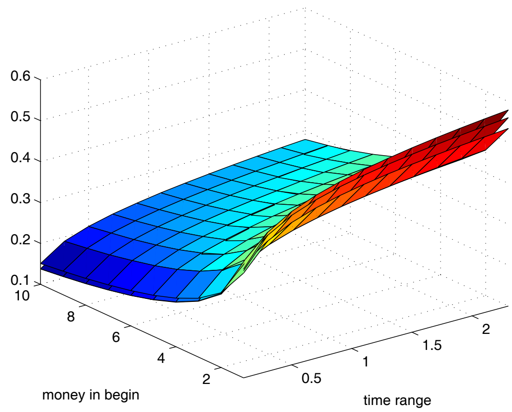

Figure 1: $\alpha = 0.5, \beta = 0.25, c = 0.1$

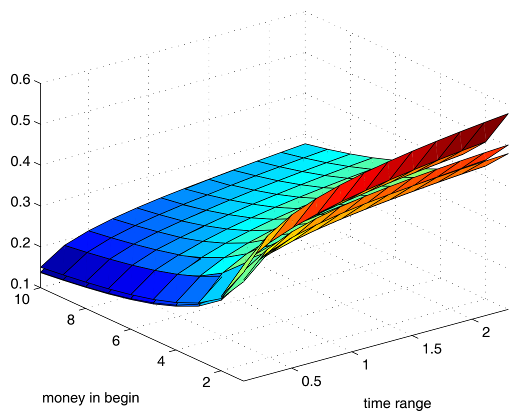

Figure 2: $\alpha = 0.5, \beta = 0.25, c = 0.5$

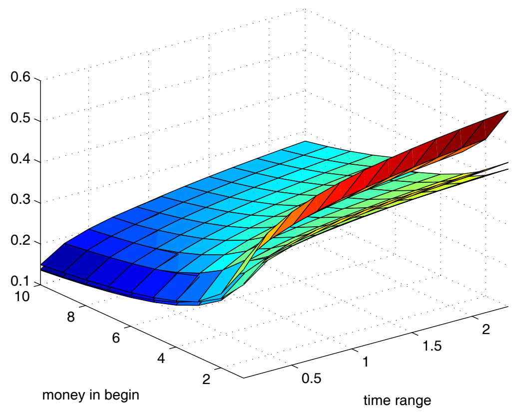

Figure 3: $\alpha = 0.5, \beta = 0.25, c = 1.0$

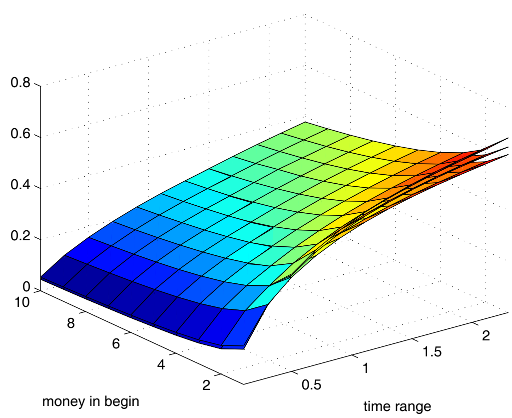

Figure 4: $\alpha = 0.5, \beta = 0.75, c = 0.1$

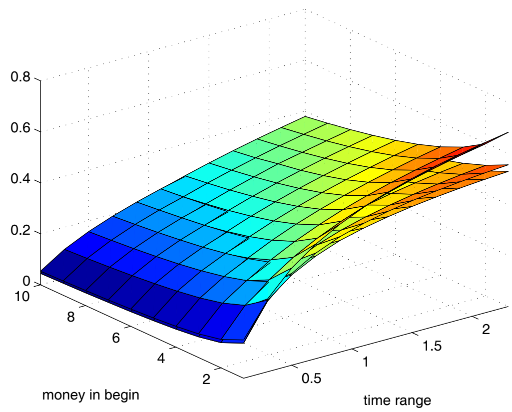

Figure 5: $\alpha = 0.5, \beta = 0.75, c = 0.5$

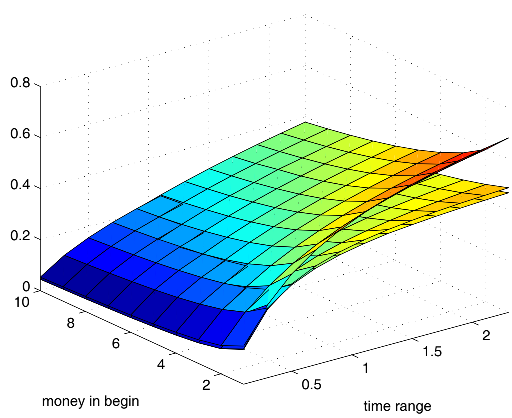

Figure 6: $\alpha = 0.5, \beta = 0.75, c = 1.0$

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