On the Semantics of Local Characterizations for Linear-Invariant Properties

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Joint work with Arnab Bhattacharyya, Elena Grigorescu, and Ning Xie

Property Testing

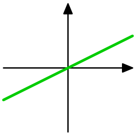
Given (huge) object, want to know if it has certain property or not

No time to read all of input, but can make (constant number of) random access queries

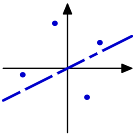
Distinguish:

- object has property (always answer "yes")
- object is far from having property (w.h.p. answer "no")

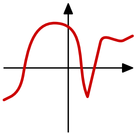
Example: Decide whether given function linear



YES (always)



DON'T CARE



NO (w.h.p.)

Some Property Testing Background

- [Rubinfeld-Sudan ’96] and [Goldreich-Goldwasser-Ron ’98] started field of property testing
- Rich literature on testing of
  - graphs (bipartiteness, k-colourability, . . .),
  - algebraic functions (linearity, low-degree polynomials, . . .),
  - other properties
- Many ingenious result, but somewhat ad hoc — want unifying explanation what makes a property testable
- Graphs well understood [Alon-Fischer-Newman-Shapira ’06]
- Algebraic functions less so [Kaufman-Sudan ’08] — starting point for this work

Invariances and Constraints

Tester (with one-sided error) must see violation of local constraint

- bipartiteness: small non-bipartite subgraph
- linearity: $\mathbf{x}$ and $\mathbf{y}$ s.t. $f(\mathbf{x}) + f(\mathbf{y}) \neq f(\mathbf{x} + \mathbf{y})$

Testable properties have invariances

- graph properties the same under relabelling of vertices
- linear functions remain linear if composed with linear transformation of domain

Many algebraic properties are linear-invariant — interesting class to study

Linear-Invariant Properties

Linear invariance

Property $\mathcal{P}$ is linear-invariant if for all linear maps $L : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ it holds that $f \in \mathcal{P} \Rightarrow f \circ L \in \mathcal{P}$

Two questions:

1. Which linear-invariant properties are testable?
2. What are these properties?

Described syntactically by local constraints, but syntactically distinct properties can collapse into semantically identical property!

Recent testability results essentially ignore this issue

This work: initiate systematic study of the semantics of linear-invariant properties

Our Results in (Very) Brief

- Develop techniques for determining whether two syntactically distinct specifications encode semantically distinct properties
- Show for fairly rich class of properties that techniques provide necessary and sufficient conditions
- Corollary: recent testability results indeed provide infinite number of new, testable properties

Outline

1. Background
   - Linear-Invariant Properties
   - Matroid Freeness
   - Previous Work
2. Our Work
   - Dichotomy Theorems
   - Homomorphisms
   - An Infinite Number of Infinite Strict Property Hierarchies
3. Concluding Remarks
   - Some Technicalities
   - Open Problems

Some Notation

- Study functions $f : \mathsf{D} \to \mathsf{R}$ from domain D to range R
- Domain vector space for linear invariance to make sense
- In this talk usually $\mathsf{D} = \mathbb{F}_2^n$ (but other base fields possible)
- Focus on range $\mathsf{R} = \{0, 1\}$ (but again other choices possible)
- L always linear transformation
- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots$ unit vectors in ambient space
- Property $\mathcal{P}$ is $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ where $\mathcal{P}_n \subseteq \{\mathbb{F}^n \to \mathsf{R}\}$
  (but customary to suppress parametrization)

Testing Linear-Invariant Properties

- Never false negatives $\Rightarrow$ must see local violation to reject
- Same answer for f and $f \circ L$ by linear invariance $\Rightarrow$ only thing that matters is linear dependencies between query points

So intuitively, it seems that what a tester has to do is:

1. Fix linearly dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{F}^r$, $r \leq k$,
2. Apply random $L : \mathbb{F}^r \to \mathbb{F}^n$ to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$
3. Reject f if pattern $\langle f(L(\mathbf{v}_1)), f(L(\mathbf{v}_2)), \ldots, f(L(\mathbf{v}_k)) \rangle$ in set of "forbidden patterns" $S \subseteq \mathsf{R}^k$; accept otherwise

A Syntactic Specification of Linear-Invariant Properties

Hence, natural to describe linear-invariant properties in terms of matroid freeness

(Linear) matroid M: bunch of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{F}^r$ for $r \leq k$

Matroid freeness property

A function $f : \mathbb{F}^n \to \mathrm{R}$ is (M, S)-free if for all $L : \mathbb{F}^r \to \mathbb{F}^n$ pattern $\langle f(L(\mathbf{v}_1)), \ldots, f(L(\mathbf{v}_k))\rangle$ is not in $S \subseteq \mathrm{R}^k$

Any linear-invariant property testable with one-sided error* can be expressed as intersection of matroid freeness properties [Bhattacharyya-Grigorescu-Shapira '10]

(*) Modulo technical assumption that tester doesn't depend in any essential way on dimension n

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Examples of Matroid Freeness Properties

1. Linearity
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
   $S = \{001, 111\}$

2. Subspace
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
   $S = \{110\}$

3. Triangle freeness
   $M = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$
   $S = \{111\}$

4. Degree-d polynomial (with zero constant term)
   $M = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d+1]\}$
   $S = \left\{\sigma \in \{0,1\}^{2^{d+1}-1} \,\middle|\, \text{parity of } \sigma \text{ odd}\right\}$

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Full Linear Matroid

Full linear matroid of dimension d

$F_d = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d]\}$

Any matroid freeness property intersection of F_d-freeness properties (forbid all labels $r \in \mathrm{R}$ for vectors we don't care about)

Also any (F_d, S)-freeness property intersection of properties forbidding each $\sigma \in S$

So understanding (F_d, σ)-freeness properties for a single pattern σ would be great!

Partial Linear Matroid

Seems a bit too hard for the moment...

So consider instead

Partial matroid of weight w

$$F_d^{\leq w} = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d], |I| \leq w\}$$

Understanding $(F_d^{\leq w}, \sigma)$-freeness properties also appears hard, but here we can at least do something

And already $w = 2$ interesting!

A Canonical Matroid Freeness Tester

Tester for (M, σ)-freeness seems obvious:

1. Consider the matroid vectors $M = \{\mathbf{v_1}, \ldots, \mathbf{v}_k\} \subseteq \mathbb{F}^r$
2. Apply random $L : \mathbb{F}^r \to \mathbb{F}^n$ to get $\{L(\mathbf{v_1}), \ldots, L(\mathbf{v}_k)\} \subseteq \mathbb{F}^n$
3. Reject f if $\big\langle f(L(\mathbf{v_1})), \ldots, f(L(\mathbf{v}_k))\big\rangle = \sigma$; accept otherwise

Clearly this test never gives false negatives (by definition)

But will it detect with high probability that f is far from (M, σ)-free?

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Testability Results for Matroid Freeness Properties

Monotone properties:

- [Green '05]:
  $(F_2, 111)$-freeness testable
- [Bhattacharyya-Chen-Sudan-Xie '09]:
  $(F_d^{\leq 2}, 1^*)$-freeness testable
- [Král'-Serra-Vena '09], [Shapira '09]:
  $(M, 1^*)$-freeness testable for any any $M \subseteq F_d$

Non-monotone properties:

- [Bhattacharyya-Chen-Sudan-Xie '09]:
  $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^k \mathbf{e}_i\}, \sigma)$-freeness testable
- [Bhattacharyya-Grigorescu-Shapira '10]
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But what are these properties?

A Property Collapse

Not too hard to show that monotone properties cannot all be the same [Bhattacharyya-Chen-Sudan-Xie '09]

But [Bhattacharyya-Chen-Sudan-Xie '09] also show for $\sigma \notin \{0^*, 1^*\}$ that all $(\{\mathbf{e}_1, \ldots, \mathbf{e}_k, \sum_{i=1}^{k} \mathbf{e}_i\}, \sigma)$-freeness properties collapse into one of 9 properties, all previously known testable!

What about properties in [Bhattacharyya-Grigorescu-Shapira '10]? Unclear. . .

Need to understand what matroid freeness properties mean!

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Some More Notation and Terminology

Matroid $M = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{F}^r$ for $r \leq k$
Forbidden pattern $\sigma = \langle \sigma_1, \ldots, \sigma_k \rangle \in \mathsf{R}^k$

Say $f : \mathbb{F}^n \to \mathsf{R}$ contains (M, σ) at L if
$\big\langle f(L(\mathbf{v}_1)), f(L(\mathbf{v}_2)), \ldots, f(L(\mathbf{v}_k)) \big\rangle = \sigma$

Other matroid $N = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\} \subseteq \mathbb{F}^s$ for $s \leq \ell$
Forbidden pattern $\tau = \langle \tau_1, \ldots, \tau_\ell \rangle \in \mathsf{R}^\ell$

Refer to (M, σ) and (N, τ) as labelled matroids with

- vector $\mathbf{v}_i$ labelled by σ_i
- vector $\mathbf{w}_j$ labelled by τ_j

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How to Relate the Structure of Two Matroids?

Matroid homomorphism $\phi : M \to N$

- linear map from $\mathbb{F}^r$ to $\mathbb{F}^s$
- sends every $\mathbf{v}_i \in M$ to some $\mathbf{w}_j \in N$

Labelled matroid homomorphism from (M, σ) to (N, τ)

- homomorphism
- label-preserving, i.e., if $\mathbf{w}_j = \phi(\mathbf{v}_i)$ then $\tau_j = \sigma_i$

Say (M, σ) embeds into (N, τ); denoted $(M, \sigma) \hookrightarrow (N, \tau)$

An Easy Observation

Homomorphisms imply property containment

Observation

If $(M, \sigma) \hookrightarrow (N, \tau)$*, then* (M, σ)*-freeness* $\subseteq$ (N, τ)*-freeness.*

Proof: If $\phi : M \to N$ is a homomorphism and f contains (N, τ) at a linear transformation L, then f contains (M, σ) at $L \circ \phi$.

What about the other direction?

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What about the other direction?

Dichotomy Theorem for Monotone Properties

Labelled homomorphisms completely determine relations between monotone matroid freeness properties!

Theorem

Let M and N be any matroids. Then one of two cases holds:

1. *If $(M, 1^*) \hookrightarrow (N, 1^*)$, then $(M, 1^*)$-freeness is contained in $(N, 1^*)$-freeness.*
2. *Otherwise, $(M, 1^*)$-freeness is* **far** *from being contained in $(N, 1^*)$-freeness.*

(2nd case means there are $(M, 1^*)$-free functions f for which a constant fraction of values needs changing to get $(N, 1^*)$-freeness)

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Dichotomy Theorem for Non-monotone Properties

For non-monotone properties things get (much) messier, but we have the following result (to be stated in more detail later)

Theorem (Informal)

For a fairly broad class of $F_d^{\leq 2}$-freeness properties we have:

1. *If $(M, \sigma) \hookrightarrow (N, \tau)$, then (M, σ)-freeness $\subseteq (N, \tau)$-freeness.*
2. *Else (M, σ)-freeness* **far** *from contained in (N, τ)-freeness.*

Corollary

The results in [BGS '10] provide an infinite number of infinite strict hierarchies of properties not previously known testable.

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Revisiting Partial Matroids of Weight 2

Recall: Intersections of (F_d, σ)-freeness properties capture all matroid freeness properties

Full linear matroid of dimension d

$F_d = \{\sum_{i \in I} \mathbf{e}_i \mid \emptyset \neq I \subseteq [d]\}$

Analogously, intersections of $(F_d^{\leq 2}, \sigma)$-freeness properties capture (almost) all matroid freeness properties currently known testable

Partial matroid of weight 2

$F_d^{\leq 2} = \{\mathbf{e}_i, \mathbf{e}_i + \mathbf{e}_j \mid 1 \leq i \neq j \leq d\}$

Some Matroid (Non-)Homomorphisms

If (M, σ) submatroid of (N, τ), then clearly $(M, \sigma) \hookrightarrow (N, \tau)$

But homomorphisms can be trickier than that — "larger" matroids can also embed into "smaller" matroids in low dimensions

For dimensions $d \geq 3$ no such homomorphism surprises, however

Lemma

If $d > c \geq 3$, then $(F_d^{\leq 2}, \sigma) \not\to (F_c^{\leq 2}, \tau)$ for any σ, τ.

Lemma

If $d \geq 3$ and σ and τ have distinct number of labels of each type, then $(F_d^{\leq 2}, \sigma) \not\to (F_d^{\leq 2}, \tau)$.

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Partial Matroids and Dichotomy Theorems

To be able to apply dichotomy theorems, focus on partial matroids with

- All non-basis vectors labelled by 1
- Basis vectors labelled 0 or 1
- So w.l.o.g. because of symmetry study labelled matroids $(F_d^{\leq 2}, 0^c1^*)$ for $c \leq d$

Denote $(F_d^{\leq 2}, 0^c1^*)$-freeness by $\mathcal{F}_d^{\leq 2}[\neg 0^c1^*]$ for brevity

(Notation $f \in \mathcal{F}_d^{\leq 2}[\neg 0^c1^*]$ means that evaluating f on any set of vectors $\{\mathbf{x}_i, \mathbf{x}_i + \mathbf{x}_j \mid 1 \leq i \neq j \leq d\} \subseteq \mathbb{F}^n$ we do **not** see pattern $\langle 0^c1^* \rangle$)

$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{e}_3$
$\mathbf{e}_4$
$\mathbf{e}_1 + \mathbf{e}_2$
$\mathbf{e}_1 + \mathbf{e}_3$
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$\mathbf{e}_1$
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Partial Matroids and Dichotomy Theorems

To be able to apply dichotomy theorems, focus on partial matroids with

- All non-basis vectors labelled by 1
- Basis vectors labelled 0 or 1
- So w.l.o.g. because of symmetry study labelled matroids $(F_d^{\leq 2}, 0^c 1^*)$ for $c \leq d$

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Denote $(F_d^{\leq 2}, 0^c 1^*)$-freeness by $\mathcal{F}_d^{\leq 2}[\neg 0^c 1^*]$ for brevity

(Notation $f \in \mathcal{F}_d^{\leq 2}[\neg 0^c 1^*]$ means that evaluating f on any set of vectors $\{\mathbf{x}_i, \mathbf{x}_i + \mathbf{x}_j \mid 1 \leq i \neq j \leq d\} \subseteq \mathbb{F}^n$ we do **not** see pattern $\langle 0^c 1^* \rangle$)

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Two Nested Hierarchies Venn Diagram-Style

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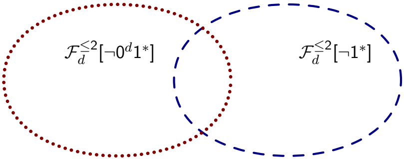
(a) $\mathcal{F}_{\overline{d}}^{\leq 2}[\neg 0^d 1^*]$: basis 0, rest 1

(b) $\mathcal{F}_{\overline{d}}^{\leq 2}[\neg 1^*]$: all labels 1



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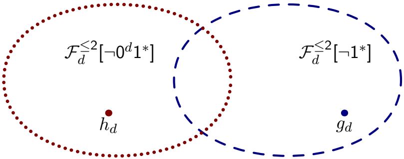

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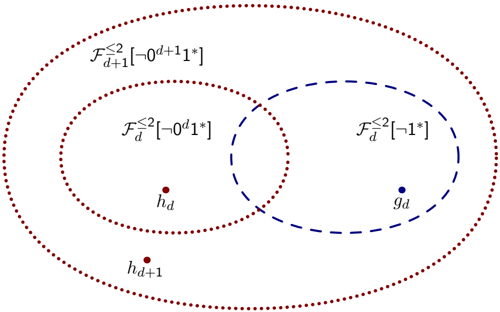

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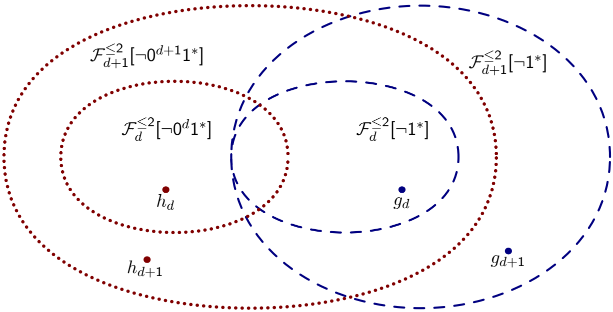

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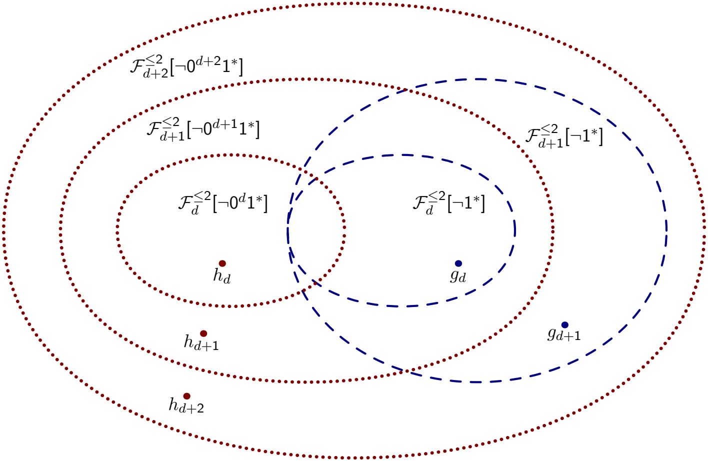

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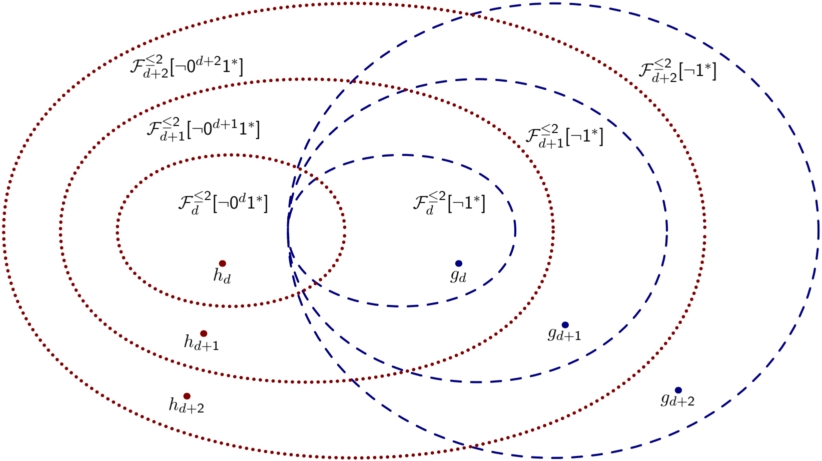

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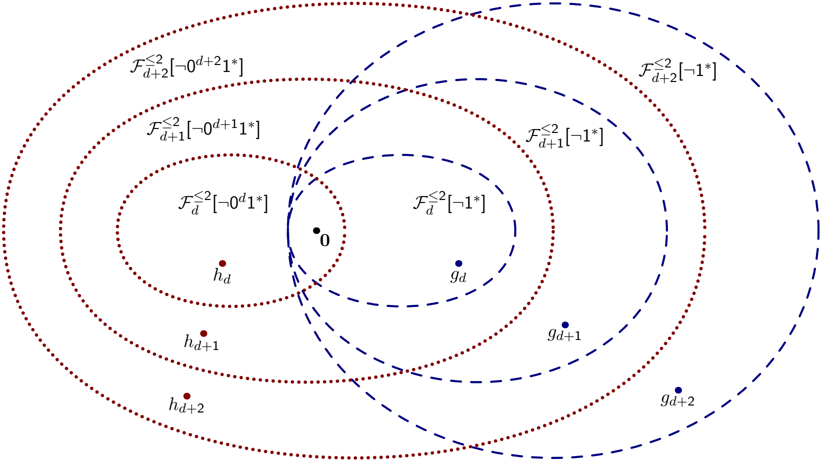

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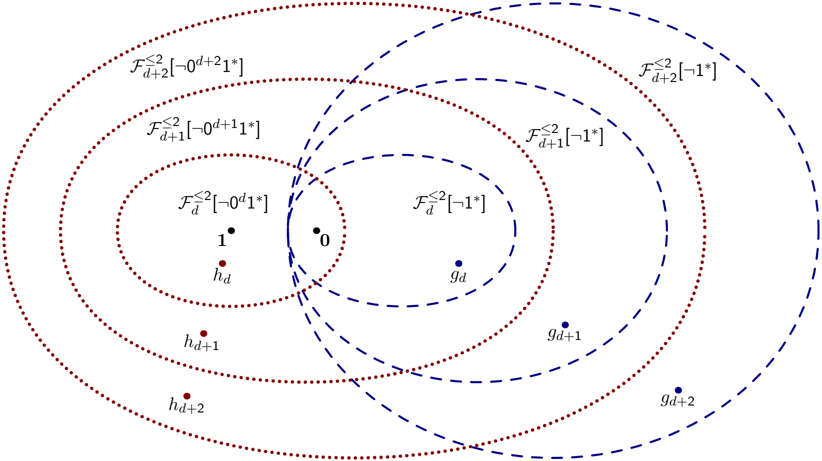

A Family of Hierarchies for Partial Matroid of Weight 2

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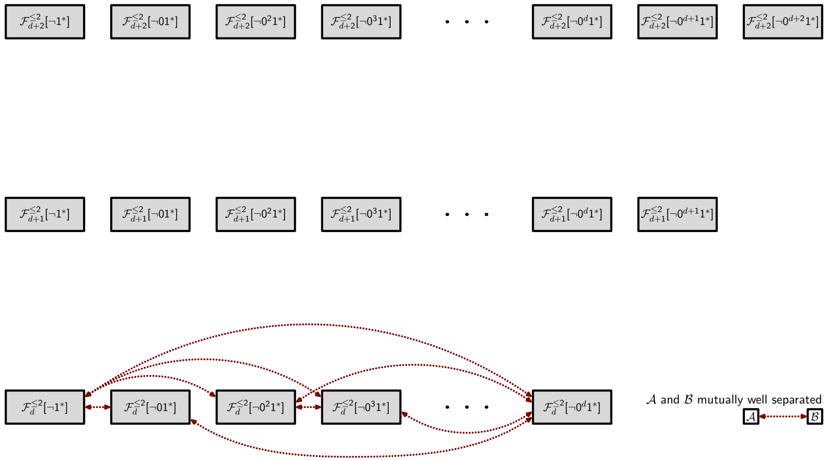

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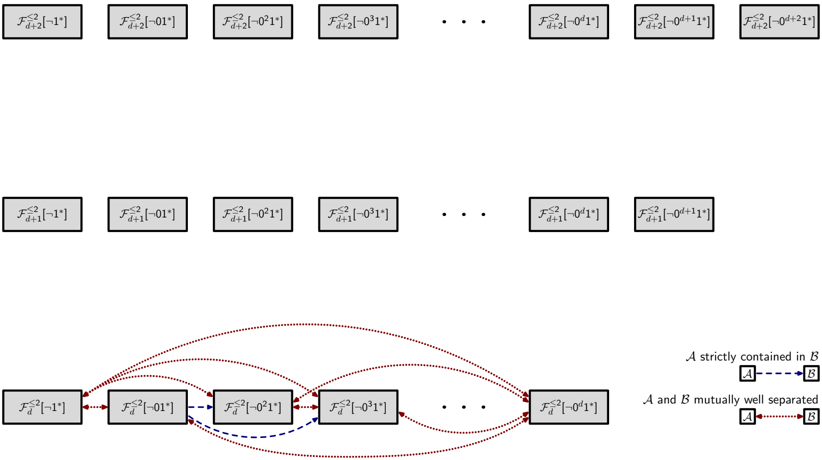

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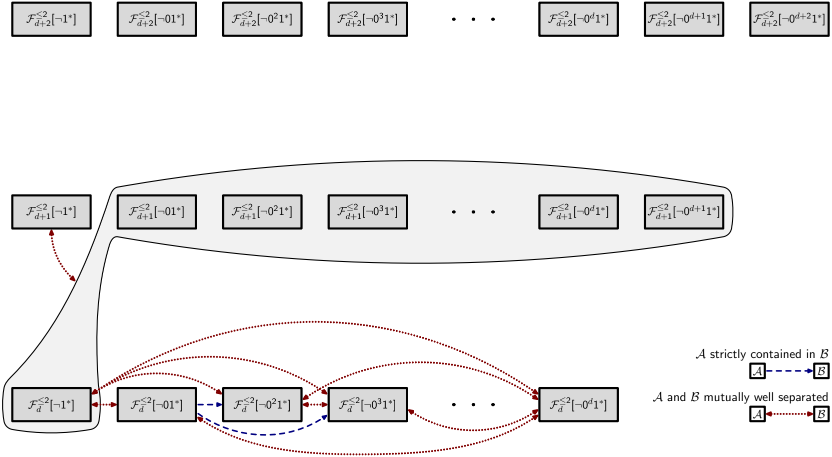

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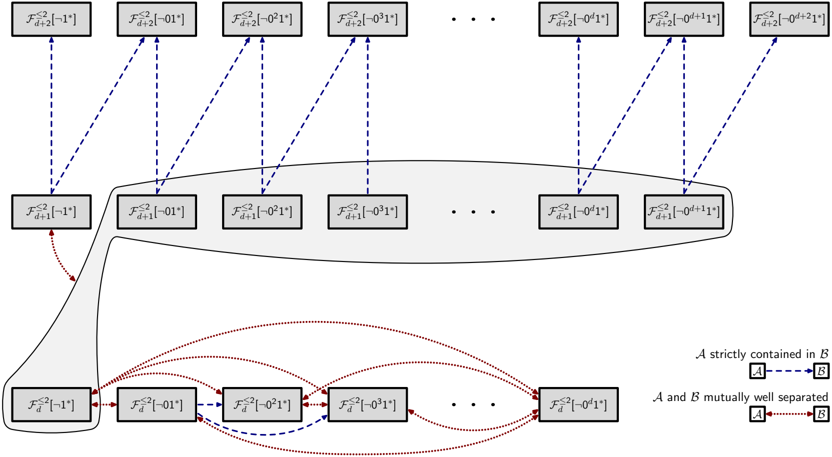

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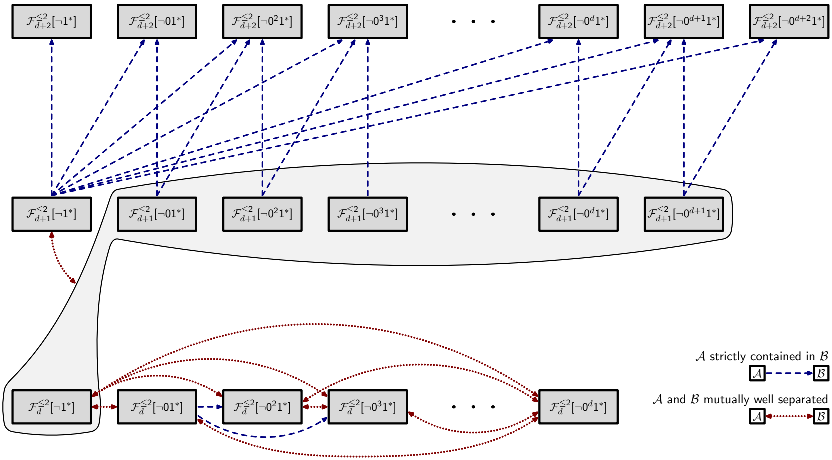

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- Results are slightly more general than stated in this talk
  (E.g. apply also to properties currently not known testable)
- A fair bit of technicalities swept under the rug
- Homomorphisms are great but don't always work
  We saw examples where (M, σ)-freeness $\subseteq$ (N, τ)-freeness although $(M, \sigma) \not\to (N, \tau)$

  (But for our examples (M, σ) "almost" embeds into (N, τ) if we are also allowed to map to $\mathbf{0}$-vector...)

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How Far Does This Approach Extend?

Open Problem 1

Can these techniques be generalized to deal with

1. any $(F_d^{\leq 2}, \sigma)$-freeness property?
2. any $(F_d^{\leq w}, \sigma)$-freeness property for $w > 2$?

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Is it true for any labelled matroids (M, σ) and (N, τ) that (M, σ)-freeness $\subseteq$ (N, τ)-freeness if and only if (M, σ) embeds into $(N, \tau) \cup \{\mathbf{0}\}$?

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When Does the Dichotomy Hold?

Open Problem 3

Does a dichotomy always hold for any two linear-invariant properties $\mathcal{P}$ and $\mathcal{Q}$ in the sense that

- either $\mathcal{P}$ is contained in $\mathcal{Q}$
- or $\mathcal{P}$ is **far** from being contained in $\mathcal{Q}$?

Summing up

- Active line of research in property testing to characterize testable properties in terms of their invariances
- If we want to understand linear-invariant properties, then matroid freeness is a fundamental concept
- However, syntactic specifications of matroid freeness properties don't say much about semantic meaning — on the contrary can be downright misleading
- This work initiates systematic study of the semantics of (local characterizations of) linear-invariant properties
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Thank you for your attention!