On the Relative Strength of Pebbling and Resolution

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The Big Picture

- Satisfiability algorithms
  - Dramatic developments last 10-15 years
  - SAT-solvers used to solve large-scale real-world problems
  - Best algorithms based on resolution proof system
  - Bottlenecks: time and memory consumption
- Pebble games
  - Used in 70s-80s to study programming languages, compiler optimization etc.
  - No developments whatsoever last 20-25(?) years
  - But has proven very useful in proof complexity last decade
- This talk
  - What can proof complexity say about time vs space?
  - Connections between resolution and pebble games?

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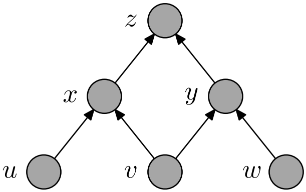
Just to Check We're on the Same Page...

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$ (assume k fixed)
- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

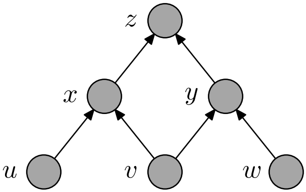


Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example CNF Formula

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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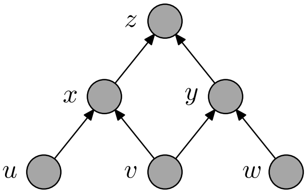

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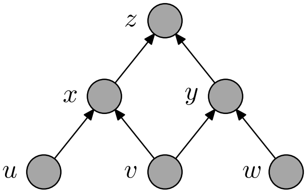

Defined in terms of directed acyclic graph (DAG):

- source vertices true
- truth propagates upwards
- but sink vertex is false

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	0
max # lines on board	0
max # literals on board	0

Can write down axioms,
erase used clauses or
infer new clauses by resolution rule

$$\frac{B \vee x \qquad C \vee \overline{x}}{B \vee C}$$

(but only from clauses currently on the board!)

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	1
max # lines on board	1
max # literals on board	1

u

Write down axiom 1: u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	2
max # lines on board	2
max # literals on board	2

u
v

Write down axiom 1: u
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	3
max # lines on board	3
max # literals on board	5

u

v

$\overline{u} \vee \overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

u
v
$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v

$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from

u and $\overline{u} \lor \overline{v} \lor x$

Erase the line $\overline{u} \lor \overline{v} \lor x$

Erase the line u

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	4
max # lines on board	4
max # literals on board	7

v

$\overline{v} \lor x$

u and $\overline{u} \lor \overline{v} \lor x$

Erase the line $\overline{u} \lor \overline{v} \lor x$

Erase the line u

Infer x from

v and $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
$\overline{v} \vee x$
x

u and $\overline{u} \vee \overline{v} \vee x$
Erase the line $\overline{u} \vee \overline{v} \vee x$
Erase the line u
Infer x from
v and $\overline{v} \vee x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
$\overline{v} \lor x$
x

Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u
Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

v
x

Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u
Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

Board:

v

x

Erase the line u

Infer x from

v and $\overline{v} \lor x$

Erase the line $\overline{v} \lor x$

Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	5
max # lines on board	4
max # literals on board	7

Board:

x

Erase the line u
Infer x from
v and $\overline{v} \vee x$
Erase the line $\overline{v} \vee x$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \lor \overline{y} \lor z$

Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$
Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	6
max # lines on board	4
max # literals on board	7

x

$\overline{x} \lor \overline{y} \lor z$

Erase the line $\overline{v} \lor x$

Erase the line v

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from

x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the line $\overline{v} \lor x$
Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x
$\overline{y} \lor z$

Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
  x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

x

$\overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	7
max # lines on board	4
max # literals on board	7

$\overline{y} \lor z$

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	8
max # lines on board	4
max # literals on board	7

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Erase the line x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{v} \vee \overline{w} \vee y$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{y} \lor z$

$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase the line $\overline{v} \lor \overline{w} \lor y$

Erase the line $\overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	9
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	10
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$
Write down axiom 2: v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	11
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	12
max # lines on board	4
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	13
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$
z

v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

w
$\overline{z}$
$\overline{w} \lor z$
z

Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the line w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$
$\overline{w} \lor z$
z

Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the line w

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

$\overline{w} \lor z$

z

Erase the line $\overline{v} \lor \overline{w} \lor z$

Infer z from

w and $\overline{w} \lor z$

Erase the line w

Erase the line $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$
z

Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the line w
Erase the line $\overline{w} \lor z$

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	14
max # lines on board	5
max # literals on board	8

$\overline{z}$

z

w and $\overline{w} \lor z$

Erase the line w

Erase the line $\overline{w} \lor z$

Infer 0 from

$\overline{z}$ and z

Example Resolution Refutation

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

Blackboard bookkeeping	
total # clauses on board	15
max # lines on board	5
max # literals on board	8

$\overline{z}$
z
0

w and $\overline{w} \lor z$
Erase the line w
Erase the line $\overline{w} \lor z$
Infer 0 from
$\overline{z}$ and z

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

x
$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length
clauses written on blackboard counted with repetitions

Space
Somewhat less straightforward — several ways of measuring

x
$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$x$$
$$\overline{y} \lor z$$
$$\overline{v} \lor \overline{w} \lor y$$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length
clauses written on blackboard counted with repetitions

Space
Somewhat less straightforward — several ways of measuring

1. x
2. $\overline{y} \lor z$
3. $\overline{v} \lor \overline{w} \lor y$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$x^{1}$$
$$\overline{y}^{2} \lor z^{3}$$
$$\overline{v}^{4} \lor \overline{w}^{5} \lor y^{6}$$

Clause space: 3

Total space: 6

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm
- Space: Lower bound on memory for proof search algorithm

Length

clauses written on blackboard counted with repetitions
(in our example resolution refutation 15)

Space

Somewhat less straightforward — several ways of measuring

x
$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Clause space:	3
(in our refutation 5)	
Total space:	6
(in our refutation 8)	

Length and Space Bounds

Let $n =$ size of formula (# symbols)

Length: at most 2^n
Lower bound $\exp(\Omega(n))$ [Urquhart '87, Chvátal & Szemerédi '88]

Clause space: at most n
Lower bound $\Omega(n)$ [Torán '99, Alekhnovich et al. '00]

Length-Space Trade-offs

Small space $\Rightarrow$ short length

$\exists$ constant clause space refutation $\Rightarrow$ $\exists$ polynomial length refutation [Atserias & Dalmau '03]

Converse not true

$\exists$ formulas refutable in linear length requiring $n/\log n$ clause space [Ben-Sasson & Nordström '08]

Severe length-space trade-offs in worst case

[Ben-Sasson & Nordström '09] showed $\exists$ formulas that are

- refutable in linear length
- refutable in (very) small space
- but any refutation in in even medium space must be superpolynomial/exponential

What We **Don't** Know About Space

Open Question

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

Open Question

3-CNF formula refutable in clause space $s \Rightarrow$ length $\mathcal{O}(n^s)$. Can you do space $\mathcal{O}(s)$ and length $n^{\mathcal{O}(s)}$ simultaneously? Fix $s = 3$ (minimum): Can a clause space-3 proof have to be superpolynomially long?

Open Question

Suppose a formula is refutable in polynomial length. Can you do polynomial length and linear space simultaneously?

What We **Don't** Know About Space

Open Question

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

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3-CNF formula refutable in clause space $s \Rightarrow$ length $\mathcal{O}(n^s)$. Can you do space $\mathcal{O}(s)$ and length $n^{\mathcal{O}(s)}$ simultaneously? Fix $s = 3$ (minimum): Can a clause space-3 proof have to be superpolynomially long?

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Open Question

Suppose a formula is refutable in polynomial length. Can you do polynomial length and linear space simultaneously?

What We **Don't** Know About Space

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Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

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Open Question

Suppose a formula is refutable in polynomial length. Can you do polynomial length and linear space simultaneously?

We Really Don't Understand Space. . .

All lower bounds on space seem to follow (with hindsight) from

- bounds for other measures that we understand better (e.g. width), or
- connections to pebble games

How to Get a Handle on Time-Space Relations?

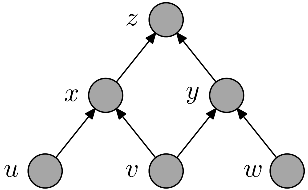
Questions about time-space trade-offs fundamental in TCS

In particular, well-studied (and well-understood) for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

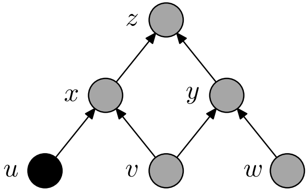


# moves	0
Current # pebbles	0
Max # pebbles so far	0

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

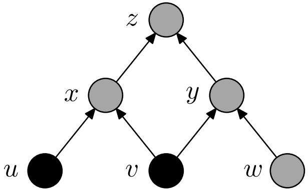


# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

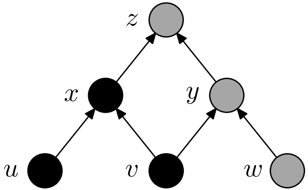


# moves	2
Current # pebbles	2
Max # pebbles so far	2

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

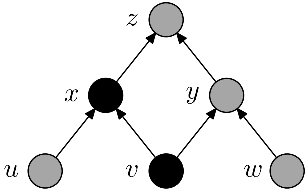


# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

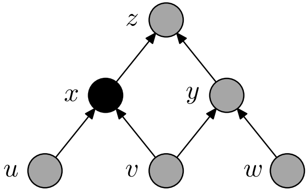


# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

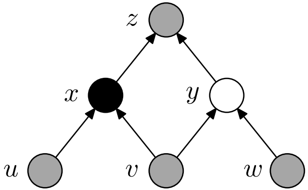


# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

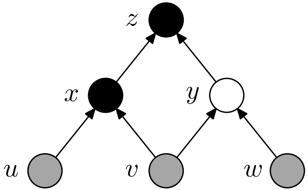


# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

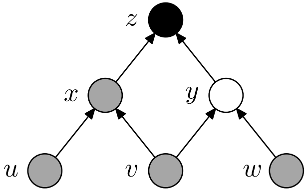


# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

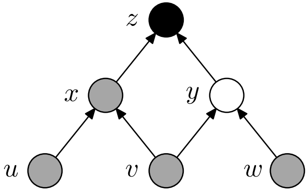


# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

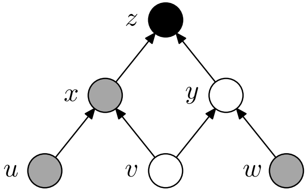


# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

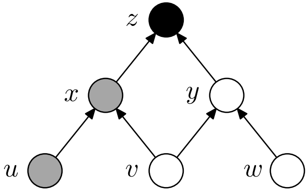


# moves	9
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

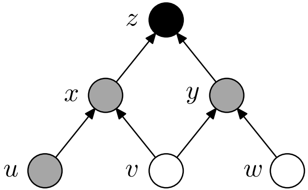


# moves	10
Current # pebbles	4
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



# moves	11
Current # pebbles	3
Max # pebbles so far	4

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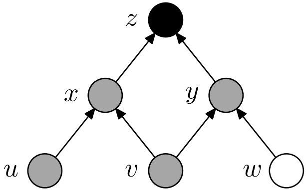

# moves	12
Current # pebbles	2
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
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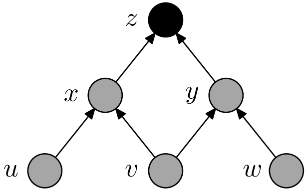
Goal: get single black pebble on sink vertex of G



# moves	13
Current # pebbles	1
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
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4. Can remove white pebble if all immediate predecessors have pebbles

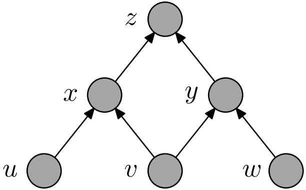
More About Pebbling

- Black pebbling: Same game but black pebbles only
- Rich literature on both black and black-white pebbling
- Black-white pebbling can save square root over black pebbling space [Wilber '85, Kalyanasundaram & Schnitger '88]
- But never more [Meyer auf der Heide '81]

Pebbling Contradictions

CNF formulas encoding pebble game on DAGs

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

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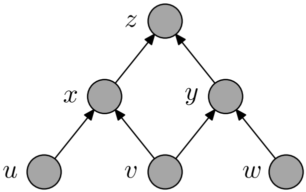

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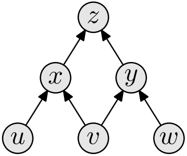
The Actual* Formulas We Need



$$
\begin{aligned}
& (u_1 \lor u_2) \\
\land\ & (v_1 \lor v_2) \\
\land\ & (w_1 \lor w_2) \\
\land\ & (\overline{u}_1 \lor \overline{v}_1 \lor x_1 \lor x_2) \\
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\land\ & \overline{z}_1 \\
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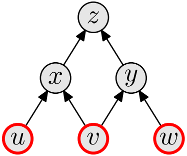

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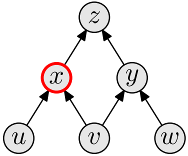

$$(u_1 \lor u_2)$$
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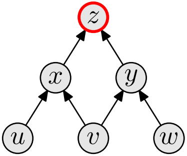

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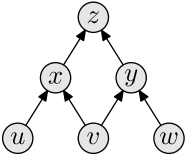

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\end{aligned}
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*) In fact, they are a bit more involved, but let's stick with this for the purposes of this talk

From Resolution to Black-White Pebbling

Black-white pebbling models non-deterministic computation

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified

"Know z assuming v, w"

Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clauses

$$\overline{v}_1 \vee \overline{w}_1 \vee z_1 \vee z_2$$
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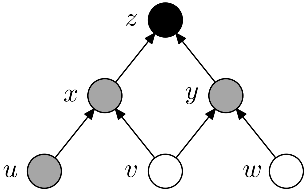

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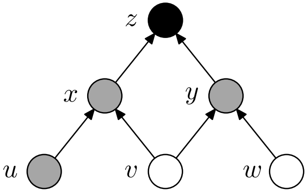

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Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson & Nordström '09)

Any refutation translates into black-white pebbling with

- *# moves $\leq$ refutation length*
- *# pebbles $\leq$ clause space*

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Proof: Just derive $v_1 \lor v_2$ inductively when vertex v is pebbled.

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A Fatal Gap and How to Close It

There is a gap in the reductions!

- From resolution to black-white pebbling
- From pebbling to resolution only for black pebbling
- Why worry — lose only square root? No, everything! (Due to exponential time blow-up)

What to do?

1. Find graphs with (essentially) same trade-off properties for black-white and black-only pebbling
2. Improve reductions between resolution and pebbling

This paper contributes in both directions

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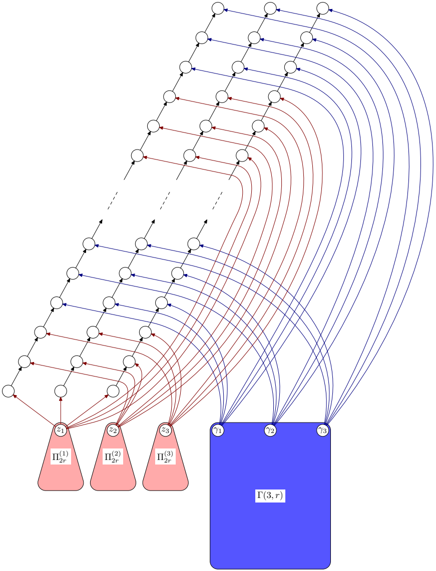
A Picture Says More Than a Thousand Words...

A couple of words about the pebbling result anyway:

- Take parametrized graph family from [Carlson & Savage '80]
- Black pebbling bounds known (upper and lower)
- Tweak graphs slightly...
- And prove matching black-white lower bounds

But remainder of this talk focuses on reductions



A Naive Idea for Simulating Black-White Pebbling

Run the intuition from [Ben-Sasson & Nordström '09] in reverse

$\overline{x}_1 \lor \overline{y}_1 \lor z_1 \lor z_2$
$\overline{x}_1 \lor \overline{y}_2 \lor z_1 \lor z_2$
$\overline{x}_2 \lor \overline{y}_1 \lor z_1 \lor z_2$
$\overline{x}_2 \lor \overline{y}_2 \lor z_1 \lor z_2$

$\overline{v}_1 \lor \overline{w}_1 \lor y_1 \lor y_2$
$\overline{v}_1 \lor \overline{w}_2 \lor y_1 \lor y_2$
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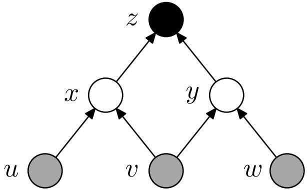

$$\overline{x}_1 \vee \overline{y}_1 \vee z_1 \vee z_2$$
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$$\overline{v}_1 \vee \overline{w}_1 \vee y_1 \vee y_2$$
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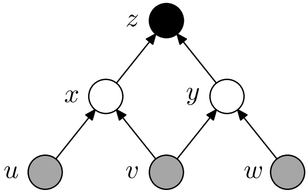

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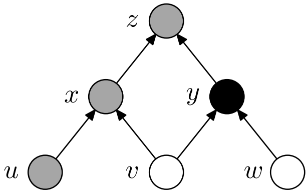

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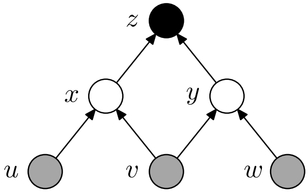
. . . And Why It Doesn't Work

What happens when we try to simulate a pebbling that "combines" these two configurations?



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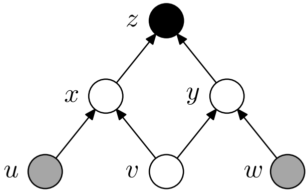

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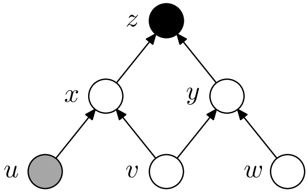

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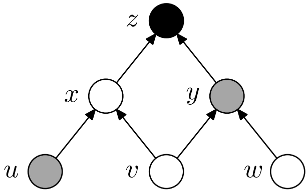

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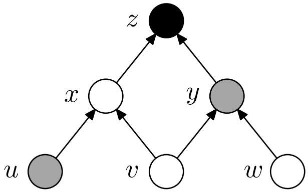

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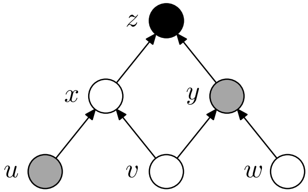

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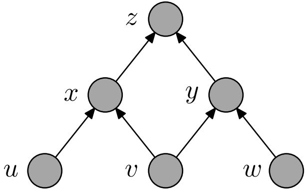
Went only from 2 to 3 white pebbles, but # clauses doubled

Exponential blow-up for naive simulation in worst case

Measure Nondeterminism More Finely

Keep track of for each black pebble which white pebbles it depends on



No black pebbles, so no dependencies
Black on z dependent on whites on $\{x, y\}$
Update dependence for z to $\{x, v, w\}$

Require that each black pebble depend on at most $\mathcal{O}(1)$ white pebbles

Black-white pebbling with "limited nondeterminism"

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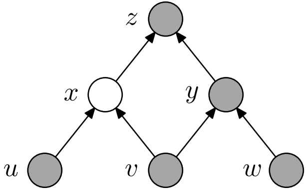

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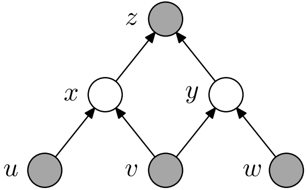

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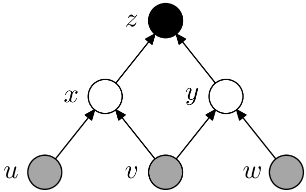

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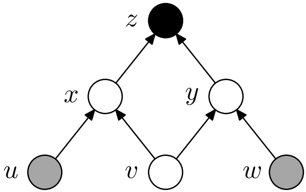

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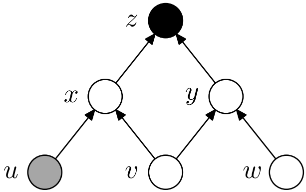

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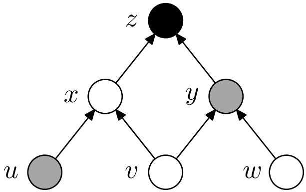

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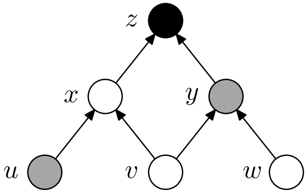

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What Does This Buy Us?

- Pebbling with limited nondeterminism easy to simulate for resolution
- Turns out all known pebbling separation results for black-white vs. black pebbling can be matched by pebblings with limited nondeterminism
- Yields tight space bounds and time-space trade-offs for pebbling formulas over such graphs
- So, in particular, not possible to reduce from resolution to black-only pebbling

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Resolution and Pebbling

Can we reduce from general black-white pebbling to resolution?

Open Question 1

Can resolution on pebbling formulas always simulate black-white pebbling?

Might or might not be true...

Pebbling with Limited Nondeterminism

Open Question 2

Can pebbling with limited nondeterminism always simulate black-white pebbling?

Affirmative answer to Question 2 would immediately answer Question 1 as well

Would be surprising, however

Candidate for refuting Question 2: Graphs in [Wilber '85]

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Space in Resolution

Open Question 3

Total space quadratic in worst case — is this tight? Not even superlinear lower bounds known!

Open Question 4

3-CNF formula refutable in clause space $s \Rightarrow$ length $\mathcal{O}(n^s)$. Can you do space $\mathcal{O}(s)$ and length $n^{\mathcal{O}(s)}$ simultaneously? Extreme case: Can a clause space-3 proof have to be superpolynomially long?

Open Question 5

Suppose a formula is refutable in polynomial length. Can you do polynomial length and linear space simultaneously?

Take-Home Message

- There are strong (and surprising!) connections between resolution and pebble games
- But still not fully clarified — how tight reductions can we get?
- Also proof space not well-understood — many (simple) remaining open questions
- See survey *Pebble Games, Proof Complexity, and Time-Space Trade-offs* at my webpage for details

Thank you for your attention!