Supplementary material for the paper Free Space of Rigid Objects: Caging, Path Non-Existence, and Narrow Passage Detection

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In this document we provide the proofs of Propositions 1, 2, and 3, and discuss the computational complexity in detail.

1 Computational Complexity

Let us now discuss the total computational complexity of the algorithm. Let s be the number of slices, n and m be the number of balls in the object's and the obstacle's spherical representation, respectively. We have pre-computed the grids over $SO(3)$ corresponding to different dispersion values, and therefore we are only interested in the complexity of the connectivity graph construction. For each slice, we execute two computationally expensive procedures: we compute a weighted Voronoi diagram of the collision space, which allows us to extract the balls representation of the free space, and then for each slice we compute its intersections with adjacent slices. In practice, each orientation in $\mathcal{Q}$ has around 20 adjacent orientation values, so each slice has around 20 neighbours[1].

In CGAL representation, the regular triangulation contains the corresponding weighted Voronoi diagram. Note that a weighted Voronoi diagram can be constructed by other means using for example the algorithm from [1]. The complexity of this step is $O(n^2m^2)$. The computation of the connected components of each slice is linear on the number of balls in the dual diagram, which makes the overall complexity of this step $O(n^2m^2)$.

The complexity of finding the intersections between two connected components belonging to different slices $O(b\log^3(b)+k)$ in the worst-case [3], where b is the number of balls in both connected components, and k the output complexity, i.e., the number of pairwise intersections of the balls.

The complexity of the final stage of the algorithm — computing connected components of the connectivity graph — is linear on the number of vertices, and can be expressed as $O(s\ c)$, where c is the average number of connected components per slice (a small number in practice).

* The first two authors contributed equally.

[1] This is the case for $SO(3)$, in the case of $SO(2)$ there are exactly 2 neighbours.

2 Proposition 1: distance-displacement correspondence

Proposition 1. *Given two unit quaternions p, q, the following equation holds:*

$$D(p\bar{q}) = 2\sin(\rho(p,q)). \tag{1}$$

Proof. We proceed by reducing both sides of the equation to the same formula, starting with the left-hand-side. To do this we once again point out that we can identify quaternions with vectors in $\mathbb{R}^4$, and that a unit quaternion $q = \cos(\frac{\theta_q}{2}) + \sin(\frac{\theta_q}{2})(q_x i + q_y j + q_z k)$ is associated to a 3D rotation of an angle of θ_q around the axis (q_x, q_y, q_z) which we will denote w_q.

$$\begin{aligned}
&D(R_{p\bar{q}}) = 2|\Im p\bar{q}| \\
&= 2\|\cos(\frac{\theta_q}{2})\sin(\frac{\theta_p}{2})w_p - \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_q - \sin(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_p \times w_q\| \\
&= 2\sqrt{\|\cos(\frac{\theta_q}{2})\sin(\frac{\theta_p}{2})w_p - \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_q\|^2 + \|\sin(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_p \times w_q\|^2}
\end{aligned}$$

Where the last equality is due to the fact that $w_p \times w_q$ is perpendicular to both w_p and w_q, and is therefore a consequence of the Pythagorean theorem. Now recall that $\|w_p \times w_q\| = \sin(\omega_{p,q})$ where $\omega_{p,q}$ is the angle between w_p, w_q and that $\langle w_p, w_q\rangle = \cos(\omega_{p,q})$, since $\|w_p\| = \|w_q\| = 1$. Recall also that $\sin^2(\theta) = 1 - \cos^2(\theta)$, whence we obtain

$$\begin{aligned}
&D(R_{p\bar{q}}) = \\
&2\sqrt{\|\cos(\frac{\theta_q}{2})\sin(\frac{\theta_p}{2})w_p - \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_q\|^2 + \sin^2(\frac{\theta_p}{2})\sin^2(\frac{\theta_q}{2})(1 - \langle w_p, w_q\rangle^2)}
\end{aligned}$$

Furthermore let $\tilde{w}_p = \frac{w_q - \langle w_p, w_q\rangle w_p}{\|w_q - \langle w_p, w_q\rangle w_p\|}$ be the component of w_q which is perpendicular to w_p, then we can rewrite $\cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_q$ as

$$\cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})w_q = \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})(\langle w_p, w_q\rangle w_p + \langle \tilde{w}_p, w_q\rangle \tilde{w}_p)$$

Substituting this into the formula, and using the pythagorean theorem to separate he w_p and $\tilde{w}_p$ components, we can proceed with

$$\begin{aligned}
D(R_{p\bar{q}}) = 2\Big(&\|(\cos(\frac{\theta_q}{2})\sin(\frac{\theta_p}{2}) - \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})\langle w_p, w_q\rangle)w_p\|^2 \\
&+ \|\cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})\langle w_q, \tilde{w}_p\rangle \tilde{w}_p\|^2 + \sin^2(\frac{\theta_p}{2})\sin^2(\frac{\theta_q}{2})(1 - \langle w_p, w_q\rangle^2)\Big)^{1/2}
\end{aligned}$$

Note that $\|w_q\|, \|w_p\|, \|\tilde{w}_p\| = 1$, and therefore $\|\langle w_p, w_q\rangle w_p + \langle w_q, \tilde{w}_p\rangle \tilde{w}_p\|^2 = 1$ which by the Pythagorean theorem gives $\langle w_q, \tilde{w}_p\rangle^2 = 1 - \langle w_p, w_q\rangle^2$, which we can substitute once again.

$$
\begin{aligned}
&D(R_{p\bar{q}}) \\
&= 2\sqrt{(\cos(\frac{\theta_q}{2})\sin(\frac{\theta_p}{2}) - \cos(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})\langle w_p, w_q\rangle)^2} \\
&\overline{+\cos^2(\frac{\theta_p}{2})\sin^2(\frac{\theta_q}{2})(1-\langle w_p, w_q\rangle^2) + \sin^2(\frac{\theta_p}{2})\sin^2(\frac{\theta_q}{2})(1-\langle w_p, w_q\rangle^2)}
\end{aligned}
$$

Now we want to deal only with a combination of tangents, therefore we divide the term inside the square root by $\cos^2(\frac{\theta_p}{2})\cos^2(\frac{\theta_q}{2})$ yielding:

$$
\begin{aligned}
D(R_{p\bar{q}}) = 2|\cos(\frac{\theta_p}{2})\cos(\frac{\theta_q}{2})|&\sqrt{(\tan(\frac{\theta_p}{2}) - \tan(\frac{\theta_q}{2})\langle w_p, w_q\rangle)^2} \\
&\overline{+\tan^2(\frac{\theta_q}{2})(1-\langle w_p, w_q\rangle^2) + \tan^2(\frac{\theta_p}{2})\tan^2(\frac{\theta_q}{2})(1-\langle w_p, w_q\rangle^2)}
\end{aligned}
$$

Now, expanding the sqares and multiplying into all the terms under the squareroot sign, as well as eliminating terms that cancel out, results in:

$$
\begin{aligned}
D(R_{p\bar{q}}) = 2|\cos(\frac{\theta_p}{2})\cos(\frac{\theta_q}{2})|&\sqrt{\tan^2(\frac{\theta_p}{2}) - 2\tan(\frac{\theta_p}{2})\tan(\frac{\theta_q}{2})\langle w_p, w_q\rangle} \\
&\overline{+\tan^2(\frac{\theta_q}{2}) + \tan^2(\frac{\theta_p}{2})\tan^2(\frac{\theta_q}{2}) - \tan^2(\frac{\theta_p}{2})\tan^2(\frac{\theta_q}{2})\langle w_p, w_q\rangle^2}
\end{aligned}
$$

By introducing an extra $1-1$ into the square root, we can use these terms to complete products in order to simplify the equation.

$$
\begin{aligned}
D(R_{p\bar{q}}) =& 2|\cos(\frac{\theta_p}{2})\cos(\frac{\theta_q}{2})|\sqrt{1+\tan^2(\frac{\theta_p}{2}) + \tan^2(\frac{\theta_q}{2}) + \tan^2(\frac{\theta_p}{2})\tan^2(\frac{\theta_q}{2})} \\
&\overline{-(1+\tan(\frac{\theta_p}{2})\tan(\frac{\theta_q}{2})\langle w_p, w_q\rangle)^2} \\
=& 2|\cos(\frac{\theta_p}{2})\cos(\frac{\theta_q}{2})|\sqrt{(1+\tan^2(\frac{\theta_p}{2}))(1+\tan^2(\frac{\theta_q}{2}))} \\
&\overline{-(1+\tan(\frac{\theta_p}{2})\tan(\frac{\theta_q}{2})\langle w_p, w_q\rangle)^2}
\end{aligned}
$$

Finally, recall that $1+\tan^2(\theta) = \frac{1}{\cos^2(\theta)}$, which gives us

$$
D(R_{p\bar{q}}) = 2\sqrt{1-\cos^2(\frac{\theta_p}{2})\cos^2(\frac{\theta_q}{2})(1+\tan(\frac{\theta_p}{2})\tan(\frac{\theta_q}{2})\langle w_p, w_q\rangle)^2}
$$

Now we begin to explore the right-hand side of the equation, by noting that when $\sin(\theta) > 0$ then $\sin(\theta) = |\sin(\theta)| = \sqrt{\sin^2(\theta)} = \sqrt{1-\cos^2(\theta)}$. Furthermore we note that $\cos^{-1}$ maps $[-1,1]$ to $[0,\pi]$ and particularly it maps $[0,1]$ to $[0,\pi/2]$ where the sine function is positive, therefore, we get

$$\begin{aligned}
2\sin(\rho(p,q)) &= 2\sin(\cos^{-1}(|\langle p,q\rangle|)) \\
&= 2\sqrt{1-\cos^2(\cos^{-1}(|\langle p,q\rangle|))} \\
&= 2\sqrt{1-\langle p,q\rangle^2} \\
&= 2\sqrt{1-(\cos(\frac{\theta_p}{2})\cos(\frac{\theta_q}{2})+\sin(\frac{\theta_p}{2})\sin(\frac{\theta_q}{2})\langle w_p,w_q\rangle)^2}
\end{aligned}$$

And finally, we get the same formula as before:

$$2\sin(\rho(p,q)) = 2\sqrt{1-\cos^2(\frac{\theta_p}{2})\cos^2(\frac{\theta_q}{2})(1+\tan(\frac{\theta_p}{2})\tan(\frac{\theta_q}{2})\langle w_p,w_q\rangle)^2}$$

Hence concluding the proof of Proposition 1.

3 Proposition 2: correctness

Proposition 2 (correctness). *Consider an object $\mathcal{O}$ and a set of obstacles $\mathcal{S}$. Let c_1, c_2 be two collision-free configurations of the object. If c_1 and c_2 are not path-connected in $\mathcal{G}(a\mathcal{C}^{free}_{\varepsilon}(\mathcal{O}))$, then they are not path-connected in $\mathcal{C}^{free}(\mathcal{O})$.*

Proof. Recall that the approximation of the free space is constructed as follows:

$$a\mathcal{C}^{free}_{\varepsilon}(\mathcal{O}) = \bigcup_{i=1}^{s} aSl^{free}_{U(\phi_i,\varepsilon)},$$

where

$$aSl^{free}_{U(\phi_i,\varepsilon)} = Dual(\mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon})) \times U(\phi_i,\varepsilon) \tag{2}$$

Now, recall that by definition $(Dual(\mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon})))^c \subset \mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon})$ [2], and that we choose ε and $U(\phi_i,\varepsilon)$ so that for any $\phi \in U(\phi_i,\varepsilon)$, $\mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon}) \subset \mathcal{C}^{col}(\mathcal{O}^{\phi})$. This implies that $(Dual(\mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon})))^c \subseteq \mathcal{C}^{col}(\mathcal{O}^{\phi})$ for any $\phi \in U(\phi_i,\varepsilon)$, and conversely that $\mathcal{C}^{free}(\mathcal{O}^{\phi}) \subset Dual(\mathcal{C}^{col}(\mathcal{O}^{\phi_i}_{\varepsilon}))$. Finally, since $Sl^{free}_{U(\phi_i,\varepsilon)} = \bigcup_{\phi} \mathcal{C}^{free}(\mathcal{O}^{\phi}) \times \{\phi\}$, we have:

$$Sl^{free}_{U(\phi_i,\varepsilon)} \subseteq aSl^{free}_{U(\phi_i,\varepsilon)}, \tag{3}$$

We now want to show that if there is no path between two vertices $v = (aC, U)$ and $v' = (aC', U')$ in $\mathcal{G}(a\mathcal{C}_\varepsilon^{free})$, then there is no path between connected components of $a\mathcal{C}_\varepsilon^{free}(\mathcal{O})$ corresponding to them. It is enough to show that if two vertices corresponding to adjacent slices are not connected by an edge, then they represent two components which are disconnected in $Sl_U^{free} \cup Sl_{U'}^{free}$.

Consider two adjacent slices $Sl_{U(\phi_i,\varepsilon)}$ and $Sl_{U(\phi_j,\varepsilon)}$, and two path-connected components $C_1 \subset aSl_{U(\phi_i,\varepsilon)}^{free}$ and $C_2 \subset aSl_{U(\phi_j,\varepsilon)}^{free}$. Let aC_1 and aC_2 be their respective representations as unions of balls.

Let v_1 and v_2 be the vertices of $\mathcal{G}(a\mathcal{C}_\varepsilon^{free}(\mathcal{O}))$ corresponding to these components: $v_1 = (aC_1, U(\phi_i, \varepsilon))$ and $v_2 = (aC_2, U(\phi_j, \varepsilon))$. By construction, these are adjacent slices, therefore $U(\phi_i, \varepsilon) \cap U(\phi_j, \varepsilon) \neq \emptyset$ and since there is no edge between v_1 and v_2, we get $aC_1 \cap aC_2 \neq \emptyset$. But, by construction $C_1 \subseteq aC_1 \times U(\phi_i, \varepsilon)$ and $C_2 \subseteq aC_2 \times U(\phi_j, \varepsilon)$, therefore, we get:

$$C_1 \cap C_2 \subseteq (aC_1 \times U(\phi_i, \varepsilon)) \cap (aC_2 \times U(\phi_j, \varepsilon)) = \emptyset$$

And so C_1 and C_2 are disjoint in the union of the corresponding slices.

4 Proposition 3: $\delta-$completeness

Proposition 3 (δ-completeness). *Let c_1, c_2 be two configurations in $\mathcal{C}^{free}(\mathcal{O})$. If they are not path-connected in $\mathcal{C}^{free}(\mathcal{O})$, then for any $\delta > 0$ there exists $\varepsilon > 0$ such that the corresponding configurations are not path-connected in $\mathcal{G}(a\mathcal{C}_\varepsilon^{free}(\mathcal{O}_{+\delta}))$, where the graph is produced according to the procedure outlined in Rem. 1.*

In proving this proposition we make used of the notion of the signed distance between two sets:

$$\mathrm{d_s}(\mathcal{O}, \mathcal{S}) = \begin{cases} \min_{p \in O} d(p, \mathcal{S}) & \text{if } \mathcal{O} \cap \mathcal{S} \neq \emptyset \\ -\max_{p \in \mathcal{O} \cap \mathcal{S}} d(p, \mathcal{S}) & \text{otherwise} \end{cases}$$

Note that $\mathrm{d_s}(A, B)$ is not necessarily the same as $\mathrm{d_s}(B, A)$.

Proof. Recall from (see Rem. 1) that there is an edge between vertices (aC_1, ϕ_1), (aC_2, ϕ_2) only if $U(\phi_1, \varepsilon)$ overlaps with $U(\phi_2, \varepsilon)$ and C_1 overlaps with C_2 where $C_i = aC_i \cap \mathcal{C}^{free}(\mathcal{O}_\varepsilon^{\phi_i})$ for $i = 1, 2$ i.e. the components of the actual free space of $\mathcal{O}_\varepsilon^{\phi_i}$ $(i = 1, 2)$ corresponding to the approximations aC_1 and aC_2. This means that we can perform the analysis in terms of collisions in workspace, rather than looking at the configuration space.

Recall now that we want to prove that for a pair of configurations c_1, c_2 which are not path-connected in $\mathcal{C}^{free}(\mathcal{O})$, then for any $\delta > 0$ they are not path-connected in $\mathcal{G}(a\mathcal{C}_\varepsilon^{free}(\mathcal{O}_{+\delta}))$ for some $\varepsilon > 0$. Therefore, we start by noting that since c_1 and c_2 are not path-connected there exists a collision configuration c in any path between them, and since collision implies regular intersection, we have $\mathrm{d_s}(c(\mathcal{O}), \mathcal{S}) < 0$. Thus, for the same configuration c we have $\mathrm{d_s}(c(\mathcal{O}_{+\delta}), \mathcal{S}) < -\delta$.

To see that this will result in path non-existence, we take an arbitrary $\varepsilon > 0$ and consider the collision space $a\mathcal{C}^{col}_{\varepsilon}(\mathcal{O}_{+\delta})$. Further, we let $c = (p, \phi) \in \mathbb{R}^3 \times SO(3)$, and for any ϕ_i such that $\phi \in U(\phi_i, \varepsilon)$, we define $c_i = (p, \phi_i)$. Finally, we restrict ourselves to the case where $\varepsilon < \delta$ and define $\delta' = \delta - \varepsilon$, then we get:

$$\begin{aligned} \mathrm{d_s}(c_i(\mathcal{O}_{+\delta'}), \mathcal{S}) &\leq \mathrm{dist}(c(\mathcal{O}_{+\delta'}), c_i(\mathcal{O}_{+\delta'})) + \mathrm{d_s}(c(\mathcal{O}_{+\delta'}), \mathcal{S}) \\ &\leq \varepsilon - \delta' \end{aligned}$$

Which implies that, as long as we choose ε such that $\varepsilon - \delta' < 0$ (i.e. $\varepsilon < \delta/2$) we obtain the required result.

References

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