A Beautiful General Survey on Hardness Condensation

Christoph Berkholz

Humboldt-Universität zu Berlin

Dagstuhl workshop 18051
Proof Complexity
Friday February 2, 2018
A Special Case of Hardness Condensation

Jakob Nordström
KTH Royal Institute of Technology

Dagstuhl workshop 18051
Proof Complexity
Friday February 2, 2018
Supercritical Space-Width Trade-offs for Resolution

Jakob Nordström

KTH Royal Institute of Technology

Dagstuhl workshop 18051
Proof Complexity
Friday February 2, 2018

Joint work with Christoph Berkholz
Proof of Complexity

\[(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})\]

**Input:** Unsatisfiable formula in conjunctive normal form (CNF)

**Output:** Polynomial-time verifiable certificate of unsatisfiability

Proof of unsatisfiability = **refutation** of formula

Want to measure efficiency of proof system in terms of different complexity measures (size, space, et cetera)

Can be viewed as proving upper and lower bounds for weak nondeterministic models of computation
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{D \lor \bar{x}} \quad \frac{C \lor x}{D \lor \bar{x}} \quad \frac{C \lor x}{D \lor \bar{x}}
\]

- Done when empty clause \( \bot \) derived
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

1. \( x \lor y \)
2. \( x \lor \overline{y} \lor z \)
3. \( \overline{x} \lor z \)
4. \( \overline{y} \lor \overline{z} \)
5. \( \overline{x} \lor \overline{z} \)
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\frac{C \lor x}{C \lor \overline{D} \lor \overline{x}} \quad \frac{D \lor \overline{x}}{C \lor D}
\]

- Done when empty clause $\perp$ derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. $x \lor y$  Axiom
2. $x \lor \overline{y} \lor z$  Axiom
3. $\overline{x} \lor z$  Axiom
4. $\overline{y} \lor \overline{z}$  Axiom
5. $\overline{x} \lor \overline{z}$  Axiom
6. $x \lor \overline{y}$  Res(2, 4)
7. $x$  Res(1, 6)
8. $\overline{x}$  Res(3, 5)
9. $\perp$  Res(7, 8)
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[ \frac{C \lor x}{D \lor \bar{x}} \quad \frac{C \lor D}{C \lor x} \]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \) Axiom
2. \( x \lor \bar{y} \lor z \) Axiom
3. \( \bar{x} \lor z \) Axiom
4. \( \bar{y} \lor \bar{z} \) Axiom
5. \( \bar{x} \lor \bar{z} \) Axiom
6. \( x \lor \bar{y} \) Res(2, 4)
7. \( x \) Res(1, 6)
8. \( \bar{x} \) Res(3, 5)
9. \( \bot \) Res(7, 8)
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\begin{align*}
C \lor x & \quad D \lor \overline{x} \\
\hline
\quad & \\
C \lor D
\end{align*}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \)  Axiom
2. \( x \lor \overline{y} \lor z \)  Axiom
3. \( \overline{x} \lor z \)  Axiom
4. \( \overline{y} \lor \overline{z} \)  Axiom
5. \( \overline{x} \lor \overline{z} \)  Axiom
6. \( x \lor \overline{y} \)  Res(2, 4)
7. \( x \)  Res(1, 6)
8. \( \overline{x} \)  Res(3, 5)
9. \( \bot \)  Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x \quad D \lor \overline{x}}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- **annotated list** or
- **directed acyclic graph** (DAG)

1. \( x \lor y \quad \text{Axiom} \)
2. \( x \lor \overline{y} \lor z \quad \text{Axiom} \)
3. \( \overline{x} \lor z \quad \text{Axiom} \)
4. \( \overline{y} \lor \overline{z} \quad \text{Axiom} \)
5. \( \overline{x} \lor \overline{z} \quad \text{Axiom} \)
6. \( x \lor \overline{y} \quad \text{Res}(2, 4) \)
7. \( x \quad \text{Res}(1, 6) \)
8. \( \overline{x} \quad \text{Res}(3, 5) \)
9. \( \bot \quad \text{Res}(7, 8) \)
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\frac{C \lor x \quad D \lor \overline{C}}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \) Axiom
2. \( x \lor \overline{y} \lor z \) Axiom
3. \( \overline{x} \lor z \) Axiom
4. \( \overline{y} \lor \overline{z} \) Axiom
5. \( \overline{x} \lor \overline{z} \) Axiom
6. \( x \lor \overline{y} \) Res(2, 4)
7. \( x \) Res(1, 6)
8. \( \overline{x} \) Res(3, 5)
9. \( \bot \) Res(7, 8)
The Resolution Proof System

Goal: refute \textbf{unsatisfiable} CNF

- Start with \textbf{axiom} clauses in formula
- Derive new clauses by \textbf{resolution rule}

\[
\frac{C \lor x \quad D \lor \overline{x}}{C \lor D}
\]

- \text{Done when empty clause } \bot \text{ derived}

Can represent refutation/proof as

- \textbf{annotated list} or
- \textbf{directed acyclic graph (DAG)}

\begin{align*}
1. & \quad x \lor y \quad \text{Axiom} \\
2. & \quad x \lor \overline{y} \lor z \quad \text{Axiom} \\
3. & \quad \overline{x} \lor z \quad \text{Axiom} \\
4. & \quad \overline{y} \lor \overline{z} \quad \text{Axiom} \\
5. & \quad \overline{x} \lor \overline{z} \quad \text{Axiom} \\
6. & \quad x \lor \overline{y} \quad \text{Res}(2, 4) \\
7. & \quad x \quad \text{Res}(1, 6) \\
8. & \quad \overline{x} \quad \text{Res}(3, 5) \\
9. & \quad \bot \quad \text{Res}(7, 8)
\end{align*}
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\frac{C \lor x}{\frac{D \lor \overline{x}}{C \lor D}}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \)  Axiom
2. \( x \lor \overline{y} \lor z \)  Axiom
3. \( \overline{x} \lor z \)  Axiom
4. \( \overline{y} \lor \overline{z} \)  Axiom
5. \( \overline{x} \lor \overline{z} \)  Axiom
6. \( x \lor \overline{y} \)  Res(2, 4)
7. \( x \)  Res(1, 6)
8. \( \overline{x} \)  Res(3, 5)
9. \( \bot \)  Res(7, 8)
The Resolution Proof System

Goal: refute \textbf{unsatisfiable} CNF

- Start with \textbf{axiom} clauses in formula
- Derive new clauses by \textbf{resolution rule}

\[
\begin{align*}
C \lor x & \quad D \lor \bar{x} \\
\hline
C \lor \bar{x} & \quad C \lor D
\end{align*}
\]

- Done when empty clause $\bot$ derived

Can represent refutation/proof as

- \textbf{annotated list} or
- \textbf{directed acyclic graph (DAG)}

\begin{align*}
1. & \quad x \lor y & \text{Axiom} \\
2. & \quad x \lor \bar{y} \lor z & \text{Axiom} \\
3. & \quad \bar{x} \lor z & \text{Axiom} \\
4. & \quad \bar{y} \lor \bar{z} & \text{Axiom} \\
5. & \quad \bar{x} \lor \bar{z} & \text{Axiom} \\
6. & \quad x \lor \bar{y} & \text{Res}(2, 4) \\
7. & \quad x & \text{Res}(1, 6) \\
8. & \quad \bar{x} & \text{Res}(3, 5) \\
9. & \quad \bot & \text{Res}(7, 8)
\end{align*}
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor \overline{x}}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \) \hspace{1cm} \text{Axiom}
2. \( x \lor \overline{y} \lor z \) \hspace{1cm} \text{Axiom}
3. \( \overline{x} \lor z \) \hspace{1cm} \text{Axiom}
4. \( \overline{y} \lor \overline{z} \) \hspace{1cm} \text{Axiom}
5. \( \overline{x} \lor \overline{z} \) \hspace{1cm} \text{Axiom}
6. \( x \lor \overline{y} \) \hspace{1cm} \text{Res}(2, 4)
7. \( x \) \hspace{1cm} \text{Res}(1, 6)
8. \( \overline{x} \) \hspace{1cm} \text{Res}(3, 5)
9. \( \bot \) \hspace{1cm} \text{Res}(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{\therefore C \lor D} \quad \frac{D \lor \bar{x}}{\therefore C \lor D}
\]

- Done when empty clause $\bot$ derived

Can represent refutation proof as

- annotated list or
- directed acyclic graph (DAG)

1. $x \lor y$ Axiom
2. $x \lor \bar{y} \lor z$ Axiom
3. $\bar{x} \lor z$ Axiom
4. $\bar{y} \lor \bar{z}$ Axiom
5. $\bar{x} \lor \bar{z}$ Axiom
6. $x \lor \bar{y}$ Res(2, 4)
7. $x$ Res(1, 6)
8. $\bar{x}$ Res(3, 5)
9. $\bot$ Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{\overline{x} \lor z} \quad \frac{C \lor x \quad D \lor \overline{x}}{C \lor \overline{D}}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \) Axiom
2. \( x \lor \overline{y} \lor z \) Axiom
3. \( \overline{x} \lor z \) Axiom
4. \( \overline{y} \lor \overline{z} \) Axiom
5. \( \overline{x} \lor \overline{z} \) Axiom
6. \( x \lor \overline{y} \) Res(2, 4)
7. \( x \) Res(1, 6)
8. \( \overline{x} \) Res(3, 5)
9. \( \bot \) Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\begin{align*}
C \lor x & \quad D \lor \overline{x} \\
\hline

\end{align*}
\]

\[
\text{Res}(2, 4)
\]

- Done when empty clause \( \bot \) derived

Can represent refutation-proof as

- annotated list or
- directed acyclic graph (DAG)

1. \( x \lor y \) \hspace{1cm} Axiom
2. \( x \lor \overline{y} \lor z \) \hspace{1cm} Axiom
3. \( \overline{x} \lor z \) \hspace{1cm} Axiom
4. \( \overline{y} \lor \overline{z} \) \hspace{1cm} Axiom
5. \( \overline{x} \lor \overline{z} \) \hspace{1cm} Axiom
6. \( x \lor \overline{y} \) \hspace{1cm} Res(2, 4)
7. \( x \) \hspace{1cm} Res(1, 6)
8. \( \overline{x} \) \hspace{1cm} Res(3, 5)
9. \( \bot \) \hspace{1cm} Res(7, 8)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D} \quad \frac{D \lor \overline{x}}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- **annotated list** or
- **directed acyclic graph (DAG)**

1. \( x \lor y \)  **Axiom**
2. \( x \lor \overline{y} \lor z \)  **Axiom**
3. \( \overline{x} \lor z \)  **Axiom**
4. \( \overline{y} \lor \overline{z} \)  **Axiom**
5. \( \overline{x} \lor \overline{z} \)  **Axiom**
6. \( x \lor \overline{y} \)  \(\text{Res}(2, 4)\)
7. \( x \)  \(\text{Res}(1, 6)\)
8. \( \overline{x} \)  \(\text{Res}(3, 5)\)
9. \( \bot \)  \(\text{Res}(7, 8)\)
The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with **axiom** clauses in formula
- Derive new clauses by **resolution rule**

\[
\frac{C \lor x}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation/proof as

- annotated list or
- **directed acyclic graph (DAG)**
The Resolution Proof System

Goal: refute unsatisfiable CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

\[
\frac{C \lor x \quad D \lor \overline{x}}{C \lor D}
\]

- Done when empty clause \( \bot \) derived

Can represent refutation-proof as

- annotated list or
- directed acyclic graph (DAG)

Tree-like resolution if DAG is tree
Resolution Size/Length and Width

**Length** of proof = \# clauses \quad (9 in our example)

Length of refuting $F = \text{min length over all proofs for } F$
Resolution Size/Length and Width

**Length** of proof \(= \# \) clauses \((9 \text{ in our example})\)

Length of refuting \(F\) \(= \text{min length over all proofs for } F\)

**Size** should strictly speaking measure \# symbols
But for resolution don’t care too much about linear factors here
Set size \(= \text{length}\)
Resolution Size/Length and Width

**Length** of proof = # clauses \((9\) in our example)\\
Length of refuting \(F\) = min length over all proofs for \(F\)

**Size** should strictly speaking measure # symbols\\
But for resolution don’t care too much about linear factors here\\
Set size = length

**Width** of proof = # literals in largest clause \((3\) in our example)\\
Width of refuting \(F\) = min width over all proofs for \(F\)

Width at most linear, so here obviously care about linear factors
Resolution Space

\textbf{Space} = \text{amount of memory needed} when performing refutation

1. \( x \lor y \)  
   \text{Axiom}

2. \( x \lor \overline{y} \lor z \)  
   \text{Axiom}

3. \( \overline{x} \lor z \)  
   \text{Axiom}

4. \( \overline{y} \lor \overline{z} \)  
   \text{Axiom}

5. \( \overline{x} \lor \overline{z} \)  
   \text{Axiom}

6. \( x \lor \overline{y} \)  
   \text{Res}(2, 4)

7. \( x \)  
   \text{Res}(1, 6)

8. \( \overline{x} \)  
   \text{Res}(3, 5)

9. \( \bot \)  
   \text{Res}(7, 8)
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:

- clause space (our focus)
- total space

1. \( x \lor y \)  Axiom
2. \( x \lor \overline{y} \lor z \)  Axiom
3. \( \overline{x} \lor z \)  Axiom
4. \( \overline{y} \lor \overline{z} \)  Axiom
5. \( \overline{x} \lor \overline{z} \)  Axiom
6. \( x \lor \overline{y} \)  Res(2, 4)
7. \( x \)  Res(1, 6)
8. \( \overline{x} \)  Res(3, 5)
9. \( \bot \)  Res(7, 8)
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:

- **clause space** (our focus)
- **total space**

Clause space at step $t$: # clauses at steps $\leq t$ used at steps $\geq t$

Total space at step $t$: Count also literals

1. $x \lor y$ Axiom
2. $x \lor \overline{y} \lor z$ Axiom
3. $\overline{x} \lor z$ Axiom
4. $\overline{y} \lor z$ Axiom
5. $\overline{x} \lor z$ Axiom
6. $x \lor \overline{y}$ Res$(2, 4)$
7. $x$ Res$(1, 6)$
8. $\overline{x}$ Res$(3, 5)$
9. $\bot$ Res$(7, 8)$
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:

- **clause space** (our focus)
- **total space**

Clause space at step \( t \): \# clauses at steps \( \leq t \) used at steps \( \geq t \)

Total space at step \( t \): Count also literals

Example: Clause space at step 7
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:

- clause space (our focus)
- total space

Clause space at step $t$: \# clauses at steps $\leq t$ used at steps $\geq t$

Total space at step $t$: Count also literals

Example: Clause space at step 7
Resolution Space

Space = amount of memory needed when performing refutation

Can be measured in different ways:

- clause space (our focus)
- total space

Clause space at step $t$: \# clauses at steps $\leq t$ used at steps $\geq t$
Total space at step $t$: Count also literals

Example: Clause space at step 7 is 5
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:
- clause space (our focus)
- total space

Clause space at step $t$: \# clauses at steps $\leq t$ used at steps $\geq t$

Total space at step $t$: Count also literals

**Example:** Clause space at step 7 is 5
Total space at step 7 is 9
Resolution Space

**Space** = amount of memory needed when performing refutation

Can be measured in different ways:
- clause space (our focus)
- total space

Clause space at step $t$: \# clauses at steps $\leq t$ used at steps $\geq t$
Total space at step $t$: Count also literals

**Example:** Clause space at step 7 is 5
Total space at step 7 is 9

Space of proof = max over all steps
Space of refuting $F$ = min over all proofs
Upper Bounds on Resolution Complexity Measures

Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

- **Size/length**: $O(2^n)$
- **Width**: $O(n)$
- **Clause space**: $O(n)$
- **Total space**: $O(n^2)$

This talk: focus on width and clause space. But results translate to total space by:

$\text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width}$
Upper Bounds on Resolution Complexity Measures

Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

- Size / length
- # derivation steps
- $O(2^n)$

This talk: focus on width and clause space

But results translate to total space by:

- clause space $\leq$ total space $\leq$ clause space $\cdot$ width
Upper Bounds on Resolution Complexity Measures

Worst-case upper bounds for resolution refutations of formula (from now on assume \( n = \#\text{variables} \)):

- **Size / length**: \# derivation steps \( \mathcal{O}(2^n) \)
- **Width**: max \# literals in a clause \( \mathcal{O}(n) \)

But results translate to total space by:

\[ \text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width} \]
Upper Bounds on Resolution Complexity Measures

Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

- **Size / length**: # derivation steps $\mathcal{O}(2^n)$
- **Width**: max # literals in a clause $\mathcal{O}(n)$
- **Clause space**: max # clauses in memory $\mathcal{O}(n)$

This talk: focus on width and clause space. But results translate to total space by:

\[ \text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width} \]
Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

<table>
<thead>
<tr>
<th>Measure</th>
<th>Description</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size / length</td>
<td># derivation steps</td>
<td>$O(2^n)$</td>
</tr>
<tr>
<td>Width</td>
<td>max # literals in a clause</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Clause space</td>
<td>max # clauses in memory</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Total space</td>
<td>total size of memory</td>
<td>$O(n^2)$</td>
</tr>
</tbody>
</table>

This talk: focus on width and clause space. But results translate to total space by:

\[
\text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width}
\]
Upper Bounds on Resolution Complexity Measures

Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

- **Size / length**: $\#$ derivation steps $O(2^n)$
- **Width**: max $\#$ literals in a clause $O(n)$
- **Clause space**: max $\#$ clauses in memory $O(n)$
- **Total space**: total size of memory $O(n^2)$

This talk: focus on width and clause space
Worst-case upper bounds for resolution refutations of formula (from now on assume $n = \#\text{variables}$):

- **Size / length**: # derivation steps $\leq O(2^n)$
- **Width**: max # literals in a clause $\leq O(n)$
- **Clause space**: max # clauses in memory $\leq O(n)$
- **Total space**: total size of memory $\leq O(n^2)$

This talk: focus on width and clause space
But results translate to total space by:

clause space $\leq$ total space $\leq$ clause space $\cdot$ width
Lower Bounds via Resolution Width

For $n$-variable $k$-CNFs ($k$ constant) it holds that:

$$\text{width} \leq \Omega(\text{clause space}) \quad [\text{Atserias & Dalmau '03}]$$
Lower Bounds via Resolution Width

For $n$-variable $k$-CNFs ($k$ constant) it holds that:

\[
\text{width} \leq \Omega(\text{clause space}) \quad \text{[Atserias & Dalmau '03]}
\]
\[
\text{width}^2 \leq \Omega(\text{total space}) \quad \text{[Bonacina '16]}
\]
Lower Bounds via Resolution Width

For $n$-variable $k$-CNFs ($k$ constant) it holds that:

\[
\begin{align*}
\text{width} & \leq \Omega(\text{clause space}) \quad \text{[Atserias & Dalmau '03]} \\
\text{width}^2 & \leq \Omega(\text{total space}) \quad \text{[Bonacina '16]} \\
\text{width}^2 & \leq \Omega(n \log(\text{size})) \quad \text{[Ben-Sasson & Widgerson '99]}
\end{align*}
\]
Lower Bounds via Resolution Width

For \( n \)-variable \( k \)-CNFs (\( k \) constant) it holds that:

\[
\begin{align*}
\text{width} & \leq \Omega(\text{clause space}) \quad \text{[Atserias & Dalmau '03]} \\
\text{width}^2 & \leq \Omega(\text{total space}) \quad \text{[Bonacina '16]} \\
\text{width}^2 & \leq \Omega(n \log(\text{size})) \quad \text{[Ben-Sasson & Widgerson '99]}
\end{align*}
\]

In particular, \( \text{width} = \Omega(n) \implies \text{size} = 2^{\Omega(n)} \)
Lower Bounds via Resolution Width

For $n$-variable $k$-CNFs ($k$ constant) it holds that:

$$\text{width} \leq \Omega(\text{clause space}) \quad [\text{Atserias & Dalmau '03}]$$

$$\text{width}^2 \leq \Omega(\text{total space}) \quad [\text{Bonacina '16}]$$

$$\text{width}^2 \leq \Omega(n \log(\text{size})) \quad [\text{Ben-Sasson & Widgerson '99}]$$

In particular, $\text{width} = \Omega(n) \implies \text{size} = 2^{\Omega(n)}$

So clearly width key measure—but not the answer to every question
Lower Bounds via Resolution Width

For $n$-variable $k$-CNFs ($k$ constant) it holds that:

\[
\text{width} \leq \Omega(\text{clause space}) \quad [\text{Atserias & Dalmau '03}]
\]
\[
\text{width}^2 \leq \Omega(\text{total space}) \quad [\text{Bonacina '16}]
\]
\[
\text{width}^2 \leq \Omega(n \log(\text{size})) \quad [\text{Ben-Sasson & Widgerson '99}]
\]

In particular, $\text{width} = \Omega(n) \implies \text{size} = 2^{\Omega(n)}$

So clearly \textbf{width key measure}—but not the answer to every question

- Can have width $\Theta(\sqrt{n})$ and still size $\text{poly}(n)$
  [Bonet & Galesi '99]
Lower Bounds via Resolution Width

For \( n \)-variable \( k \)-CNFs \((k \text{ constant})\) it holds that:

\[
\begin{align*}
\text{width} & \leq \Omega(\text{clause space}) & \text{[Atserias & Dalmau ’03]} \\
\text{width}^2 & \leq \Omega(\text{total space}) & \text{[Bonacina ’16]} \\
\text{width}^2 & \leq \Omega(n \log(\text{size})) & \text{[Ben-Sasson & Widgerson ’99]}
\end{align*}
\]

In particular, \( \text{width} = \Omega(n) \implies \text{size} = 2^{\Omega(n)} \)

So clearly \textbf{width key measure}—but not the answer to every question

- Can have width \( \Theta(\sqrt{n}) \) and still size \( \text{poly}(n) \)
  \[\text{[Bonet & Galesi ’99]}\]
- Can have width \( O(1) \) and still clause space \( \Omega(n/\log n) \)
  \[\text{[Ben-Sasson & Nordström ’08]}\]
Upper Bounds via Resolution Width

\[ \text{size} \leq n^{O(\text{width})} \]
Upper Bounds via Resolution Width

\[
\begin{align*}
\text{size} & \leq n^{O(\text{width})} \\
\text{time to find refutation} & \leq n^{O(\text{width})}
\end{align*}
\]

for \( w \leftarrow 3 \ldots n \) do

Resolve all clauses & keep resolvents with at most \( w \) literals

If \( \bot \) has been derived, then output UNSAT

end for

Output SAT
Upper Bounds via Resolution Width

\[
\text{size} \leq n^{O(\text{width})}
\]
\[
\text{time to find refutation} \leq n^{O(\text{width})}
\]

\textbf{for } w \leftarrow 3 \ldots n \textbf{ do}

\hspace{1em} \text{Resolve all clauses \& keep resolvents with at most } w \text{ literals}

\hspace{1em} \text{If } \bot \text{ has been derived, then output } \text{UNSAT}

\textbf{end for}

\text{Output SAT}

\text{Algorithm (and resolution proof) requires time/size } n^{O(\text{width})}

\text{Cannot do better in general [Atserias, Lauria, \& Nordström '14]}

\text{What is the space of a small-width proof? Trivially at most } n^{O(\text{width})}
Upper Bounds via Resolution Width

\[ \text{size} \leq n^{O(\text{width})} \]
\[ \text{time to find refutation} \leq n^{O(\text{width})} \]

\textbf{for } w \leftarrow 3 \ldots n \textbf{ do}

Resolve all clauses & keep resolvents with at most \( w \) literals
If \( \bot \) has been derived, then output \text{UNSAT}

\textbf{end for}

Output \text{SAT}

Algorithm (and resolution proof) requires time/size \( n^{O(\text{width})} \)
Cannot do better in general [Atserias, Lauria, & Nordström '14]

What is the space of a small-width proof? Trivially at most \( n^{O(\text{width})} \)

[Ben-Sasson '02] exhibited formulas

- refutable in width \( O(1) \) and clause space \( O(1) \)
- width \( O(1) \) \( \implies \) clause space \( \Omega(n/\log n) \)
Upper Bounds via Resolution Width

\begin{align*}
\text{size} & \leq n^{O(\text{width})} \\
\text{time to find refutation} & \leq n^{O(\text{width})}
\end{align*}

\textbf{for } w \leftarrow 3 \ldots n \textbf{ do}

Resolve all clauses & keep resolvents with at most \( w \) literals

\text{If } \bot \text{ has been derived, then output UNSAT}

\textbf{end for}

Output SAT

Algorithm (and resolution proof) requires time/size \( n^{O(\text{width})} \)

Cannot do better in general [Atserias, Lauria, & Nordström ’14]

What is the space of a small-width proof? Trivially at most \( n^{O(\text{width})} \)

[Ben-Sasson ’02] exhibited formulas

\begin{itemize}
\item refutable in width \( O(1) \) and clause space \( O(1) \)
\item width \( O(1) \) \(\implies\) clause space \( \Omega(n/\log n) \)
\end{itemize}

Which bound is closer to the truth?
Upper Bounds via Resolution Width

\[ \text{size} \leq n^{O(\text{width})} \]
\[ \text{time to find refutation} \leq n^{O(\text{width})} \]

\begin{verbatim}
for \( w \leftarrow 3 \ldots n \) do
    Resolve all clauses & keep resolvents with at most \( w \) literals
    If \( \bot \) has been derived, then output \text{UNSAT}
end for
Output \text{SAT}
\end{verbatim}

Algorithm (and resolution proof) requires time/size \( n^{O(\text{width})} \)
Cannot do better in general [Atserias, Lauria, & Nordström '14]

What is the space of a small-width proof? Trivially at most \( n^{O(\text{width})} \)

[Ben-Sasson '02] exhibited formulas

\begin{itemize}
    \item refutable in width \( O(1) \) and clause space \( O(1) \)
    \item width \( O(1) \) \( \Rightarrow \) clause space \( \Omega(n/\log n) \)
\end{itemize}

Which bound is closer to the truth?
Recall: can always do clause space \( O(n) \)
A Supercritical Space-Width Tradeoff

Theorem

For any \( \varepsilon > 0 \) and \( 6 \leq w \leq n^{\frac{1}{2} - \varepsilon} \) exist \( n \)-variable CNFs \( F_n \) s.t.

1. Resolution can refute \( F_n \) in width \( w \)
2. Any width-\( w \) refutation of \( F_n \) requires clause space \( n^{\Omega(w)} \)
A Supercritical Space-Width Tradeoff

Theorem

For any $\varepsilon > 0$ and $6 \leq w \leq n^{1/2-\varepsilon}$ exist $n$-variable CNFs $F_n$ s.t.

1. Resolution can refute $F_n$ in width $w$
2. Any width-$w$ refutation of $F_n$ requires clause space $n^{\Omega(w)}$

Space lower bound $n^{\Omega(w)}$ holds for all proofs up to width $o(w \log n)$
A Supercritical Space-Width Tradeoff

Theorem
For any $\varepsilon > 0$ and $6 \leq w \leq n^{1/2-\varepsilon}$ exist $n$-variable CNFs $F_n$ s.t.

1. Resolution can refute $F_n$ in width $w$
2. Any width-$w$ refutation of $F_n$ requires clause space $n^{\Omega(w)}$

Space lower bound $n^{\Omega(w)}$ holds for all proofs up to width $o(w \log n)$

Proof outline
Use hardness condensation approach in [Razborov '16]:

1. Start with formula that requires nearly linear clause space
2. Reduce the number of variables from $n$ to $n^{1/w}$
3. But maintain space lower bound for small-width proofs
A Supercritical Space-Width Tradeoff

Theorem
For any $\varepsilon > 0$ and $6 \leq w \leq n^{1/2 - \varepsilon}$ exist $n$-variable CNFs $F_n$ s.t.

1. Resolution can refute $F_n$ in width $w$
2. Any width-$w$ refutation of $F_n$ requires clause space $n^{\Omega(w)}$

Space lower bound $n^{\Omega(w)}$ holds for all proofs up to width $o(w \log n)$

Proof outline
Use hardness condensation approach in [Razborov '16]:
1. Start with formula that requires nearly linear clause space
2. Reduce the number of variables from $n$ to $n^{1/w}$
3. But maintain space lower bound for small-width proofs

Key components:
- Expander graphs
- XORification (substitution with exclusive or)
What Do You Mean “Supercritical”?!?

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$
- Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case!
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

▶ Have two complexity measures $\varphi$ and $\psi$
▶ Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
▶ There are instances $I_n$ such that:
  ▶ $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small}'$ and $\psi(S_2) = \text{small}''$

Supercritical setting for trade-offs:

▶ Any $S$ with $\varphi(S)$ medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$
▶ Conversely, $\psi(S)$ medium-small $= \Rightarrow \varphi(S) \approx \varphi_{\text{crit}}$

▶ Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case!
▶ Very strong trade-os Razborov refers to them as ultimate
▶ We feel supercritical is more descriptive
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small'}$ and $\psi(S_2) = \text{small''}$
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$

Conversely, $\psi(S) \approx \text{medium-small} \Rightarrow \varphi(S) \approx \varphi_{\text{crit}}$

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$
- Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case!
- Very strong trade-offs Razborov refers to them as ultimate
- We feel supercritical is more descriptive

Jakob Nordström (KTH) Supercritical Space-Width Trade-offs Dagstuhl Feb '18 10/21
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small'}$ and $\psi(S_2) = \text{small''}$
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$
  - Conversely, $\psi(S)$ medium-small $\implies \varphi(S) \approx \varphi_{\text{crit}}$
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small}'$ and $\psi(S_2) = \text{small}''$
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$
  - Conversely, $\psi(S)$ medium-small $\implies \varphi(S) \approx \varphi_{\text{crit}}$

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small}'$ and $\psi(S_2) = \text{small}''$
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$
  - Conversely, $\psi(S)$ medium-small $\implies \varphi(S) \approx \varphi_{\text{crit}}$

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$
- Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case!
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1$, $S_2$ with $\varphi(S_1) = \text{small'}$ and $\psi(S_2) = \text{small''}$
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$
  - Conversely, $\psi(S)$ medium-small $\implies \varphi(S) \approx \varphi_{\text{crit}}$

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$
- Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case!
- Very strong trade-offs—Razborov refers to them as “ultimate”
What Do You Mean “Supercritical”?! 

Typical setting for trade-off results:

- Have two complexity measures $\varphi$ and $\psi$.
- Worst-case (usually trivial) upper bounds $\varphi_{\text{crit}}$ and $\psi_{\text{crit}}$.
- There are instances $I_n$ such that:
  - $\exists$ solutions $S_1, S_2$ with $\varphi(S_1) = \text{small'}$ and $\psi(S_2) = \text{small''}$.
  - Any solution $S$ with $\varphi(S)$ even medium-small must have $\psi(S)$ approach critical value $\psi_{\text{crit}}$.
  - Conversely, $\psi(S)$ medium-small $\implies \varphi(S) \approx \varphi_{\text{crit}}$.

Supercritical setting for trade-offs:

- Any $S$ with $\varphi(S)$ medium-small must have $\psi(S) \gg \psi_{\text{crit}}$.
- Optimizing $\varphi$ pushes $\psi$ up into supercritical regime above worst case.
- Very strong trade-offs—Razborov refers to them as “ultimate”.
- We feel “supercritical” is more descriptive.
Expander graphs play a leading role in many proof complexity lower bounds.
Expanders

Very well-connected so-called expander graphs play leading role in many proof complexity lower bounds

Clause-variable incidence graph (CVIG)

- Clauses on the left
- Variables on the right
- Edge if variable $\in$ clause (ignore signs)
Expanders

Very well-connected so-called expander graphs play leading role in many proof complexity lower bounds.

Clause-variable incidence graph (CVIG)
- Clauses on the left
- Variables on the right
- Edge if variable ∈ clause (ignore signs)

If CVIG well-connected, then lower bounds for
- width, size, and space in resolution
  [Ben-Sasson & Wigderson ’99, Ben-Sasson & Galesi ’03]
- degree and size in polynomial calculus
  [Impagliazzo et al. ’99, Alekhnovich & Razborov ’01]
Expander graphs play a leading role in many proof complexity lower bounds.

**Clause-variable incidence graph (CVIG)**
- Clauses on the left
- Variables on the right
- Edge if variable \( \in \) clause (ignore signs)

If CVIG well-connected, then lower bounds for
- width, size, and space in resolution
  \([\text{Ben-Sasson & Wigderson ’99, Ben-Sasson & Galesi ’03}]\)
- degree and size in polynomial calculus
  \([\text{Impagliazzo et al. ’99, Alekhnovich & Razborov ’01}]\)

Can also define more general graphs that capture “underlying combinatorial structure” and extend results \([\text{Mikša & Nordström ’15}]\).
Modifying $F$ to $F[⊕_2]$ by substituting $x_1 ⊕ x_2$ for every variable $x$
**XORification**

Modify $F$ to $F[\oplus_2]$ by substituting $x_1 \oplus x_2$ for every variable $x$

\[ \overline{x} \lor y \]
\[ \downarrow \]
\[ \neg (x_1 \oplus x_2) \lor (y_1 \oplus y_2) \]
\[ \downarrow \]
\[ (x_1 \lor \overline{x}_2 \lor y_1 \lor y_2) \]
\[ \land (x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2) \]
\[ \land (\overline{x}_1 \lor x_2 \lor y_1 \lor y_2) \]
\[ \land (\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2) \]
XORification

Modify $F$ to $F[⊕_2]$ by substituting $x_1 ⊕ x_2$ for every variable $x$

$$\overline{x} \lor y$$

$$\Downarrow$$

$$\neg (x_1 \oplus x_2) \lor (y_1 \oplus y_2)$$

$$\Downarrow$$

$$(x_1 \lor \overline{x}_2 \lor y_1 \lor y_2)$$

$$\land (x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2)$$

$$\land (\overline{x}_1 \lor x_2 \lor y_1 \lor y_2)$$

$$\land (\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2)$$

Used to prove, e.g.:

- $\text{width} \geq w$ for $F \implies \text{size} \geq \exp(\Omega(w))$ for $F[⊕_2]$
  
  [Ben-Sasson '02] (credited to [Alekhnovich & Razborov])
XORification

Modify $F$ to $F[⊕_2]$ by substituting $x_1 ⊕ x_2$ for every variable $x$

$$\overline{x} ∨ y$$
$$\Downarrow$$
$$\neg (x_1 ⊕ x_2) ∨ (y_1 ⊕ y_2)$$
$$\Downarrow$$
$$(x_1 ∨ \overline{x}_2 ∨ y_1 ∨ y_2)$$
$$∧ (x_1 ∨ \overline{x}_2 ∨ \overline{y}_1 ∨ \overline{y}_2)$$
$$∧ (\overline{x}_1 ∨ x_2 ∨ y_1 ∨ y_2)$$
$$∧ (\overline{x}_1 ∨ x_2 ∨ \overline{y}_1 ∨ \overline{y}_2)$$

Used to prove, e.g.:

- width $\geq w$ for $F$ $\implies$ size $\geq \exp(Ω(w))$ for $F[⊕_2]$ [Ben-Sasson '02] (credited to [Alekhnovich & Razborov])
- # vars in memory $\geq s$ for $F$ $\implies$ clause space $\geq Ω(s)$ for $F[⊕_2]$ [Ben-Sasson & Nordström '08]
Intuition for XORification Lower Bounds

How to construct resolution refutation $\pi$ of $F[\oplus_2]$?

Naive idea: Simulate resolution refutation $\pi'$ of $F$ (using substitution on previous slide)

Seems like a bad idea

- Linear in # variables in memory
- Exponential in width

Nevertheless, can prove (sort of) this is the best resolution can do

Intuition behind proof

- Given resolution refutation $\pi$ of $F[\oplus_2]$
- Extract the refutation $\pi'$ of $F$ that $\pi$ is simulating
- Prove that extraction preserves complexity measures of interest
Intuition for XORification Lower Bounds

How to construct resolution refutation $\pi$ of $F[\oplus_2]$?

Naive idea: *Simulate resolution refutation $\pi'$ of $F$* (using substitution on previous slide)
Intuition for XORification Lower Bounds

How to construct resolution refutation $\pi$ of $F[\oplus_2]$?

Naive idea: Simulate resolution refutation $\pi'$ of $F$
(using substitution on previous slide)

 Seems like a bad idea—XORification causes bad blow-up
  - linear in $\#$ variables in memory
  - exponential in width
Intuition for XORification Lower Bounds

How to construct resolution refutation $\pi$ of $F[\oplus_2]$?

Naive idea: Simulate resolution refutation $\pi'$ of $F$
(using substitution on previous slide)

Seems like a bad idea—XORification causes bad blow-up
  ▶ linear in $\#$ variables in memory
  ▶ exponential in width

Nevertheless, can prove (sort of) this is the best resolution can do
Intuition for XORification Lower Bounds

How to construct resolution refutation $\pi$ of $F[\oplus_2]$?

Naive idea: Simulate resolution refutation $\pi'$ of $F$
(using substitution on previous slide)

Seems like a bad idea—XORification causes bad blow-up
  - linear in $\#$ variables in memory
  - exponential in width

Nevertheless, can prove (sort of) this is the best resolution can do

Intuition behind proof
  - Given resolution refutation $\pi$ of $F[\oplus_2]$
  - Extract the refutation $\pi'$ of $F$ that $\pi$ is simulating
  - Prove that extraction preserves complexity measures of interest
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson ’99]

1. \( u_1 \oplus u_2 \)
2. \( v_1 \oplus v_2 \)
3. \( w_1 \oplus w_2 \)
4. \( (u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2) \)
5. \( (v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2) \)
6. \( (x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2) \)
7. \( \neg (z_1 \oplus z_2) \)

▶ sources are true
▶ truth propagates upwards
▶ but sink is false

\( z \)
\( x \)
\( y \)

\( u \)
\( v \)
\( w \)
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

1. \( u_1 \oplus u_2 \)
2. \( v_1 \oplus v_2 \)
3. \( w_1 \oplus w_2 \)
4. \((u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)\)
5. \((v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)\)
6. \((x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)\)
7. \(\neg(z_1 \oplus z_2)\)

- sources are true
- truth propagates upwards
- but sink is false
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg(z_1 \oplus z_2)$

- sources are true
- truth propagates upwards
- but sink is false
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg(z_1 \oplus z_2)$

- sources are true
- truth propagates upwards
- but sink is false
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson ’99]

1. \( u_1 \oplus u_2 \)
2. \( v_1 \oplus v_2 \)
3. \( w_1 \oplus w_2 \)
4. \((u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)\)
5. \((v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)\)
6. \((x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)\)
7. \(\neg(z_1 \oplus z_2)\)

- sources are true
- truth propagates upwards
- but sink is false
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson ’99]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg (z_1 \oplus z_2)$

- sources are true
- truth propagates upwards
- but sink is false

Written in CNF as explained before, e.g.

$u_1 \oplus u_2 = (u_1 \lor u_2) \land (u_1 \lor u_2)$

$\neg (z_1 \oplus z_2) = (z_1 \lor z_2) \land (z_1 \lor z_2)$

Easy to refute pebbling formulas in size $O(n)$ and width $O(1)$

Pebbling space lower bounds $\Rightarrow$ clause space lower bounds

[Ben-Sasson & Nordström ‘08, ‘11]
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson ’99]

1. \( u_1 \oplus u_2 \)
2. \( v_1 \oplus v_2 \)
3. \( w_1 \oplus w_2 \)
4. \((u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2) \)
5. \((v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2) \)
6. \((x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2) \)
7. \(\neg(z_1 \oplus z_2) \)

Written in CNF as explained before, e.g.

\[
\begin{align*}
    u_1 \oplus u_2 &= (u_1 \lor u_2) \land (\overline{u}_1 \lor \overline{u}_2) \\
    \neg(z_1 \oplus z_2) &= (z_1 \lor \overline{z}_2) \land (\overline{z}_1 \lor z_2)
\end{align*}
\]

\[\text{sources are true} \]
\[\text{truth propagates upwards} \]
\[\text{but sink is false} \]
Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

1. \( u_1 \oplus u_2 \)
2. \( v_1 \oplus v_2 \)
3. \( w_1 \oplus w_2 \)
4. \( (u_1 \oplus u_2) \land (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2) \)
5. \( (v_1 \oplus v_2) \land (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2) \)
6. \( (x_1 \oplus x_2) \land (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2) \)
7. \( \neg(z_1 \oplus z_2) \)

Written in CNF as explained before, e.g.

\[
\begin{align*}
 u_1 \oplus u_2 &= (u_1 \lor u_2) \land (\overline{u_1} \lor \overline{u_2}) \\
 \neg(z_1 \oplus z_2) &= (z_1 \lor \overline{z_2}) \land (\overline{z_1} \lor z_2)
\end{align*}
\]

Easy to refute pebbling formulas in size \( O(n) \) and width \( O(1) \)

Pebbling space lower bounds \( \Rightarrow \) clause space lower bounds
[Ben-Sasson & Nordström '08, '11]
XOR Substitution with Recycling (1/2)

Suppose

- $F$ CNF formula over variables $U$
- $G = (U \cup V, E)$ bipartite graph

**Substituted formula $F[G]$ over variables $V$:**

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$
XOR Substitution with Recycling (1/2)

Suppose
- $F$ CNF formula over variables $U$
- $G = (U \cup V, E)$ bipartite graph

**Substituted formula $F[G]$ over variables $V$:**
- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$
Suppose

- $F$ CNF formula over variables $U$
- $G = (U \cup V, E)$ bipartite graph

**Substituted formula** $F[G]$ over variables $V$:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

\[ \overline{u_1} \lor u_3 \quad \rightarrow \quad \neg (v_1 \oplus v_2) \lor (v_5 \oplus v_6) \]
XOR Substitution with Recycling (1/2)

Suppose

- $F$ CNF formula over variables $U$
- $G = (U \cup V, E)$ bipartite graph

Substituted formula $F[G]$ over variables $V$:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

![Diagram of XOR substitution with recycling]

$F \quad F[G]$
XOR Substitution with Recycling (1/2)

Suppose

- $F$ CNF formula over variables $U$
- $G = (U \cup V, E)$ bipartite graph

Substituted formula $F[G]$ over variables $V$:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

\[
\begin{align*}
\overline{u_2} \lor u_5 & \quad \rightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5)
\end{align*}
\]
XOR Substitution with Recycling (2/2)

\[ \overline{u_2} \lor u_5 \quad \longrightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5) \]
XOR Substitution with Recycling (2/2)

\[ u_2 \vee u_5 \quad \rightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \vee (v_3 \oplus v_5) \]

- Apply to pebbling formulas \( F \) in [Ben-Sasson & Nordström '08]
  - refutable in width 6
  - require space \( \Omega(n/\log n) \)
XOR Substitution with Recycling (2/2)

\[
\overline{u}_2 \lor u_5 \quad \rightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5)
\]

- Apply to pebbling formulas \( F \) in [Ben-Sasson & Nordström '08]
  - refutable in width 6
  - require space \( \Omega(n/\log n) \)
- \( G \) with left-degree \( \leq w/6 \), \(|U| = n\), and \(|V| = n^{O(1/w)}\)
XOR Substitution with Recycling (2/2)

\[
\overline{u}_2 \lor u_5 \implies \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5)
\]

- Apply to pebbling formulas $F$ in [Ben-Sasson & Nordström ’08]
  - refutable in width 6
  - require space $\Omega(n / \log n)$
- $G$ with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$
  - $F[G]$ refutable in width $\leq w$
XOR Substitution with Recycling (2/2)

\[ \overline{u_2} \lor u_5 \quad \rightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5) \]

- **Apply to pebbling formulas** $F$ in [Ben-Sasson & Nordström ’08]
  - refutable in width 6
  - require space $\Omega(n/\log n)$
- **$G$** with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$
  - $F[G]$ refutable in width $\leq w$ ✅
XOR Substitution with Recycling (2/2)

Apply to pebbling formulas $F$ in [Ben-Sasson & Nordström ’08]

- refutable in width 6
- require space $\Omega(n/\log n)$

$G$ with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$

- $F[G]$ refutable in width $\leq w$ ✓
- space of width-$w$ refutation of $F[G] \geq$
  
  space of refutation of $F = \Omega(n/\log n) = |V|^{\Omega(w)}$
XOR Substitution with Recycling (2/2)

- Apply to pebbling formulas $F$ in [Ben-Sasson & Nordström ’08]
  - refutable in width 6
  - require space $\Omega(n/\log n)$
- $G$ with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$
  - $F[G]$ refutable in width $\leq w$ ✓
  - space of width-$w$ refutation of $F[G] \gtrapprox$
    space of refutation of $F = \Omega(n/\log n) = |V|^{\Omega(w)}$ ?
XOR Substitution with Recycling (2/2)

- Apply to pebbling formulas $F$ in [Ben-Sasson & Nordström ’08]
  - refutable in width 6
  - require space $\Omega(n/\log n)$
- $G$ with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$
  - $F[G]$ refutable in width $\leq w \checkmark$
  - space of width-$w$ refutation of $F[G] \gtrapprox$
    - space of refutation of $F = \Omega(n/\log n) = |V|^{\Omega(w)}$ ?
XOR Substitution with Recycling (2/2)

Apply to pebbling formulas $F$ in [Ben-Sasson & Nordström '08]
- refutable in width 6
- require space $\Omega(n / \log n)$
- $G$ with left-degree $\leq w/6$, $|U| = n$, and $|V| = n^{O(1/w)}$
  - $F[G]$ refutable in width $\leq w$ ✓
  - space of width-$w$ refutation of $F[G] \gtrapprox$
    space of refutation of $F = \Omega(n / \log n) = |V|^{\Omega(w)}$ ?
XOR Substitution with Recycling (2/2)

\[
\begin{align*}
\bar{u}_2 \lor u_5 &\quad \rightarrow \quad \neg (v_1 \oplus v_2 \oplus v_3) \lor (v_3 \oplus v_5) \\
u_6 &\quad \rightarrow \quad (v_4 \oplus v_5) \\
\bar{u}_7 &\quad \rightarrow \quad \neg (v_4 \oplus v_5)
\end{align*}
\]

**Solution:** Use expander graphs!

- Apply to pebbling formulas \( F \) in [Ben-Sasson & Nordström ’08]
  - refutable in width 6
  - require space \( \Omega(n/\log n) \)
- \( G \) expander with left-degree \( \leq w/6 \), \( |U| = n \), and \( |V| = n^{O(1/w)} \)
  - \( F[G] \) refutable in width \( \leq w \)
  - space of width-\( w \) refutation of \( F[G] \) \( \approx \) space of refutation of \( F = \Omega(n/\log n) = |V|^{\Omega(w)} \)
Bipartite Boundary Expander

\[ G = (U \cup V, E) \] is \((d, r, c)\)-boundary expander if

- left-degree \(\leq d\)
- for every \(U' \subseteq U\), \(|U'| \leq r\) it holds that \(|\partial(U')| \geq c|U'|\)

\[ \partial(U') := \{ v \in N(U') : |N(v) \cap U'| = 1 \} \]
Bipartite Boundary Expander

\[ G = (U \cup V, E) \] is \((d, r, c)\)-boundary expander if

- left-degree \( \leq d \)
- for every \( U' \subseteq U, |U'| \leq r \) it holds that \( |\partial(U')| \geq c|U'| \)

\[ \partial(U') := \{ v \in N(U') : |N(v) \cap U'| = 1 \} \]

Example

- left-degree \( d = 3 \)
- expanding set size \( r = 3 \)
- expansion factor \( c = 1 \)
Bipartite Boundary Expander

\[ G = (U \cup V, E) \] is \((d, r, c)\)-boundary expander if

- left-degree \( \leq d \)
- for every \( U' \subseteq U \), \(|U'| \leq r \) it holds that \(|\partial(U')| \geq c|U'|\)

\( \partial(U') := \{ v \in N(U') : |N(v) \cap U'| = 1 \} \)

Example

- left-degree \( d = 3 \)
- expanding set size \( r = 3 \)
- expansion factor \( c = 1 \)
Bipartite Boundary Expander

\[
\mathcal{G} = (U \cup V, E) \text{ is } (d, r, c)\text{-boundary expander if}
\]

- left-degree \( \leq d \)
- for every \( U' \subseteq U, |U'| \leq r \) it holds that \(|\partial(U')| \geq c|U'|\)

\[
\partial(U') := \{ v \in N(U') : |N(v) \cap U'| = 1 \}
\]

Example

- left-degree \( d = 3 \)
- expanding set size \( r = 3 \)
- expansion factor \( c = 1 \)
Bipartite Boundary Expander

\[ G = (U \cup V, E) \text{ is } (d, r, c)\text{-boundary expander if} \]

- left-degree \( \leq d \)
- for every \( U' \subseteq U \), \( |U'| \leq r \) it holds that \( |\partial(U')| \geq c|U'| \)

\( \partial(U') := \{ v \in N(U') : |N(v) \cap U'| = 1 \} \)

Example
- left-degree \( d = 3 \)
- expanding set size \( r = 3 \)
- expansion factor \( c = 1 \)

Lemma ([Razborov '16])

For \( \varepsilon > 0 \) and \( n, d \) with \( d \leq |V|^{\frac{1}{2} - \varepsilon} \), \( |U| = n \), \( |V| = n^{O(1/d)} \) there are \( (d, r, 2)\)-boundary expanders \( G \) with \( r = d \log n \)
Sketch of Proof Sketch

Look at clauses $C$ in memory in width-$w$ refutation of $F[G]$
Recover clauses $D$ in memory in “simulated refutation” of $F$
Sketch of Proof Sketch

Look at clauses $\mathcal{C}$ in memory in width-$w$ refutation of $F[G]

Recover clauses $\mathcal{D}$ in memory in “simulated refutation” of $F$

Must have $N(\text{Vars}(\mathcal{D})) \subseteq \text{Vars}(\mathcal{C})$

$\text{Ker}(V') := \{u \in U : N(u) \subseteq V'\}$

$|V'| \leq r \implies |\text{Ker}(V')| \leq |V'|$

(since left vertex sets expand a lot)
Sketch of Proof Sketch

Look at clauses $C$ in memory in width-$w$ refutation of $F[G]$
Recover clauses $D$ in memory in “simulated refutation” of $F$

Must have $N(Vars(D)) \subseteq Vars(C)$

$$\text{Ker}(V') := \{ u \in U : N(u) \subseteq V' \}$$

$$|V'| \leq r \implies |\text{Ker}(V')| \leq |V'|$$

(since left vertex sets expand a lot)

Example

$V' = \{ v_3, \ldots, v_8 \}$, $\text{Ker}(V') = \{ u_6, u_7, u_{12} \}$
Sketch of Proof Sketch

Look at clauses $\mathcal{C}$ in memory in width-$w$ refutation of $F[\mathcal{G}]$
Recover clauses $\mathcal{D}$ in memory in “simulated refutation” of $F$

Must have $N(Vars(\mathcal{D})) \subseteq Vars(\mathcal{C})$

$\text{Ker}(V') := \{u \in U : N(u) \subseteq V'\}$

$|V'| \leq r \implies |\text{Ker}(V')| \leq |V'|$

(since left vertex sets expand a lot)

Example

$V' = \{v_3, \ldots, v_8\}$, $\text{Ker}(V') = \{u_6, u_7, u_{12}\}$
Sketch of Proof Sketch

Look at clauses $C$ in memory in width-$w$ refutation of $F[G]$.
Recover clauses $D$ in memory in “simulated refutation” of $F$.

Must have $N(Vars(D)) \subseteq Vars(C)$

$\text{Ker}(V') := \{u \in U : N(u) \subseteq V'\}$

$|V'| \leq r \implies |\text{Ker}(V')| \leq |V'|$

(since left vertex sets expand a lot)

Example

$V' = \{v_3, \ldots, v_8\}, \text{Ker}(V') = \{u_6, u_7, u_{12}\}$

Locally looks almost like XORification without recycling, so previous proof might work...

And give bound in terms of $|U| \gg |V|$
Sketch of Proof Sketch

Look at clauses $C$ in memory in width-$w$ refutation of $F[G]$.
Recover clauses $D$ in memory in "simulated refutation" of $F$.

Must have $N(Vars(D)) \subseteq Vars(C)$.

$Ker(V') := \{ u \in U : N(u) \subseteq V' \}$

$|V'| \leq r \implies |Ker(V')| \leq |V'|$
(since left vertex sets expand a lot)

Example
$V' = \{v_3, \ldots, v_8\}$, $Ker(V') = \{u_6, u_7, u_{12}\}$

Locally looks almost like XORification without recycling, so previous proof might work... And give bound in terms of $|U| \gg |V|$.

Actual details very different.
Some More Details

\( F \) and \( G \) simultaneously falsifiable if \( \exists \alpha \) s.t. \( \alpha(F) = \alpha(G) = 0 \)
Some More Details

$F$ and $G$ simultaneously falsifiable if $\exists \alpha$ s.t. $\alpha(F) = \alpha(G) = 0$

Associate “substituted clause” $C$ over $\text{Vars}(F[G])$ with all consistent “original clauses” $D$ over $\text{Vars}(F)$

$$G^{-1}(C) = \left\{ D \mid \begin{array}{l}
\text{Vars}(D) = \text{Ker}(\text{Vars}(C)) \\
D[G] \text{ and } C \text{ simultaneously falsifiable}
\end{array} \right\}$$
Some More Details

$F$ and $G$ simultaneously falsifiable if $\exists \alpha$ s.t. $\alpha(F) = \alpha(G) = 0$

Associate “substituted clause” $C$ over $\text{Vars}(F[G])$ with all consistent “original clauses” $D$ over $\text{Vars}(F)$

$$G^{-1}(C) = \left\{ D \bigg| \begin{array}{c} \text{Vars}(D) = \text{Ker}(\text{Vars}(C)) \\ D[G] \text{ and } C \text{ simultaneously falsifiable} \end{array} \right\}$$

Let $\pi = (C_1, C_2, \ldots, C_L)$ width-$w$ refutation of $F[G]$ and argue
Some More Details

\( F\) and \( G\) simultaneously falsifiable if \( \exists \alpha \) s.t. \( \alpha(F) = \alpha(G) = 0 \)

Associate “substituted clause” \( C \) over \( \text{Vars}(F[G]) \) with all consistent “original clauses” \( D \) over \( \text{Vars}(F) \)

\[
G^{-1}(C) = \left\{ D \mid \begin{align*}
\text{Vars}(D) &= \text{Ker}(\text{Vars}(C)) \\
D[G] \text{ and } C \text{ simultaneously falsifiable}
\end{align*} \right\}
\]

Let \( \pi = (C_1, C_2, \ldots, C_L) \) width-\( w \) refutation of \( F[G] \) and argue

1. \(|D| \leq |C| \leq w\) because of expansion
Some More Details

$F$ and $G$ simultaneously falsifiable if $\exists \alpha$ s.t. $\alpha(F) = \alpha(G) = 0$

Associate “substituted clause” $C$ over $\text{Vars}(F[G])$ with all consistent “original clauses” $D$ over $\text{Vars}(F)$

$$G^{-1}(C) = \left\{ D \mid \begin{align*} \text{Vars}(D) &= \text{Ker}(\text{Vars}(C)) \\ D[G] \text{ and } C \text{ simultaneously falsifiable} \end{align*} \right\}$$

Let $\pi = (C_1, C_2, \ldots, C_L)$ width-$w$ refutation of $F[G]$ and argue

1. $|D| \leq |C| \leq w$ because of expansion
2. $|G^{-1}(C)| \leq 2^{|C|} \leq 2^w$ because of simultaneous satisfiability
Some More Details

$F$ and $G$ simultaneously falsifiable if $\exists \alpha \text{ s.t. } \alpha(F) = \alpha(G) = 0$

Associate “substituted clause” $C$ over $\text{Vars}(F[G])$ with all consistent “original clauses” $D$ over $\text{Vars}(F)$

$$G^{-1}(C) = \left\{ D \mid \begin{array}{c} \text{Vars}(D) = \text{Ker}(\text{Vars}(C)) \\ D[G] \text{ and } C \text{ simultaneously falsifiable} \end{array} \right\}$$

Let $\pi = (C_1, C_2, \ldots, C_L)$ width-$w$ refutation of $F[G]$ and argue

1. $|D| \leq |C| \leq w$ because of expansion
2. $|G^{-1}(C)| \leq 2^{|C|} \leq 2^w$ because of simultaneous satisfiability
3. $(G^{-1}(C_1), G^{-1}(C_2), \ldots, G^{-1}(C_L))$ “backbone” of refutation of $F$ in clause space roughly $s2^w$
Some More Details

\[ F \text{ and } G \text{ simultaneously falsifiable if } \exists \alpha \text{ s.t. } \alpha(F) = \alpha(G) = 0 \]

Associate “substituted clause” \( C \) over \( Vars(F[G]) \) with all consistent “original clauses” \( D \) over \( Vars(F) \)

\[
G^{-1}(C) = \left\{ D \mid \begin{array}{l}
Vars(D) = \text{Ker}(Vars(C)) \\
D[G] \text{ and } C \text{ simultaneously falsifiable}
\end{array} \right\}
\]

Let \( \pi = (C_1, C_2, \ldots, C_L) \) width-\( w \) refutation of \( F[G] \) and argue

1. \( |D| \leq |C| \leq w \) because of expansion
2. \( |G^{-1}(C)| \leq 2^{\|C\|} \leq 2^w \) because of simultaneous satisfiability
3. \( (G^{-1}(C_1), G^{-1}(C_2), \ldots, G^{-1}(C_L)) \) “backbone” of refutation of \( F \) in clause space roughly \( s2^w \)

Some further technical twists needed, but this is main idea of proof
On the Method of Hardness Condensation

Introduced in [Razborov JACM '16] to show that treelike resolution refutations of width $w$ can require doubly exponential size $2^{n\Omega(w)}$.
On the Method of Hardness Condensation

Introduced in [Razborov JACM '16] to show that treelike resolution refutations of width $w$ can require doubly exponential size $2^{n^{\Omega(w)}}$

Has also been used to establish

- Tradeoffs between width and rank for Lovász-Schrijver linear programming hierarchy [Razborov ECCC TR16-010]
On the Method of Hardness Condensation

Introduced in [Razborov JACM '16] to show that treelike resolution refutations of width $w$ can require doubly exponential size $2^{\Omega(w)}$.

Has also been used to establish

- Tradeoffs between width and rank for Lovász-Schrijver linear programming hierarchy [Razborov ECCC TR16-010]
- Relation between depth and space for general proof systems [Razborov ECCC TR16-184]

Where else can this technique be useful?

Jakob Nordström (KTH)  Supercritical Space-Width Trade-offs  Dagstuhl Feb '18  20/21
On the Method of Hardness Condensation

Introduced in [Razborov JACM ’16] to show that treelike resolution refutations of width $w$ can require doubly exponential size $2^{\Omega(w)}$

Has also been used to establish

- Tradeoffs between width and rank for Lovász-Schrijver linear programming hierarchy [Razborov ECCC TR16-010]
- Relation between depth and space for general proof systems [Razborov ECCC TR16-184]
- Quantifier depth lower bounds for finite variable fragments of first-order logic [Berkholz & Nordström LICS ’16]
On the Method of Hardness Condensation

Introduced in [Razborov JACM '16] to show that treelike resolution refutations of width $w$ can require doubly exponential size $2^{n^{\Omega(w)}}$.

Has also been used to establish

- Tradeoffs between width and rank for Lovász-Schrijver linear programming hierarchy [Razborov ECCC TR16-010]
- Relation between depth and space for general proof systems [Razborov ECCC TR16-184]
- Quantifier depth lower bounds for finite variable fragments of first-order logic [Berkholz & Nordström LICS '16]

Where else can this technique be useful?
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear
  (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and huge size blow-up
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and huge size blow-up
- Inherent for XORification with large arity
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and huge size blow-up
- Inherent for XORification with large arity

Open question 2
Are there supercritical tradeoffs for 3-CNFs?

Thank you for your attention!
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1

Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and huge size blow-up
- Inherent for XORification with large arity

Open question 2

Are there supercritical tradeoffs for 3-CNFs?

- Probably yes, unless PSPACE = EXPTIME
Concluding Remarks

- We exhibit supercritical space-width trade-offs for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and huge size blow-up
- Inherent for XORification with large arity

Open question 2
Are there supercritical tradeoffs for 3-CNFs?

- Probably yes, unless PSPACE = EXPTIME
- Can search for small-space refutations in PSPACE, but finding refutations in given width EXPTIME-complete [Berkholz '12]
Concluding Remarks

- We exhibit **supercritical space-width trade-offs** for resolution
- Minimizing width can make space go way above linear (worst-case “critical” bound)

Open question 1
Similar tradeoffs for degree vs. space in polynomial calculus?

- Weaknesses: non-constant width and **huge** size blow-up
- Inherent for XORification with large arity

Open question 2
Are there supercritical tradeoffs for 3-CNFs?

- Probably yes, unless PSPACE = EXPTIME
- Can search for small-space refutations in PSPACE, but finding refutations in given width EXPTIME-complete [Berkholz ’12]

Thank you for your attention!