Relating Proof Complexity Measures and Practical Hardness of SAT

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Joint work with Matti Järvisalo, Arie Matsliah, and Stanislav Živný

Proof Complexity and SAT Solving

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What (if anything) can proof complexity say about this?

Resolution Proof System

Refute unsatisfiable formulas in conjunctive normal form (CNF):

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
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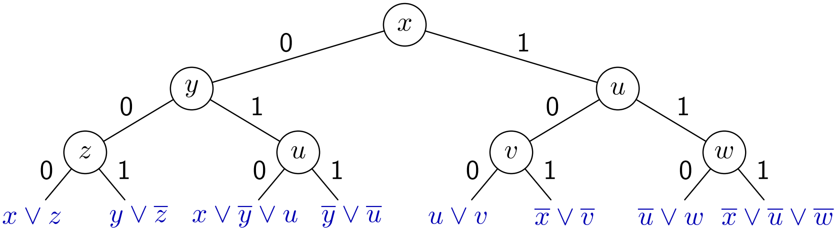
So prove CNF formula unsatisfiable by deriving contradiction by resolution

CDCL Solvers Generate Resolution Proofs

Simple example for DPLL:



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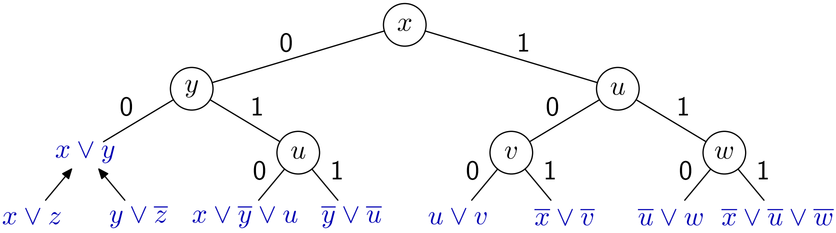

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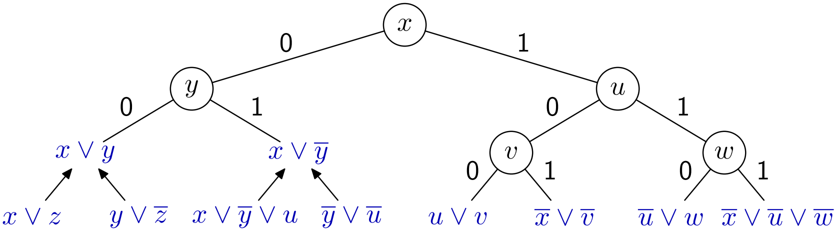

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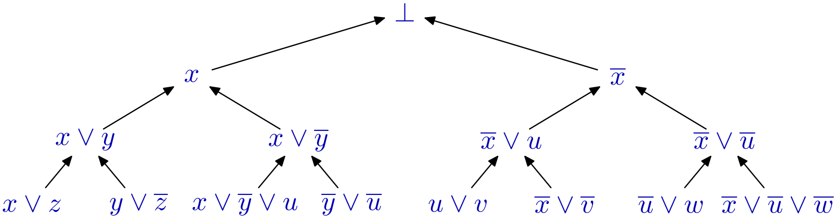

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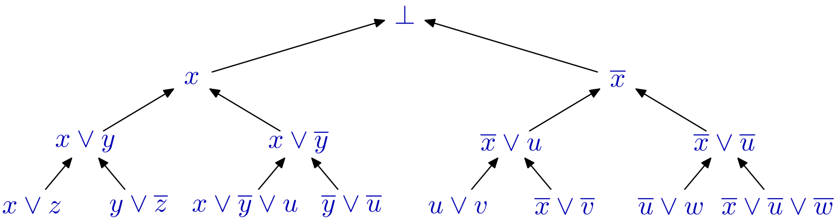

- Conflict-driven clause learning adds “shortcut edges” in tree
- But still yields resolution proof
- True also for (most) preprocessing techniques

Complexity Measures for Resolution

Let n = size of formula

Length

clauses in refutation — at most $\exp(n)$

Width

Size of largest clause in refutation — at most n

Space

Max # clauses one needs to remember when "verifying correctness of refutation on blackboard" — at most n (!)

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- So small length upper bound might be much too optimistic
- Not the right measure of "hardness in practice"

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  (but space model even more idealized)

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This work can be viewed as implementing program outlined in [ABLM08]

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We prove first asymptotic separation of space and tree-like space

Theorem

There are formulas requiring space $\mathcal{O}(1)$ for which tree-like space grows like $\Omega(\log n)$

Only constant-factor separation known before [Esteban & Torán '03]

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We show connections between backdoors and space complexity (elaborating on [ABLM08])

Theorem (Informal)

If a formula has a small backdoor set, then it requires small space

Result 3: Hardness in Practice Correlates with Space

Recall

$$\log \text{length} \leq \text{width} \leq \text{space} \leq \text{tree-like space}$$

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(*) But such formulas are nontrivial to find
(**) With some caveats to be discussed later

Use Pebbling Formulas...

CNF formulas encoding so-called pebble games on DAGs

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2. v
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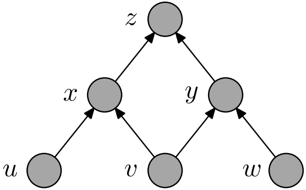

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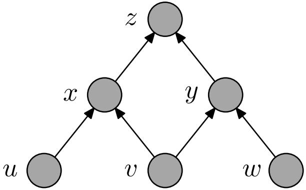

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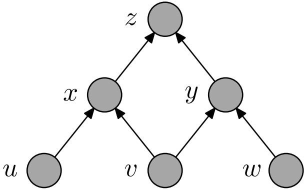

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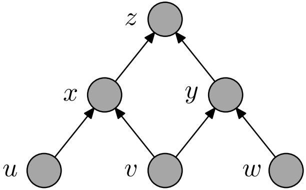

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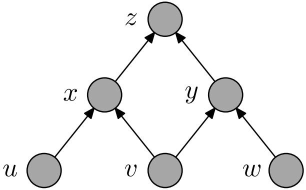

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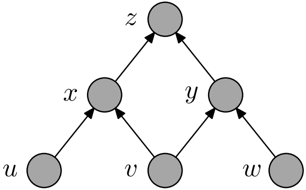

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...with Functions Substituted for Variables

Won't work — pebbling formulas solved by unit propagation, so supereasy

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \lor y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)$$

$$\Downarrow$$

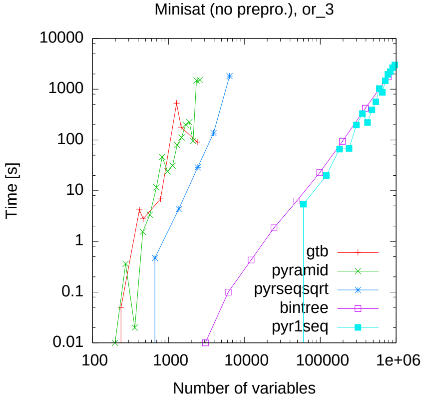
$$\begin{aligned} & (x_1 \lor \overline{x}_2 \lor y_1 \lor y_2) \\ \land\ & (x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor y_1 \lor y_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2) \end{aligned}$$

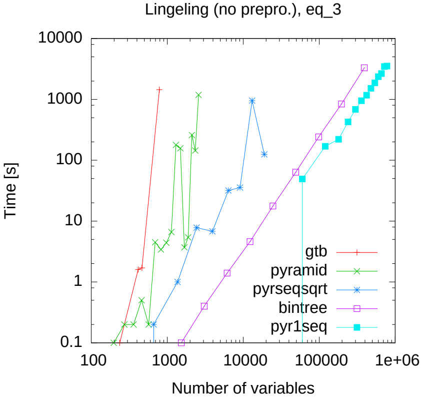
Now CNF formula inherits pebbling graph properties!

About the Experiments

- 12 graph families with varying space complexity
- 8 different substitution functions
- Total of 96 formula families with around 50 instances per family
- CDCL solvers Minisat 2.2.0 and Lingeling version 774
- Experiments
  - with and without preprocessing
  - with and without random shuffling of clauses and variables
- Intel Core i5-2500 3.3-GHz quad-core CPU with 8 GB of memory
- Time-out 1 hour per instance
- Massive amounts of data...

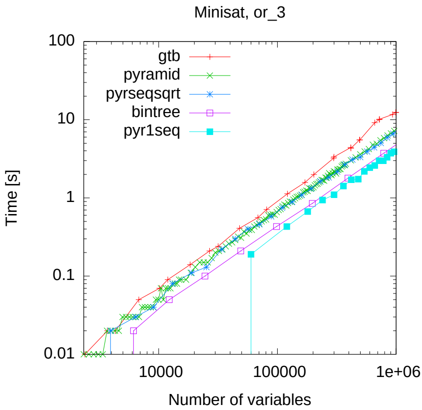
Example Results Without Preprocessing

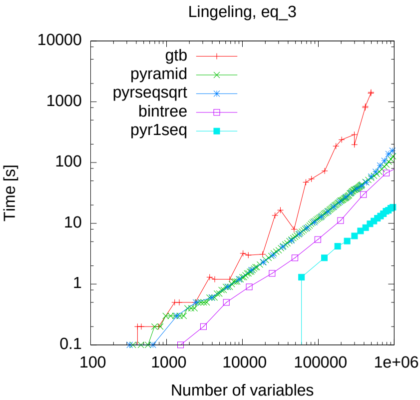




Looks nice. . . Easy formulas solved fast and hard formulas take longer time

Example Results with Preprocessing





Less nice... Which is not surprising

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Varying width and space independently would be more convincing

- Very true, but provably impossible since space $\geq$ width
- Want to see if space is "more fine-grained" hardness indicator

Summing up

- Modern CDCL SAT solvers amazingly successful in practice
- But poorly understood which formulas are easy or hard
- We propose space complexity as a measure of hardness in practice
- Don't claim conclusive evidence, but nontrivial correlations
- Would like to get similar results also with preprocessing
- Would like to study if theoretical time-space trade-offs show up in practice
- Believe there are more connections between proof complexity and SAT solving worth exploring

Thank you for your attention!