KTH Computer Science
and Communication

School of Computer Science and Communication

An immersed finite element method and its convergence for elliptic interface problems with discontinuous coefficients and singular sources

Michael Hanke, Alexei Loubenets

TRITA-NA-2007:3

Michael Hanke, Alexei Loubenets
An immersed finite element method
and its convergence for elliptic interface problems
with discontinuous coefficients and singular sources

Report number: TRITA-NA-2007:3
ISSN 0348-2952
ISRN KTH/NA--07/03--SE
Publication date: September 2007
E-mail of author: alexei@nada.kth.se

Reports can be ordered from:

School of Computer Science and Communication (CSC)
Numerical Analysis
Royal Institute of Technology (KTH)
SE-100 44 Stockholm
SWEDEN

telefax: +46 8 790 09 30
http://www.csc.kth.se/

An immersed finite element method and its convergence for elliptic interface problems with discontinuous coefficients and singular sources

Michael Hanke, Alexei Loubenets

September 14, 2007

Abstract

This paper is concerned with the analysis of an immersed finite element method for two dimensional elliptic interface problems. The main idea of the method is to use specifically designed macro elements in the vicinity of the interface, such that the jump conditions are well approximated. In general, the resulting immersed finite element space is non-conforming. It is shown that the presented method is second order accurate in L^2 norm. The provided numerical results agree with the theoretical estimates.

1 Introduction

Numerical solutions of second order elliptic interface problems are encountered in many engineering and scientific applications, most commonly related to fluid dynamics or material sciences. Such problems arise when two or more different fluids (or materials) with discontinuous or even singular physical properties are involved. In the majority of cases, the solution to these kind of problems is characterized by high degree of regularity in the separate subregions occupied by different fluids (materials), but the global regularity of the solution is usually very low.

In this article, we will analyze the error of an immersed finite element solution formed by first degree polynomials to the following two dimensional elliptic interface problem. Denote by $\Omega \subset \mathbb{R}^2$ a bounded domain with the boundary $\partial\Omega$ that is sufficiently smooth such that the divergence theorem applies and let $\Omega^- \subset \Omega$ be an open domain with a smooth closed double-point free boundary $\Gamma = \partial\Omega^- \subset \Omega$. Then the problem reads:

$$-\nabla \cdot (\beta \nabla u) = f \quad \text{in} \quad \Omega^-, \qquad -\nabla \cdot (\beta \nabla u) = f \quad \text{in} \quad \Omega^+ \tag{1}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega$$

and jump conditions across the interface Γ

$$[u] = 0, \qquad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = q, \tag{2}$$

Here, $\Omega^+ = \Omega \backslash \overline{\Omega}^-$ and $[v]$ is the jump of a quantity v across the interface Γ and $\mathbf{n}$ is the unit outward normal to Γ. For definiteness, we take

$$[v] = v^-(x,y) - v^+(x,y), \quad x \in \Gamma$$

with v^- and v^+ denoting the restrictions of v on Ω^- and Ω^+, respectively. For the sake of simplicity, we assume that the coefficient function β is positive and piecewise constant, i.e.

$$\beta(x,y) = \beta^- \quad \text{for} \quad x \in \Omega^-, \qquad \beta(x,y) = \beta^+ \quad \text{for} \quad x \in \Omega^+.$$

It is a well known fact ([2],[3]) that in order for the standard Galerkin method to achieve an optimal order of accuracy its elements are required, in some way, to be aligned with the interface. In the applications where the interface is moving and deforming with time this constraint proves to be quite restrictive. In addition, in some applications it might be advantageous to use uniform partitions, thus preventing the use of Galerkin methods based on body-fitted grids.

In this paper we investigate an alternative approach, the immersed finite element method. This approach, originally proposed by Li in [5] ,[10] and extended in [7], [8] uses a triangulation that is independent of the interface. The interface itself is represented by an additional structure (t.ex. Lagrangian markers with a parametric description) that is continuously updated using some information obtained from the uniform partition.

The elements of the partition are separated in two classes, the one that are intersected by the interface and the rest. On the non-intersected elements we use the standard linear polynomials. On the intersected elements we use a strategy similar to that of the Hsieh-Clough-Tocher macro-element [1]. That is, each intersected element is subdivided by the interface in two subdomains. Then, we construct a C^0 function consisting of piecewise linear polynomials such that the element has a total of 6 degrees of freedom. At the vertices of the original element, we specify the function values. The additional degrees of freedom are satisfied by the approximation of the jump conditions. Since this procedure involves subpartion of the original triangle, we can regard the intersected elements as macro-elements. The resulting immersed finite element space over the entire domain is, in general, non-conforming. At this point, we should note, that the approximation capability of the immersed finite element space in the case of $q = 0$ has been earlier studied by Li and coauthors in [6].

The main goal of this article is to investigate the error of an immersed finite element solution to the two dimensional elliptic interface problem (1)-(2). This investigation should be regarded as a critical step towards analyzing the error of the immersed finite element method applied to the Stokes and Navier-Stokes equations. Numerical results are also presented, showing good agreement with the theoretical results.

Let us now introduce some notation that is going to be used in this paper. The following standard function spaces will be employed:

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R}| \quad u \text{ is } k\text{-times continuously differentiable}\},$$
$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}| \quad u \text{ is Lebesgue measurable}, ||u||_{L^p(\Omega)} < \infty\},$$
$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}| \quad u \text{ is Lebesgue measurable}, ||u||_{L^\infty(\Omega)} < \infty\}$$

where k is a non-negative integer, $p \geq 1$ is a real number and

$$||u||_{L^\infty(\Omega)} = ess\sup_{\Omega}|u| \quad \text{and} \quad ||u||_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx\Big)^{1/p}, \qquad (1 \leq p < \infty).$$

The support of a function $\phi : \Omega \to \mathbb{R}$ is denoted by

$$\text{supp}(\phi) := \overline{\{x \in \Omega|\phi(x) \neq 0\}}$$

where the overline denotes the closure. The functions with compact support that belong to $C^k(\Omega)$ are denoted by $C_0^k(\Omega)$. In order to introduce the Sobolev spaces we also define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}} = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}u(x)$$

for a given a multi-index α. Then, we define the standard Sobolev space of integer $k \geq 0$ as follows

$$H^k(\Omega) = \{u : \Omega \to \mathbb{R}| \quad \forall\alpha : |\alpha| \leq k, D^\alpha u \in L^2(\Omega)\}$$

with the norm given by

$$||u||_{k,\Omega} = \Big(\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha u|^2 dx\Big)^{1/2}$$

and seminorm defined by

$$|u|_{k,\Omega} = \Big(\sum_{|\alpha|=k}\int_\Omega |D^\alpha u|^2 dx\Big)^{1/2}.$$

As usual, $H_0^k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$. Finally, we denote

$$\tilde{H}^k(\Omega) = \{u|u|_{\Omega^i} \in H^k(\Omega^i), \quad i = \pm\} \tag{3}$$

with the corresponding norm defined by

$$||u||_{k,\Omega}^2 = \Big(\sum_{i=\pm}||u||_{k,\Omega^i}^2\Big)^{1/2} \tag{4}$$

and seminorm given by

$$|u|_{k,\Omega}^2 = \Big(\sum_{i=\pm}|u|_{k,\Omega^i}^2\Big)^{1/2}. \tag{5}$$

We also use $(\cdot,\cdot)_\Omega$ and $\langle\cdot,\cdot\rangle_\Gamma$ to denote the scalar products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively.

The rest of the paper is organized in the following way. The weak formulation and the approximate problem are presented in Section 2. The main result of the article and its proof is given in Section 3. This is followed by some numerical examples and conclusions.

2 Weak formulation and approximate problem

As Γ and $\partial\Omega$ is of class C^2, there exists a function (cf. [9]) $\tilde{u} \in H_0^1(\Omega) \cap \tilde{H}^2(\Omega)$ such that

$$-\nabla \cdot (\beta \nabla \tilde{u}) = 0 \quad \text{in} \quad \Omega, \qquad \tilde{u} = 0 \quad \text{on} \quad \partial\Omega \tag{6}$$

together with the following jump conditions across the interface Γ

$$[\tilde{u}] = 0, \qquad \left[\beta \frac{\partial \tilde{u}}{\partial \mathbf{n}}\right] = q$$

where $q \in H^2(\Gamma)$. The function $\tilde{u}$ is characterized by the following variational formulation

$$a(\tilde{u}, v) = \langle q, v \rangle_\Gamma, \quad \forall v \in H_0^1(\Omega)$$

where $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form defined as

$$a(u, v) = \int_\Omega \beta(x, y) \nabla u \cdot \nabla v dxdy, \quad \forall u, v \in H^1(\Omega).$$

Let $\hat{u} = u - \tilde{u}$, then the weak formulation to the interface problem (1) reads : For $f \in L^2(\Omega)$ find $\hat{u} \in H_0^1(\Omega)$ such that

$$a(\hat{u}, v) = (f, v)_\Omega + \langle q, v \rangle_\Gamma - a(\tilde{u}, v), \quad v \in H_0^1(\Omega). \tag{7}$$

Obviously, $\hat{u}$ fulfills the following jump conditions

$$[\hat{u}] = 0, \qquad \left[\beta \frac{\partial \hat{u}}{\partial \mathbf{n}}\right] = 0$$

across the interface Γ. Note, that the last two terms in (7) are kept only for the sake of the discrete formulation of the problem. In addition, for the purposes of this article, the C^2 regularity of the the domain boundary $\partial\Omega$ is unimportant and thus we will consider a polygonial domain Ω.

We introduce the triangulation $\mathcal{T}_h = \{T\}$ of the domain Ω where h denotes the diameter for its elements. This triangulation satisfies the following standard conditions:

- $\bar{\Omega} = \cup_{T \in \mathcal{T}_h} T$,
- If $T_1, T_2 \in \mathcal{T}_h$ and $T_1 \neq T_2$, then either $T_1 \cap T_2 = \varnothing$ or $T_1 \cap T_2$ is a common vertex or edge of both triangles,
- The triangulation is assumed to be uniform i.e. there are two positive constants independent of h such that
$$C_0 \rho_T \leq h \leq C_1 \bar{\rho}_T$$
where ρ_T and $\bar{\rho}_T$ stands for the diameters of inscribed and circumscribed circles, respectively,

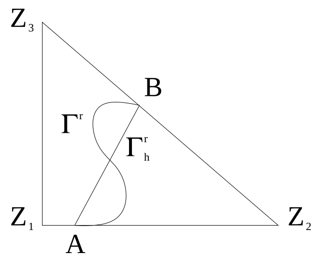


Figure 1: The arc segment Γ^r and its linear approximation Γ^r_h.

together with the following compatibility conditions with the interface Γ:

- If Γ meets a triangle T at two points, then these points must be on different edges of this triangle,
- If Γ meets one edge of a triangle at more than two points, then this edge is a part of Γ.

Denote by $\mathcal{T}'_h$ the set of all elements in the triangulation $\mathcal{T}_h$ that are intersected by the interface Γ. By the construction, the interface Γ can meet the triangle T at most at two edges. Denote these intersection points by $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and use $\mathbf{Z}_1 = (x_1, y_1)$, $\mathbf{Z}_2 = (x_2, y_2)$ and $\mathbf{Z}_3 = (x_3, y_3)$ to denote the vertices of the triangle. Let m_h denote the total number of the intersected triangles. Then, we represent the interface Γ and its piecewise linear approximation Γ_h as

$$\Gamma = \cup_{r=1}^{m_h} \Gamma^r \quad \text{and} \quad \Gamma_h = \cup_{r=1}^{m_h} \Gamma^r_h$$

where the arc segment $\Gamma^r = \Gamma \cap T^r$ for some $T^r \in \mathcal{T}'_h$ and $\Gamma^r_h = \overline{AB}$ is the linear approximation of Γ^r, see Fig. 1.

Each $T \in \mathcal{T}'_h$ is subdivided by the corresponding arc segment into two subdomains

$$\hat{T}^+ = T \cap \Omega^+ \quad \text{and} \quad \hat{T}^- = T \cap \Omega^-$$

see Fig. 2(a). In addition, denote by T^- and T^+ the subdomains that are formed by the partition of T by the line approximation of the corresponding arc segment see Fig. 2(b).

For the future convenience, we also introduce the following domains

$$T^{-*} = T^- \cap \hat{T}^+ \quad \text{and} \quad T^{+*} = T^+ \cap \hat{T}^-$$

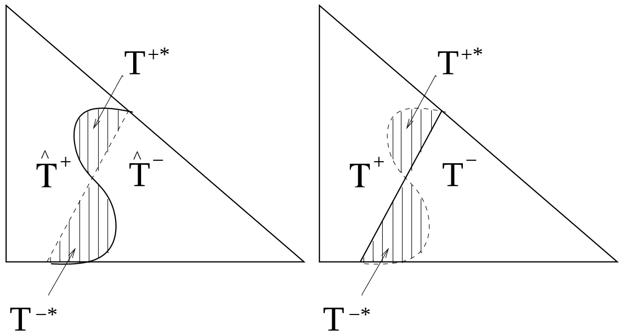
and curve segments

$$\Gamma^{r,-} = \Gamma^r \cap T^- \quad \text{and} \quad \Gamma^{r,+} = \Gamma^r \cap T^+$$

see Figures 2 and 3, correspondingly.

Introduce a local coordinates x^r_1, x^r_2 for each Γ^r. We take the x^r_2-axis along Γ^r_h and x^r_1-axis in the normal direction to Γ^r_h, see Fig. 3. Then the arc Γ^r can be expressed in the parametric form

$$\Gamma^r = \{(x^r_1, x^r_2); \quad x^r_1 = \psi^r(x^r_2), \quad x^r_2 \in [0, s^r_h]\}$$



(a) Subdomains $\hat{T}^+$ and $\hat{T}^-$ are formed by Γ^r.
(b) Subdomains T^+ and T^- are formed by Γ_h^r.

Figure 2: The partition of the interface element T by the arc segment Γ^r and the line segment Γ_h^r. The shaded domains correspond to T^{+*} and T^{-*}, respectively.

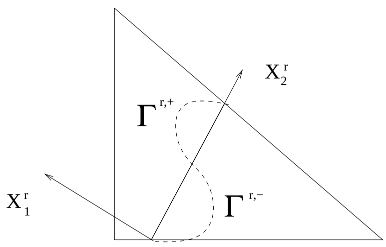


Figure 3: Local coordinates for arc segment Γ^r.

where s_h^r is the length of Γ_h^r. Due to the fact that the interface Γ is of class C^2, we have $\psi^r \in C^2([0, s_h^r])$ for $r = 1, .., m_h$. It was shown in [4] that

$$|\psi^r(x_2^r)| \leq C(s_h^r)^2 \leq Ch^2, \quad \forall x_2^r \in [0, s_h^r], \tag{8}$$

$$\left|\frac{\partial}{\partial x_2^r}\psi^r(x_2^r)\right| \leq Ch, \quad \forall x_2^r \in [0, s_h^r]. \tag{9}$$

Similar to (3) denote by $\tilde{H}^2(T) = \{u | u|_{\hat{T}^i} \in H^k(\hat{T}^i), \quad i = \pm\}$ and let $\mathcal{M}(T) = C(T) \cap \tilde{H}^2(T)$. Note, that on each interface triangle the solution $u = \hat{u} + \tilde{u}$ to (7) forms a set of functions that belong to $\mathcal{M}(T)$ and satisfies

$$\begin{cases} u^i = u|_{T^i} \in H^2(\hat{T}^i), \quad i = \pm, \\ u^+(A) = u^-(A), \quad u^+(B) = u^-(B), \\ Q_{\Gamma^r}(u) := \int_{\Gamma^r} (\beta^- \nabla u^- - \beta^+ \nabla u^+) \cdot \mathbf{n}_{\Gamma^r} ds = \int_{\Gamma^r} q ds \end{cases}$$

where $\mathbf{n}_{\Gamma^r}$ is the unit normal vector to the arc segment Γ^r. We are now ready to consider the construction of the finite element functions on an interface triangle. The strategy is to use the partition of $T \in \mathcal{T}_h'$ generated by the approximation of the interface Γ_h in order to approximate the set $\mathcal{M}(T)$. That is, for each interface triangle we form a finite element function by two polynomials defined separately on T^- and T^+

$$\phi(x,y) = \begin{cases} \phi^-(x,y) = a_1 x + a_2 y + a_3, & (x,y) \in T^-, \\ \phi^+(x,y) = b_1 x + b_2 y + b_3, & (x,y) \in T^+. \end{cases} \tag{10}$$

Then for all $T \in \mathcal{T}_h'$ define a linear space

$$S_h(T) = \begin{cases} \phi \text{ is defined by (10)}, \\ \phi^-(A) = \phi^+(A), \quad \phi^-(B) = \phi^+(B), \end{cases}$$

that consists of the the piecewise linear functions satisfying the regularity conditions along Γ_h. Define

$$Q_{\Gamma_h^r}(\phi) := \int_{\Gamma_h^r} (\beta_h^- \nabla\phi^- - \beta_h^+ \nabla\phi^+) \cdot \mathbf{n}_{\Gamma_h^r} ds$$

where $\mathbf{n}_{\Gamma_h^r}$ is the unit normal vector to line segment Γ_h^r and β_h stands for the approximation of the coefficient function $\beta(x,y)$ that is equal to β on the non-interface elements and is defined as follows

$$\beta_h(x,y) = \begin{cases} \beta^-, & \forall \quad (x,y) \in T^-, \\ \beta^+, & \forall \quad (x,y) \in T^+, \end{cases}$$

for any $T \in \mathcal{T}_h'$. Note, that β_h is just a restriction of the piecewise constant function. We now denote by $S_h^0(T)$ the following linear space

$$S_h^0(T) = \{\phi \in S_h(T) |, Q_{\Gamma_h^r}(\phi) = 0\}$$

and refer to it as an immersed interface spaces on the interface triangle T. Then, for all $u \in \mathcal{M}(T)$, we denote by $I_h u \in S_h(T)$ its interpolant as follows

$$I_h : \mathcal{M}(T) \to S_h(T)$$

that satisfies the standard interpolation conditions

$$I_h u(\mathbf{Z}_i) = u(\mathbf{Z}_i), \quad 1 \le i \le 3, \tag{11}$$

and the interpolation condition on $Q_{\Gamma_h^r}(\phi)$

$$Q_{\Gamma_h^r}(I_h u) = Q_{\Gamma^r}(u). \tag{12}$$

We now use the partition $\mathcal{T}_h$ to define an immersed finite element space $S_h(\Omega)$ and $S_h^0(\Omega)$ on the whole of Ω. Namely,

$$S_h(\Omega) = \begin{cases} \phi \in S_h(T), & \forall T \in \mathcal{T}_h', \\ \phi \text{ standard linear function }, & \forall T \in \mathcal{T}_h \backslash \mathcal{T}_h', \end{cases}$$

$$S_h^0(\Omega) = \begin{cases} \phi \in S_h^0(T), & \forall T \in \mathcal{T}_h', \\ \phi \text{ standard linear function }, & \forall T \in \mathcal{T}_h \backslash \mathcal{T}_h'. \end{cases}$$

Since neither $S_h(\Omega)$ nor $S_h^0(\Omega)$ are subspaces of $H_0^1(\Omega)$, the resulting finite element method is non-conforming. The possible discontinuities of the functions in $\in S_h(\Omega)$ and $\in S_h^0(\Omega)$ are located along the edges of $T \in \mathcal{T}_h'$ and the across the interface Γ. However, note that across the piecewise linear approximation Γ_h these functions are continuous. We use $S_{h,0}^0$ to denote the subspace of S_h^0 with its functions vanishing on the boundary $\partial\Omega$. For the later analysis, we also introduce the following space

$$\mathbf{X} = H_0^1(\Omega) + S_h(\Omega).$$

Corresponding to the bilinear form $a(\cdot,\cdot)$ defined previously, we introduce its discrete form $a_h(\cdot,\cdot) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$

$$a_h(u,v) = \sum_{T\in\mathcal{T}_h} \int_T \beta_h \nabla u \cdot \nabla v dxdy, \quad \forall u, v \in X. \tag{13}$$

Obviously, if $u, v \in H_0^1(\Omega)$, $a_h(u,v) = a(u,v)$. Furthermore, with a given partition $\mathcal{T}_h$ and $m \geq 1$ we define the discrete energy norm

$$||u||_h := \sqrt{a_h(u,u)},$$

together with the following discrete norms

$$||u||_{m,T} := ||u||_{m,T^+\cap\hat{T}^+} + ||u||_{m,T^-\cap\hat{T}^-} + ||u||_{m,T^{+*}} + ||u||_{m,T^{-*}},$$
$$||u||_{m,h} := \sqrt{\sum_{T\in\mathcal{T}_h} ||u||_{m,T}^2}.$$

with the seminorm $|\cdot|_{m,T}$ defined in the similar way. These norms are a common quantities used in the error estimation of non-conforming finite element methods. It is obvious that

$$||u||_h \leq C\sqrt{\sum_{T\in\mathcal{T}_h} |u|_{1,T}^2} \leq C||u||_{1,h} \tag{14}$$

together with

$$|a_h(u,v)| \leq C||u||_h||v||_h \quad \text{and} \quad a_h(u,u) \geq C||u||_h^2.$$

We are now in a position to define the finite element approximation to the interface problem (1): Find $\hat{u}_h \in S_{h,0}^0(\Omega)$ such that

$$a_h(\hat{u}_h, v_h) = (f, v_h)_\Omega + \langle q_h, v_h\rangle_{\Gamma_h} - a_h(\tilde{u}_h, v_h), \quad \forall v_h \in S_{h,0}^0(\Omega) \tag{15}$$

where $\tilde{u}_h \in S_h(\Omega)$ is an appropriately chosen approximation of $\tilde{u}$ and q_h is the linear restriction of the interface function q on Γ_h, such that on each segment Γ_h^r of length s_h^r, q_h is given by

$$q_h(0,x_2^r) = \frac{x_2^r}{s_h^r} q(0,s_h^r) + \frac{s_h^r - x_2^r}{s_h^r} q(0,0), \quad \forall x_2^r \in [0, s_h^r]. \tag{16}$$

3 Theoretical Analysis

In this section we prove that the solution obtained from the immersed interface finite element method with the linear basis functions is second order accurate in L^2 norm. The strategy of the proof resembles the idea behind the proof in the case of smooth solutions [1]:

- determine the order of approximation of the interpolant of the solution in the given finite element space;
- use Strang's lemma for the non-conforming finite element methods to find the order of the convergence in the discrete energy norm;
- apply generalized Aubin-Nitsche's lemma to gain an $L^2(\Omega)$ error estimate.

The critical point is that, in the present case, the solution does not fulfill the necessary regularity assumptions. Therefore, it will be crucial to use the details of the construction of our finite element space. Note, that throughout the paper C denotes some generic constant that is independent of the finite element mesh parameter h.

3.1 An interpolation result

We now discuss the approximation capability of the linear space $S_h(T)$ when $T \in \mathcal{T}_h'$. Let $\mathcal{M}^{ext}(T) = \mathcal{M}(T) + S_h(T)$ and endow this space the norm

$$|||u|||_{2,T} = |u|_{2,T} + \sum_{i=1}^{3} |u(\mathbf{Z}_i)| + \begin{cases} |Q_{\Gamma^r}(u)|, & u \in \mathcal{M}(T) \\ |Q_{\Gamma_h^r}(u)|, & u \in S_h(T) \end{cases} , \quad \forall u \in \mathcal{M}^{ext}(T)$$

where $\mathbf{Z}_i = (x_i, y_i)$ are the vertex coordinates of the interface triangle.

Lemma 3.1. *The norms $|||\cdot|||_{2,T}$ and $||\cdot||_{2,T}$ are equivalent on $\mathcal{M}^{ext}(T)$.*

Proof. By the imbedding and the trace theorems of Sobolev spaces

$$|||u|||_{2,T} \leq C||u||_{2,T}.$$

Suppose now that the converse statement

$$||u||_{2,T} \leq C|||u|||_{2,T}, \quad \text{for } \forall u \in \mathcal{M}^{ext}(T)$$

fails for every positive number C. Then there exists a sequence $\{u_k\} \in \mathcal{M}^{ext}(T)$ such that

$$||u_k||_{2,T} = 1, \quad |||u_k|||_{2,T} \leq \frac{1}{k}.$$

Invoking the compactness of the imbedding of $\tilde{H}^2(T)$ in $\tilde{H}^1(T)$, we may further assume that this sequence converges in $||\cdot||_{1,T}$. Since

$$||u_l - u_m||_{2,T}^2 \leq ||u_l - u_m||_{1,T}^2 + (|u_l|_{2,T} + |u_m|_{2,T})^2,$$

we conclude that $\{u_k\}$ is a Cauchy sequence in $\mathcal{M}^{ext}(T)$ with respect to the $||\cdot||_{2,T}$ norm. Thus, there exists a $u^* \in \mathcal{M}^{ext}(T)$ such that $\lim_{k\to\infty} u_k = u^*$. By continuity considerations we have

$$||u^*||_{2,T} = 1, \quad |||u^*|||_{2,T} = 0$$

which is a contradiction.

$\square$

Lemma 3.2. *Let T be an interface triangle and the interpolation operator $I_h : \mathcal{M}(T) \to S_h(T)$ defined by (11) and (12). Then we have*

$$||u - I_h u||_{2,T} \leq C|u|_{2,T}, \quad \forall u \in \mathcal{M}^{ext}(T).$$

Proof. By Lemma 3.1

$$||u - I_h u||_{2,T} \leq C|||u - Iu|||_{2,T} = C\Big(|u - Iu|_{2,T} + \sum_{i=1}^{3} |(u - Iu)(\mathbf{Z}_i)| + |Q_{\Gamma^r}(u) - Q_{\Gamma^r_h}(I_h u)|\Big). \quad (17)$$

Due to the definition of $Q_{\Gamma^r}(I_h u)$ and the linearity of the interpolant we get

$$||u - Iu||_{2,T} \leq C|u - Iu|_{2,T} \leq C|u|_{2,T}$$

where we have also used the fact that the interpolant $I_h u$ coincides with u at the interpolation points $\mathbf{Z}_i = (x_i, y_i)$. $\square$

Lemma 3.3. *Let Σ and $\hat{\Sigma}$ be affine equivalent, i.e. there exists a bijective affine mapping*

$$F : \hat{\Sigma} \to \Sigma, \quad F\hat{\xi} = \xi_0 + B\hat{\xi}$$

with a nonsingular transformation matrix B. If $v \in \tilde{H}^m(\Sigma)$, then $\hat{v} := v \circ F \in \tilde{H}^m(\hat{\Sigma})$, and there exists a constant C such that

$$|v|_{k,\Sigma} \leq C||B||^{-k}|\det B|^{1/2}|\hat{v}|_{k,\hat{\Sigma}}, \quad (18)$$

$$|\hat{v}|_{k,\hat{\Sigma}} \leq C||B||^{k}|\det B|^{-1/2}|v|_{k,\hat{\Sigma}}. \quad (19)$$

Proof. See [1] for the proof and further details on the affine transformations. $\square$

Consider now a function u satisfying

$$u \in C(\Omega), \quad u|_{\Omega^i} \in H^2(\Omega^i), \quad i = \pm$$

together with

$$(\beta^- \nabla u^- - \beta^+ \nabla u^+) \cdot \mathbf{n} = q$$

across the interface Γ. Then, the main approximation result of this chapter reads

Theorem 3.4. *Under the assumptions of Lemma 3.2 and for the linear interpolation operator $I_h : \mathcal{M}(T) \to S_h(T)$ we have*

$$||u - I_h u||_{0,h} + h||u - I_h u||_{1,h} \leq Ch^2||u||_{2,\Omega}, \quad \forall u \in \mathcal{M}(T).$$

Proof. It suffices to proof only the following inequality

$$||u - I_h u||_{l,T} \leq Ch^{2-l}|u|_{2,T}, \quad l = 0, 1.$$

Apply Lemma 3.2 on the reference element T_{ref} and use the transformation formulas from Lemma 3.3 in both directions to obtain

$$\begin{aligned} |u - I_h u|_{l,T} &\leq C||B||^{-l}|\det B|^{1/2}|u_{ref} - I_h u_{ref}|_{l,T_{ref}} \\ &\leq C||B||^{-l}|\det B|^{1/2}|u_{ref}|_{2,T_{ref}} \leq C||B||^{-l}|\det B|^{1/2}||B||^2|\det B|^{-1/2}|u|_{2,T} \\ &\leq C(||B||||B^{-1}||^l)||B||^{2-l}|u|_{2,T}. \end{aligned} \tag{20}$$

By the shape regularity we get

$$|u - I_h u|_{l,T} \leq Ch^{2-l}|u|_{2,T}.$$

Finally, squaring and summing over l establishes the assertion. □

3.2 Convergence in discrete energy norm

The main result of this section is stated in the following theorem:

Theorem 3.5. *Assume that the solution $\hat{u}$ of the interface problem (7) is in $C(\Omega) \cap \tilde{H}^2(\Omega)$. Then the immersed finite element solution $\hat{u}_h$ given by (15) has the following estimate for a constant $C > 0$*

$$||\hat{u} - \hat{u}_h||_h \leq C\Big[h(|\hat{u}|_{2,h} + |\tilde{u}|_{2,h}) + h^{3/2}|q|_{2,\Gamma}\Big].$$

where $\tilde{u}$ is the solution to (6).

Before proving Theorem 3.5, we first show several additional results that we will require. For w in $C(\Omega) \cap \tilde{H}^2(\Omega)$ and $v_h \in S^0_{h,0}(\Omega)$ we can write the bilinear form (13) as follows

$$\begin{aligned} a_h(w, v_h) = &\sum_{T \in \mathcal{T}_h \setminus \mathcal{T}'_h} \beta \nabla w \cdot \nabla v_h dxdy \\ &+ \sum_{T \in \mathcal{T}'_h} \Big(\int_{T^-} \beta^- \nabla w \cdot \nabla v_h^- dxdy + \int_{T^+} \beta^+ \nabla w \cdot \nabla v_h^+ dxdy \Big) \end{aligned} \tag{21}$$

where $v_h^i = v_h|_{T^i}$ for $i = \pm$. Note, that the second sum represents the bilinear form on the interface triangles $T \in \mathcal{T}'_h$. Without loss of generality, consider a typical interface element, see Figures 1 and 2. Then

$$
\begin{aligned}
a_h(w, v_h) = &\sum_{T\in\mathcal{T}_h\setminus\mathcal{T}_h'} \beta\nabla w\cdot\nabla v_h dxdy + \\
&\sum_{T\in\mathcal{T}_h'}\Big(\int_{T^-\setminus T^{-*}} \beta^-\nabla w^-\cdot\nabla v_h^- dxdy + \int_{T^+\setminus T^{+*}} \beta^+\nabla w^+\cdot\nabla v_h^+ dxdy \\
&\qquad + \int_{T^{-*}} \beta^-\nabla w^+\cdot\nabla v_h^- dxdy + \int_{T^{+*}} \beta^+\nabla w^-\cdot\nabla v_h^+ dxdy\Big) \qquad (22)
\end{aligned}
$$

where $w^i = w|_{\Omega^i}$ for $i=\pm$. Add and subtract

$$
\int_{T^{-*}} \beta^+\nabla w^+\cdot\nabla v_h^+ dxdy \qquad \text{and} \qquad \int_{T^{+*}} \beta^-\nabla w^-\cdot\nabla v_h^- dxdy
$$

to obtain

$$
\begin{aligned}
&\int_{T^-} \beta^-\nabla w\cdot\nabla v_h^- dxdy + \int_{T^+} \beta^+\nabla w\cdot\nabla v_h^+ dxdy \\
&\qquad = \int_{\hat{T}^-} \beta^-\nabla w^-\cdot\nabla v_h^- dxdy + \int_{\hat{T}^+} \beta^+\nabla w^+\cdot\nabla v_h^+ dxdy \\
&\quad + \int_{T^{-*}} \nabla w^+\cdot(\beta^-\nabla v_h^- - \beta^+\nabla v_h^+) dxdy + \int_{T^{+*}} \nabla w^-\cdot(\beta^+\nabla v_h^+ - \beta^-\nabla v_h^-) dxdy \qquad (23)
\end{aligned}
$$

for any $T\in\mathcal{T}_h'$. Note, that v_h^+ on T^{-*} and v_h^- on T^{+*} should be understood as general extension of v_h^+ from T^+ and v_h^- from T^-, correspondingly. Integrating by parts the terms over $\hat{T}^+$ and $\hat{T}^-$ we get

$$
\begin{aligned}
&\int_{T^-} \beta_h^-\nabla w\cdot\nabla v_h^- dxdy + \int_{T^+} \beta_h^+\nabla w\cdot\nabla v_h^+ dxdy \\
&\qquad = -\int_T \nabla\cdot(\beta\nabla w) v_h dxdy + \int_{\partial\hat{T}^+} \beta^+\Big(\frac{\partial w}{\partial n^+}\Big) v_h^+ ds + \int_{\partial\hat{T}^-} \beta^-\Big(\frac{\partial w}{\partial n^-}\Big) v_h^- ds \\
&\quad + \int_{T^{-*}} \nabla w^+\cdot(\beta^-\nabla v_h^- - \beta^+\nabla v_h^+) dxdy + \int_{T^{+*}} \nabla w^-\cdot(\beta^+\nabla v_h^+ - \beta^-\nabla v_h^-) dxdy \qquad (24)
\end{aligned}
$$

where n^- and n^+ are the outward normal vectors to T^{-*} and T^{+*} correspondingly. Insert

(24) in (21) and Take $n = n^- = -n^+$ and sum over $T \in \mathcal{T}_h$ to obtain

$$
\begin{aligned}
-\int_\Omega \nabla\cdot(\beta\nabla w)v_h dxdy &+ \sum_{e_j\in\mathcal{E}'_h}\Big(\int_{e_j^-}\Big[\Big[\beta\frac{\partial w}{\partial n}v_h\Big]\Big]ds + \int_{e_j^+}\Big[\Big[\beta\frac{\partial w}{\partial n}v_h\Big]\Big]ds\Big)\\
&+\sum_{\Gamma^r\in\Gamma}\int_{\Gamma^r}\Big[\beta\frac{\partial w}{\partial n}v_h\Big]ds + \sum_{T^{-*}\in\mathcal{T}'_h}\int_{T^{-*}}\nabla w^+\cdot(\beta^-\nabla v_h^- - \beta^+\nabla v_h^+)dxdy\\
&+\sum_{T^{+*}\in\mathcal{T}'_h}\int_{T^{+*}}\nabla w^-\cdot(\beta^+\nabla v_h^+ - \beta^-\nabla v_h^-)dxdy\\
&= -\int_\Omega\nabla\cdot(\beta\nabla w)v_h dxdy + \sum_{e_i\in\mathcal{E}'_h}((I)_1+(I)_2)\\
&\quad + \sum_{\Gamma^r\in\Gamma}(I)_3 + \sum_{T^{-*}\in\mathcal{T}'_h}(I)_4 + \sum_{T^{+*}\in\mathcal{T}'_h}(I)_5. \qquad (25)
\end{aligned}
$$

Here $\mathcal{E}'_h$ is the set of all edges e_j that meet the interface Γ between their vertices and $[[\cdot]]$ denotes the jump between two neighboring elements along the common edge e.

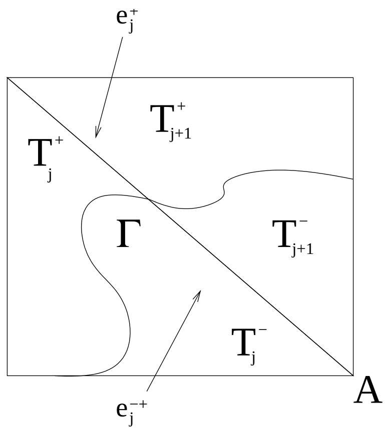


Figure 4: An example of two adjacent interface triangles.

Note, that the first sum in (25) represents the consistency error across the neighboring edges of the interface triangles and the second summation represents the error across the interface itself, see the sketch in Figure 4. The choice of normal direction in the first summation does not affect the results. The strategy now is to take $w = \tilde{u}$ and address each of the integral terms in (25) separately. Note, that in this case , the first integral term vanishes due to (6).

Lemma 3.6. *Assume that the solution $\tilde{u}$ of the interface problem (6) is in $C(\Omega)\cap\tilde{H}^2(\Omega)$. Then there exists a constant C such that for any $v_h \in S^0_{h,0}(\Omega)$ we have*

$$
|\sum_{e_j\in\mathcal{E}'_h}((I)_1+(I)_2)| \le Ch|\tilde{u}|_{2,h}||v_h||_h. \qquad (26)
$$

Proof. Assume that e_j^- is the common edge between T_j and T_{j+1} elements. Since $[\beta\frac{\partial \tilde{u}}{\partial n}]=0$ in $(T_j\cup T_{j+1})\cap\Omega^-$ and $(T_j\cup T_{j+1})\cap\Omega^+$ we have

$$(I)_1=\int_{e_j^-}\beta\frac{\partial \tilde{u}}{\partial n}[[v_h]]ds.$$

We also note that that for a common vertex A of T_j^- and T_{j+1}^- (see Figure 4) we have

$$v_{h,j}^-(A)=v_{h,j+1}^-(A).$$

Thus

$$\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}[[v_h]]ds\Big|\le\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}(v_{h,j+1}^--v_{h,j+1}^-(A))ds\Big|+\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}(v_{h,j}^-(A)-v_{h,j}^-)ds\Big|.$$

Denote by $\mathbf{X}_s=(x(s),y(s))$ some point on e_j^-, then the difference $v_{h,j+1}^-(\mathbf{X}_s)-v_{h,j+1}^-(A)$ can be written in the following form

$$v_{h,j+1}^-(\mathbf{X}_s)-v_{h,j+1}^-(A)=\int_{\overline{A\mathbf{X}_s}}\frac{\partial v_{h,j+1}^-}{\partial t}d\eta$$

where $\frac{\partial v_{h,j+1}}{\partial t}$ is the tangential derivative along $\overline{A\mathbf{X}_s}$. Then, by Cauchy-Schwartz inequality together with the trace theorem we get

$$\begin{aligned}\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}(v_{h,j+1}^--v_{h,j+1}^-(A))ds\Big|&\le\beta^-h\Big(\int_{e_j^-}\Big|\frac{\partial v_{h,j+1}^-}{\partial t}\Big|^2ds\Big)^{1/2}\Big(\int_{e_j^-}\Big|\frac{\partial \tilde{u}}{\partial n}\Big|^2ds\Big)^{1/2}\\&\le C\beta^-h||\nabla v_{h,j+1}^-||_{1,T_{j+1}^-}||\nabla\tilde{u}||_{1,T_{j+1}^-}.\end{aligned}$$

Finally, using the fact that $v_{h,j+1}^-$ is a linear polynomial we arrive at

$$\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}(v_{h,j+1}^--v_{h,j+1}^-(A))ds\Big|\le C\beta^-h|v_{h,j+1}^-|_{1,T_{j+1}^-}|\tilde{u}|_{2,T_{j+1}^-}.$$

Similarly, we obtain

$$\Big|\int_{e_j^-}\beta^-\frac{\partial \tilde{u}}{\partial n}(v_{h,j}^-(A)-v_{h,j}^-)ds\Big|\le C\beta^-h|v_{h,j}^-|_{1,T_j^-}|\tilde{u}|_{2,T_j^-}$$

that combined with the previous estimate gives us

$$\Big|(I)_1\Big|\le C\beta^-h\Big(|v_{h,j+1}|_{1,T_{j+1}^-}|\tilde{u}|_{2,T_{j+1}^-}+|v_{h,j}|_{1,T_j^-}|\tilde{u}|_{2,T_j^-}\Big). \tag{27}$$

By the same argument, we arrive at

$$\Big|(I)_2\Big|\le C\beta^+h\Big(|v_{h,j+1}|_{1,T_{j+1}^+}|\tilde{u}|_{2,T_{j+1}^+}+|v_{h,j}|_{1,T_j^+}|\tilde{u}|_{2,T_j^+}\Big). \tag{28}$$

Then, the result of the lemma is obtained by inserting (27) and (28) in the left hand side of (26), using the triangular inequality and summing over the edges e_j. □

We now consider the consistency error across the interface.

Lemma 3.7. *Under the assumptions of Lemma 3.6 we have*

$$\Big|\sum_{\Gamma^r\in\Gamma}\Big(\int_{\Gamma^r}\Big[\beta\frac{\partial\tilde{u}}{\partial n}v_h\Big]ds-\langle q,v_h\rangle_{\Gamma^r}\Big)\Big|\le Ch\max(\beta^+,\beta^-)|\tilde{u}|_{2,h}||v_h||_h$$

where

$$\langle q,v_h\rangle_{\Gamma^r}=\int_{\Gamma^{r,+}}qv_h^+ds+\int_{\Gamma^{r,-}}qv_h^-ds.$$

Proof. Consider

$$\begin{aligned}(I)_3=\int_{\Gamma^r}\Big[\beta\frac{\partial\tilde{u}}{\partial n}v_h\Big]ds=\int_{\Gamma^{r,+}}\Big(\beta^-\frac{\partial\tilde{u}^-}{\partial n}v_h^-ds-\beta^+\frac{\partial\tilde{u}^+}{\partial n}v_h^+\Big)ds&\\ +\int_{\Gamma^{r,-}}\Big(\beta^-\frac{\partial\tilde{u}^-}{\partial n}v_h^-ds-\beta^+\frac{\partial\tilde{u}^+}{\partial n}v_h^+\Big)ds.&\end{aligned}$$

Add and subtract

$$\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}v_h^-ds\qquad\text{and}\qquad\int_{\Gamma^{r,+}}\beta^-\frac{\partial\tilde{u}^-}{\partial n}v_h^+ds$$

to obtain

$$\begin{aligned}(I)_3&=\int_{\Gamma^{r,-}}\Big(\beta^-\frac{\partial\tilde{u}^-}{\partial n}-\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big)v_h^-ds+\int_{\Gamma^{r,+}}\Big(\beta^-\frac{\partial\tilde{u}^-}{\partial n}-\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big)v_h^+ds\\ &\quad+\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}(v_h^--v_h^+)ds+\int_{\Gamma^{r,+}}\beta^-\frac{\partial\tilde{u}^-}{\partial n}(v_h^--v_h^+)ds\\ &=\langle q,v_h\rangle_{\Gamma^r}+\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}(v_h^--v_h^+)ds+\int_{\Gamma^{r,+}}\beta^-\frac{\partial\tilde{u}^-}{\partial n}(v_h^--v_h^+)ds.\end{aligned}$$

We now proceed in the similar fashion as we did in (27). That is, assume that the interface segment Γ^r meets the edge of the interface triangle T^r at a point E. Then by construction of the functions in $S_{h,0}^0$ we have

$$v_h^+(E)=v_h^-(E).$$

Therefore

$$\begin{aligned}&\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}(v_h^--v_h^+)ds=\\ &\qquad\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big(v_h^--v_h^-(E)\Big)ds+\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big(v_h^+(E)-v_h^+\Big)ds\\ &\qquad=\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big(\int_{\widehat{EX_s}}\frac{\partial v_h^+}{\partial t}d\eta\Big)ds+\int_{\Gamma^{r,-}}\beta^+\frac{\partial\tilde{u}^+}{\partial n}\Big(\int_{\widehat{X_sE}}\frac{\partial v_h^-}{\partial t}d\eta\Big)ds\end{aligned}$$

for some point $\mathbf{X}_s = (x(s), y(s))$ on $\Gamma^{r,-}$. Repeating the arguments of (27) and (28) we obtain

$$\Big|\int_{\Gamma^{r,-}} \beta^+ \frac{\partial \tilde{u}^+}{\partial n}\Big(\int_{\widehat{EX_s}} \frac{\partial v_h^+}{\partial t} d\eta\Big) ds\Big| \le C\beta^+ h |v_h^+|_{1,T^{-*}} |\tilde{u}^+|_{2,T^{-*}}$$

and

$$\Big|\int_{\Gamma^{r,-}} \beta^+ \frac{\partial \tilde{u}^+}{\partial n}\Big(\int_{\widehat{EX_s}} \frac{\partial v_h^-}{\partial t} d\eta\Big) ds\Big| \le C\beta^+ h |v_h^-|_{1,T^{-*}} |\tilde{u}^+|_{2,T^{-*}}$$

Thus

$$\begin{aligned}\Big|\int_{\Gamma^{r,-}} \beta^+ \frac{\partial \tilde{u}^+}{\partial n}(v_h^- - v_h^+) ds\Big| &\le C\beta^+ h |\tilde{u}^+|_{2,T^{-*}}(|v_h^-|_{1,T^{-*}} + |v_h^+|_{1,T^{-*}}) \\ &\le C\beta^+ h |\tilde{u}^+|_{2,\hat{T}^+} ||v_h||_{h,T}. \qquad (29)\end{aligned}$$

Similar, we obtain

$$\begin{aligned}\Big|\int_{\Gamma^{r,+}} \beta^- \frac{\partial \tilde{u}^-}{\partial n}(v_h^- - v_h^+) ds\Big| &\le C\beta^- h |\tilde{u}^-|_{2,T^{+*}}(|v_h^-|_{1,T^{+*}} + |v_h^+|_{1,T^{+*}}) \\ &\le C\beta^- h |\tilde{u}^-|_{2,\hat{T}^-} ||v_h||_{h,T}. \qquad (30)\end{aligned}$$

Finally, the result of the lemma is obtained by summing over the interface arcs Γ^r. □

Lemma 3.8. *Under the assumptions of Lemma 3.6 we have*

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h'}\Big(\int_{T^{-*}} \nabla\tilde{u}^+ \cdot (\beta^-\nabla v_h^- - \beta^+\nabla v_h^+) dxdy + \int_{T^{+*}} \nabla\tilde{u}^- \cdot (\beta^+\nabla v_h^+ - \beta^-\nabla v_h^-) dxdy\Big)\Big| \\ \le Ch|\tilde{u}|_{2,h}||v_h||_h.\end{aligned}$$

Proof. By Greens theorem we get

$$\begin{aligned}\int_{T^{-*}} \nabla\tilde{u}^+ \cdot (\beta^-\nabla v_h^- - \beta^+\nabla v_h^+) dxdy &= -\int_{\partial T^{-*}} \tilde{u}^+ (\beta^- \frac{\partial v_h^-}{\partial n} - \beta^+ \frac{\partial v_h^+}{\partial n}) ds \\ &= -\Big(\int_{\Gamma^{r,-}} \tilde{u}^+ (\beta^- \frac{\partial v_h^-}{\partial n} - \beta^+ \frac{\partial v_h^+}{\partial n}) ds + \int_{\Gamma_h^{r,-}} \tilde{u}^+ (\beta^- \frac{\partial v_h^-}{\partial n} - \beta^+ \frac{\partial v_h^+}{\partial n}) ds\Big)\end{aligned}$$

where $\Gamma_h^{r,-}$ is a linear approximation of $\Gamma^{r,-}$. Following the same arguments as in (29) we obtain

$$\begin{aligned}\Big|\int_{T^{-*}} \nabla\tilde{u}^+ \cdot (\beta^-\nabla v_h^- - \beta^+\nabla v_h^+) dxdy\Big| \\ \le Ch\max(\beta^-, \beta^+)|\tilde{u}^+|_{2,T^{-*}}(|v_h^-|_{1,T^{+*}} + |v_h^+|_{1,T^{+*}}) \\ \le Ch\max(\beta^-, \beta^+)|\tilde{u}^+|_{2,\hat{T}^-}||v_h||_{h,T}.\end{aligned}$$

Similar

$$\Big|\int_{T^{+*}} \nabla\tilde{u}^- \cdot (\beta^+\nabla v_h^+ - \beta^-\nabla v_h^-) dxdy\Big| \le Ch\max(\beta^-, \beta^+)|\tilde{u}^-|_{2,\hat{T}^+}||v_h||_{h,T}.$$

Summing over $T \in \mathcal{T}_h'$ establishes the claim. □

Theorem 3.9. *Assume that the solution $\tilde{u}$ of the interface problem (6) is in $C(\Omega) \cap \tilde{H}^2(\Omega)$. Then there exists a constant C such that for any $v_h \in S^0_{h,0}(\Omega)$ we have*

$$|a_h(\tilde{u}, v_h) - \langle q, v_h \rangle_\Gamma| \leq Ch|\tilde{u}|_{2,h}||v_h||_h.$$

Proof. The result is obtained by representing the bilinear form as in (25) and using Lemmas 3.6, 3.7 and 3.8. □

Theorem 3.10. *Assume that the solution $\hat{u}$ of the interface problem (7) is in $H^1_0(\Omega) \cap \tilde{H}^2(\Omega)$. Then there exists a constant C such that for any $v_h \in S^0_{h,0}(\Omega)$ we have*

$$|a_h(\hat{u}, v_h) - (f, v_h)_\Omega| \leq Ch|\hat{u}|_{2,h}||v_h||_h.$$

Proof. Similar to Theorem 3.9 we use the representation (25) of the discrete bilinear form. Due to (1) the first integral terms in (25) will cancel out with $(f, v_h)_\Omega$. Then the assertion of the theorem is established by applying Lemmas 3.6, 3.7 and 3.8.

□

Lemma 3.11. *Let $q \in H^2(\Gamma)$ and q_h be its a piecewise-linear approximation given by (16). Then we have*

$$|\langle q_h, v_h \rangle_{\Gamma_h} - \langle q, v_h \rangle_\Gamma| \leq Ch^{3/2}|q|_{2,\Gamma}||v_h||_h, \quad \forall v_h \in S^0_{h,0}.$$

Proof. For the sake of simplicity, we will prove the lemma for the transformed problem in the local (x_1^r, x_2^r) coordinates. This slight abuse of notation will not affect the results. The arguments used in this proof are similar to those used in [3].

For $r = 1, .., m_h$ we have

$$\int_{\Gamma^r} qv_h ds - \int_{\Gamma_h^r} q_h v_h ds = \int_0^{s_h^r} q(\psi^r(x_2^r), x_2^r) v_h(\psi^r(x_2^r), x_2^r) \sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} dx_2^r - \int_0^{s_h^r} q_h(0, x_2^r) v_h(0, x_2^r) dx_2^r \quad (31)$$

By adding and subtracting

$$\int_0^{s_h^r} q(\psi(x_2^r), x_2^r) v_h(0, x_2^r) \sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} dx_2^r$$

and

$$\int_0^{s_h^r} q_h(0, x_2^r) v_h(0, x_2^r) \sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} dx_2^r$$

terms we can rewrite (31) as follows

$$
\begin{aligned}
\int_{\Gamma^r} q v_h ds - \int_{\Gamma_h^r} q_h v_h ds = \\
& \int_0^{s_h^r} q(\psi^r(x_2^r), x_2^r)\Big(v_h(\psi^r(x_2^r), x_2^r) - v_h(0, x_2^r)\Big)\sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} dx_2^r \\
& + \int_0^{s_h^r} \Big(q(\psi^r(x_2^r), x_2^r) - q_h(0, x_2^r)\Big) v_h(0, x_2^r)\sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} dx_2^r \\
& + \int_0^{s_h^r} q_h(0, x_2^r) v_h(0, x_2^r)\Big(\sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} - 1\Big) dx_2^r \\
& = (II)_1 + (II)_2 + (II)_3. \quad (32)
\end{aligned}
$$

The strategy now is to estimate each of the integral terms separately. Taylor expanding $v_h(\psi^r(x_2^r), x_2^r)$ around $v_h(0, x_2^r)$ and recalling (8) we obtain

$$(II)_1 \leq Ch^3 ||q||_{L^\infty(\Gamma^r)} ||\nabla v_h||_{L^\infty(T)}$$

and by the inverse inequality we get

$$(II)_1 \leq Ch^2 ||q||_{L^\infty(\Gamma^r)} ||\nabla v_h||_{0,T}. \quad (33)$$

For the second integral, the Cauchy-Schwartz inequality imply

$$(II)_2 \leq Ch^{1/2} ||v_h||_{L^\infty(\Gamma_h^r)} ||q(\psi^r(\cdot), \cdot) - q_h(0, \cdot)||_{0,\Gamma_h^r} \leq Ch^{5/2} ||v_h||_{L^\infty(\Gamma_h^r)} ||q||_{2,\Gamma_h^r}$$

where in the last inequality we used the standard one-dimensional interpolation results. By the inverse inequality we obtain

$$(II)_2 \leq Ch^{3/2} ||v_h||_{0,T} ||q||_{2,\Gamma_h^r}. \quad (34)$$

For the third term we note, that by binomial theorem and by (8)

$$\Big|\sqrt{1 + |\frac{d}{dx_2^r}\psi^r(x_2^r)|^2} - 1\Big| \leq \frac{1}{2}\Big|\frac{d}{dx_2^r}\psi^r(x_2^r)\Big|^2 \leq Ch^2.$$

This result together with the inverse inequality yield

$$(II)_3 \leq Ch^2 ||q||_{L^\infty(\Gamma_h^r)} ||v_h||_{0,T}. \quad (35)$$

Inserting (33),(34) and (35) in (32) and summing over the interface elements completes the proof. □

Lemma 3.12. *Assume that the solution $\tilde{u}$ of the interface problem (6) is in $C(\Omega) \cap \tilde{H}^2(\Omega)$. In addition, denote by $I_h\tilde{u} \in S_h$ its interpolant. Then there exists a constant C such that for any $v_h \in S^0_{h,0}(\Omega)$ we have*

$$|a_h(\tilde{u}, v_h) - a_h(I_h\tilde{u}, v_h)| \leq Ch|\tilde{u}|_{2,h}||v_h||_h.$$

Proof. By continuity of the discrete bilinear form we get

$$|a_h(\tilde{u}, v_h) - a_h(I_h\tilde{u}, v_h)| \leq C||\tilde{u} - I_h\tilde{u}||_h||v_h||_h \leq Ch||\tilde{u}||_{2,h}||v_h||_h$$

where in the last inequality we used (14) and the approximation result from Theorem 3.4. □

We are now ready to proof Theorem 3.5. Namely

Proof. By the second Strang's lemma [1], we have

$$||\hat{u} - \hat{u}_h||_h \leq C\Big(\inf_{v_h \in S^0_{h,0}} ||\hat{u} - v_h||_h + \sup_{w_h \in S^0_{h,0}} \frac{|L_{\hat{u}}(w_h)|}{||w_h||_h}\Big).$$

where the consistency error term is given by

$$L_{\hat{u}}(v_h) = (f, v_h)_\Omega + \langle q, v_h\rangle_{\Gamma_h} - a_h(\hat{u}, v_h) - a_h(I_h\tilde{u}, v_h).$$

Using Theorem 3.9 we rewrite $L_{\hat{u}}(v_h)$ as follows

$$\begin{aligned} |L_{\hat{u}}(v_h)| \leq |(f, v_h)_\Omega - a_h(\hat{u}, v_h)| + |\langle q_h, v_h\rangle_{\Gamma_h} - \langle q, v_h\rangle_\Gamma| \\ + |a_h(\tilde{u}, v_h) - a_h(I_h\tilde{u}, v_h)| + Ch|\tilde{u}|_{2,h}||v_h||_h. \end{aligned} \tag{36}$$

By (14) and Theorem 3.4 we also have

$$||\hat{u} - v_h||_h \leq Ch||\hat{u}||_{2,h}.$$

Then the result the theorem follows Theorem 3.10 together with the results from Lemmas 3.11 and 3.12. □

3.3 Convergence in L^2 norm

We now turn our attention to the L^2 estimates. Note that for any $g \in L^2(\Omega)$, the dual problem

$$a(\phi, v) = (g, v)_\Omega, \quad \forall v \in H^1(\Omega)$$

has a unique solution in $C(\Omega) \cap \tilde{H}^2(\Omega)$ satisfying

$$||\phi_g||_{2,h} \leq C||g||_{0,\Omega}.$$

Let $\phi_{g,h}$ denote the immersed finite element solution of this dual problem. Then, from Theorem 3.5 with $q = 0$ we have

$$||\phi_g - \phi_{g,h}||_h \leq Ch||\phi_g||_{2,h} \leq Ch||g||_{0,\Omega} \tag{37}$$

In addition, we apply Theorem 3.10 to get

$$\begin{aligned} |a_h(\hat{u} - \hat{u}_h, \psi_g) - (\hat{u} - \hat{u}_h, g)_\Omega| &\leq Ch|\psi_g|_{2,h}||\hat{u} - \hat{u}_h||_h \\ &\leq Ch^2||g||_{0,\Omega}|u|_{2,h}. \end{aligned} \tag{38}$$

$$\begin{aligned} |a_h(\hat{u}, \psi_g - \psi_{g,h}) - (f, \psi_g - \psi_{g,h})_\Omega| &\leq Ch|\hat{u}|_{2,h}||\psi_g - \psi_{g,h}||_h \\ &\leq Ch^2||g||_{0,\Omega}|u|_{2,h}. \end{aligned} \tag{39}$$

These estimates leads us to the following result

Theorem 3.13. *Assume that the conditions in the Theorem 3.10 are satisfied. Then the error of the immersed interface finite element solution $\hat{u}_h$ has the following estimate in the L^2 norm*

$$||\hat{u} - \hat{u}_h||_0 \leq Ch^2|\hat{u}|_{2,h} \tag{40}$$

for a constant $C > 0$.

Proof. By the generalized Aubin-Nitsche lemma [1], we have

$$\begin{aligned} ||\hat{u} - \hat{u}_h||_0 \leq \sup_{g \in L^2(\Omega)} \frac{1}{||g||_{0,\Omega}} \Big(&||\hat{u} - \hat{u}_h||_h||\psi_g - \psi_{g,h}||_h \\ &+ \Big|a_h(\hat{u} - \hat{u}_h, \psi_g) - (\hat{u} - \hat{u}_h, g)_\Omega\Big| + \Big|a_h(\hat{u}, \psi_g - \psi_{g,h}) - (f, \psi_g - \psi_{g,h})_\Omega\Big|\Big) \end{aligned} \tag{41}$$

The estimate (40) is obtained by applying Theorem 3.5, (37),(38) and (39) to the above. □

4 Numerical results

Here, we numerically investigate the performance of our immersed finite element method for two-dimensional elliptic problems. For the sake of simplicity, for all the test cases the computational domain Ω is the rectangle $-1 \leq x, y \leq 1$ and the interface Γ is represented by a circle with the center at the origin and with some radius r_0. For every problem the source term and the Dirichlet boundary conditions are determined from the exact solution. The main emphasis will be to investigate the performance of our approach and compare it to the results obtained with the standard finite element method, that is a standard conforming Galerkin finite element whose piecewise linear basis functions has not been modified using the jump conditions. In all the test problems, the solution is approximated on a uniform $n \times n$. For the performance analysis we employ the discrete L^2 norm defined by

$$||E_n||_{L^2} = h\sqrt{\sum_{i,j} e_{ij}^2},$$

where $e_{ij} = u(x_i, y_j) - u_{ij}$ is the error in the grid point (x_i, y_j) between the exact solution $u(x_i, y_j)$ and the approximate solution u_{ij}. We also display the ratios between the successive errors

$$ratio = ||E_n||_{L^2}/||E_{2n}||_{L^2}.$$

A ratio of 2 corresponds to first order accuracy, while a ratio of 4 indicates second order of accuracy.

4.1 Test problem 1

In this example we compare the results from our method and the standard FEM for a problem with the piecewise constant coefficient β. The problem reads

$$\nabla \cdot (\beta \nabla u) = 9\sqrt{x^2 + y^2}, \quad \text{on} \quad \Omega,$$
$$\beta(x, y) = \begin{cases} \beta^-, & \text{if} \quad r \leq r0, \\ \beta^+, & \text{otherwise} \ , \end{cases}$$

where $r = \sqrt{x^2 + y^2}$ is the radius, $r0 = \pi/6.28$ and the Dirichlet boundary conditions are given by the exact solution

$$u(x, y) = \begin{cases} r^3/\beta^-, & \text{if} \quad r \leq r0, \\ r^3/\beta^+ + (1/\beta^- - 1/\beta^+)r_0^3, & \text{otherwise.} \end{cases}$$

It is easy to check that in this case the solution and its flux are continuous ($[u]|_\Gamma = 0$ and $[\beta \partial u/\partial n]|_\Gamma = 0$). In this test problem, the emphasis is to investigate how well the modified and the standard schemes can handle the jump in the β coefficient. The solution for the case when $\beta_+ = 1000, \beta_- = 1$ is presented in Figure 5. Note, that the jump in the normal derivative of the solution caused by the large difference in the coefficients is captured sharply. The convergence studies for the cases when $\beta_+ = 1000, \beta_- = 1$ and $\beta_+ = 1000, \beta_- = 1$ are presented in Tables 1 and 2, correspondingly.

Note, that the immersed finite element method exhibits second order convergence in L^2. The standard FEM is at most first order accurate.

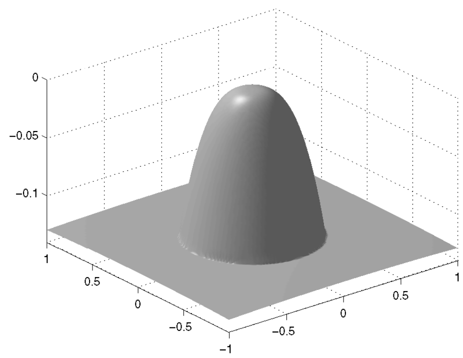


Figure 5: The inverse of the solution for the test problem 1 with piecewise constant coefficients $\beta_+ = 1000$ and $\beta_- = 1$ obtained by the immersed interface FEM.

n	Standard FEM		Modified FEM	
	$\|\|E_n\|\|_{L^2}$	ratio	$\|\|E_n\|\|_{L^2}$	ratio
20	3.94e-02	-	1.48e-03	-
40	2.00e-02	1.97	4.84e-04	3.06
80	9.71e-03	2.05	1.07e-04	4.49
160	4.31e-03	2.25	2.45e-05	4.40

Table 1: Grid refinement study for test problem 1 with $\beta_+ = 1000$ and $\beta_- = 1$ using both modified approach and standard FEM for $n \times n$ grids.

n	Standard FEM		Modified FEM	
	$\|\|E_n\|\|_{L^2}$	ratio	$\|\|E_n\|\|_{L^2}$	ratio
20	3.24e-02	-	1.63e-03	-
40	2.04e-02	1.58	4.38e-04	3.72
80	1.03e-02	1.97	1.04e-04	4.20
160	5.60e-03	1.84	2.49e-05	4.17

Table 2: Grid refinement study for test problem 2 with $\beta_+ = 1$ and $\beta_- = 1000$ using both modified approach and standard FEM for $n \times n$ grids .

4.2 Test problem 2

Here, we consider more elaborate problem with both discontinuous coefficients and a singular source function

$$\nabla \cdot (\beta \nabla u) = f(x,y) + C\delta_\Gamma, \quad \text{on} \quad \Omega,$$
$$f(x,y) = 8(x^2+y^2) + 4,$$
$$\beta(x,y) = \begin{cases} x^2+y^2+1, & \text{if} \quad r \le 0.5, \\ b, & \text{otherwise,} \end{cases}$$

with $C = 0.1$, $b = 10$ and the boundary conditions given by the exact solution

$$u(x,y) = \begin{cases} r^2, & \text{if} \quad r \le 0.5, \\ (1 - 1/(8b) - 1/b)/4 + (r^4/2 + r^2)/b + C\log(2r)/b, & \text{otherwise.} \end{cases}$$

Here, δ_Γ denotes the Dirac delta-functional with the support on the interface Γ. Table 3 gives the convergence analysis for the modified FEM and the standard FEM. The computed solution is presented in Figure 6. As expected, the solution is second order accurate in the L^2 norm.

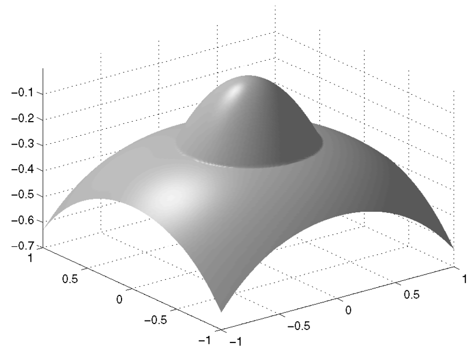


Figure 6: The inverse of the solution for the test problem 2 obtained by the immersed interface FEM.

n	Standard FEM		Modified FEM	
	$\|\|E_n\|\|_{L^2}$	ratio	$\|\|E_n\|\|_{L^2}$	ratio
20	2.69e-02	-	1.56e-02	-
40	1.46e-02	1.84	4.28e-03	3.65
80	6.85e-03	2.12	1.13e-03	3.76
160	3.74e-03	1.83	2.69e-04	4.22

Table 3: Grid refinement study for the test problem 2 with $\beta = 10$ and $C = 0.1$ using both modified FEM and standard FEM for $n \times n$.

5 Conclusions

In this paper, we have analyzed a new second order accurate finite element based method for the solution of the two-dimensional elliptic interface problems involving discontinuous coefficients and singular source functions on the uniform Cartesian grid that is not aligned with the interface. The interface jump conditions associated with the discontinuities in the coefficients and singularities of the forces have been derived and used to appropriately modify the basis functions such that the jump conditions are well approximated. The approximation capabilities of the method has been studied both theoretically and numerically. The numerical experiments also confirmed that, for the considered test problems, the investigated approach is superior to the standard finite element method.

References

[1] D. Braess. Finite elements. theory, fast solvers, and applications in solid mechanics. *Cambridge university press*, 1997.

[2] J. Bramble and J. King. A finite element method for interface problems in domains with smooth boundaries and interfaces. *Advances in Comput. Math.*, 6:109–138, 1996.

[3] Z. Chen and J. Zou. Finite element methods and their convergence for elliptic and parabolic interface problems. *Numer. Math.*, 79:175–202, 1998.

[4] M. Feistauer and A. Ženišek. Finite element solutions of nonlinear problems. *Numer. Math.*, 50:451–475, 1987.

[5] Z. Li. The immersed interface method using a finite element formulation. *Appl. Num. Math.*, 27:253–267, 1998.

[6] Z. Li, T. Lin, Y. Lin, and R.C. Rogers. An immersed finite element space and its approximation capability. *Numerical Methods of PDEs*, 20:338–367, 2004.

[7] A. Loubenets. A new finite element method for elliptic interface problems. *Licenciate thesis, Royal Institute of Technology, NADA*, 2006.

[8] A. Loubenets, T. Ali, and M. Hanke. Highly accurate finite element method for one-dimensional elliptic interface problems. *submitted in Applied Numerical Mathematics*, 2006.

[9] J.A. Roitberg and Z.G. Seftel. A theorem on homeomorphisms for elliptic systems and its applications. *Math. USSR-Sb.*, 7:439–465, 1987.

[10] T. Lin Z. Li and X. Wu. New cartesian grid methods for interface problems using finite element formulation. *NSCU CRSC-TR99-5*, 1999.