A Generalized Method for Resolution and Polynomial Calculus Lower Bounds

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Computational Complexity of Discrete Problems
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Based on joint work with Massimo Lauria and Mladen Mikša

Proof Complexity and Expansion

- **General goal:** Prove that concrete proof systems cannot efficiently certify unsatisfiability of concrete CNF formulas
- **General theme:**

$$\text{CNF formula } \mathcal{F} \text{ “expanding”}$$
$$\Downarrow$$
$$\text{Large proofs needed to refute } \mathcal{F}$$

- Paradigm implemented for
  - resolution: well-developed machinery
  - polynomial calculus: very much less so

  (Will define these proof systems shortly)
- What “expanding” means is usually a formula-specific hack

A General Expansion Criterion for Hardness

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$, build bipartite graph

- Left vertex set partition of clauses into $\mathcal{F} = \bigcup_{i=1}^{m} F_i$
- Right vertex set division of variables $\mathcal{V} = \bigcup_{j=1}^{n} V_j$
- Edge (F_i, V_j) if $Vars(F_i) \cap V_j \neq \emptyset$

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Lower bound on proof size if

1. Bipartite graph is an expander (very well-connected)
2. We can win the edge game on every edge (F_i, V_j)

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Edge game on (F_i, V_j)

- Adversary assigns all variables $\mathcal{V} \setminus V_j$
- We assign V_j
- We win if F_i true

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Consequences

- Extends techniques in [BW01] and [AR03]
- Unifies many previous lower bounds
- And yields some new ones

Outline

1. Proof Complexity Overview
   - Preliminaries
   - Resolution
   - Polynomial Calculus
2. Lower Bounds from Expansion
   - Resolution Width
   - Polynomial Calculus Degree
   - New Polynomial Calculus Lower Bounds
3. Open Problems

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $\mathcal{F} = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  $k = \mathcal{O}(1)$ constant in this talk
- $\mathit{true} = 1$; $\mathit{false} = 0$
- M = size of formula = # literals ($\approx$ # clauses for k-CNF)
- N = # variables $\leq M$

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

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Can represent refutation as

- annotated list or
- directed acyclic graph

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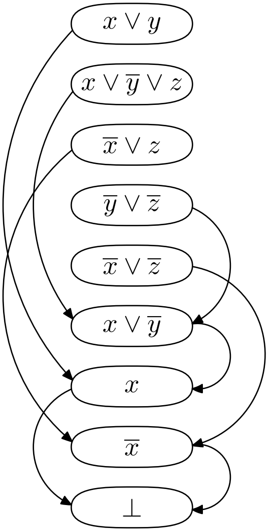

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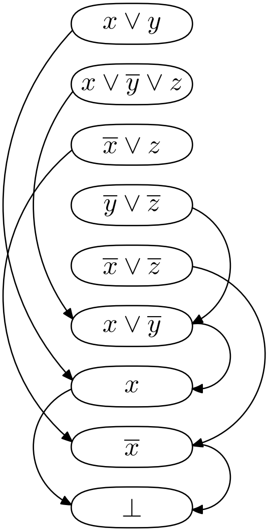
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $= \#$ clauses in refutation [9 in our example]

Most fundamental measure in proof complexity

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(M))$ lower bounds known

(Recall $N = \#$ variables $\leq$ formula size $= M$)

Examples of Hard Formulas w.r.t Resolution Size (1/3)

Pigeonhole principle (PHP) [Hak85]
"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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Even onto functional PHP formulas are hard for resolution
"Resolution cannot count"

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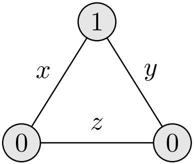
But only lower bound $\exp\left(\Omega\left(\sqrt[3]{M}\right)\right)$ in terms of formula size

Examples of Hard Formulas w.r.t Resolution Size (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



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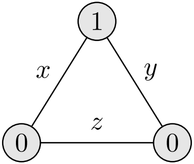

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Requires size $\exp\big(\Omega(M)\big)$ on bounded-degree edge expanders

"Resolution cannot count $\mathrm{mod}\ 2$"

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Random k-CNF formulas [CS88, BKPS02]
Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(M))$

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And more...

- k-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Subset cardinality formulas [Spe10, VS10, MN14]
- Et cetera...

Resolution Width

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Proof: at most $(2N)^{\text{width}}$ distinct clauses
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Width lower bound $\Rightarrow$ size lower bound

Much less obvious. . .

Width Lower Bounds Imply Size Lower Bounds

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$$\textit{proof size} \geq \exp\left(\Omega\left(\frac{(\textit{proof width})^2}{N}\right)\right)$$

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For tree-like resolution have proof size $\geq 2^{\text{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no size lower bounds — possible to tighten analysis? **No!**

Optimality of the Size-Width Lower Bound

Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

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Refutable in resolution in size $\mathcal{O}\big(N^{3/2}\big) = \mathcal{O}(M)$
Requires resolution width $\Omega\big(\sqrt{N}\big)$

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$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

Refutable in resolution in size $\mathcal{O}(N^{3/2}) = \mathcal{O}(M)$
Requires resolution width $\Omega(\sqrt{N})$

But initial clauses have width $\Omega(n) = \Omega(\sqrt{N})$ — a bit more work needed to make the width lower bound meaningful...

Conversion to k-CNF "Graph Versions" of Formulas

- Need bounded-width CNFs to use lower bound in [BW01]
- But PHP and ordering principle formulas have wide clauses
- **Solution:** Restrict formulas to bounded-degree graphs

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For (onto functional) PHP, pigeons can fly only to neighbour holes:

$\bigvee_{j \in \mathcal{N}(i)} p_{i,j}$	pigeon i goes into hole in $\mathcal{N}(i)$
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- Now strong width lower bounds $\Rightarrow$ strong size lower bounds
- And size lower bounds hold for original, unrestricted formulas

Polynomial Calculus (PC)

From [CEI96]; with adjustment in [ABRW02]

Clauses interpreted as polynomial equations over field $\mathbb{F}$

Example: $x \lor y \lor \overline{z}$ gets translated to $\overline{x}\,\overline{y}z = 0$

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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Formalizes Gröbner basis computation

Polynomial Calculus Size and Degree

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Write out all polynomials as sums of monomials
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Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently

- Can mimic resolution refutation step by step
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Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas (over $\mathrm{GF}(2)$ can do Gaussian elimination)
- Onto functional pigeonhole principle (over any field) [Rii93]
- Also other examples

Size vs. Degree

- Degree upper bound ⇒ size upper bound [CEI96]
  Similar to resolution bound; argument a bit more involved
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- **Examples of open problems:**
  - Hardness of functional PHP and onto PHP formulas?
  - Hardness of k-colouring formulas?

Lower Bounds via Graph Expansion

Standard approach:

Lower bounds from expansion

Simplest example is the clause-variable incidence graph (CVIG)

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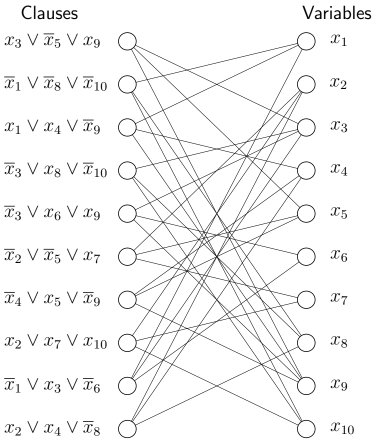

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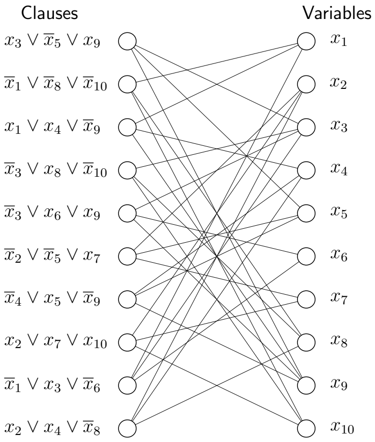
Boundary expansion:
Subsets of left vertices have many unique right neighbours



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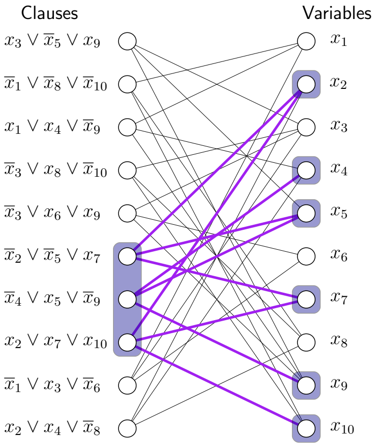

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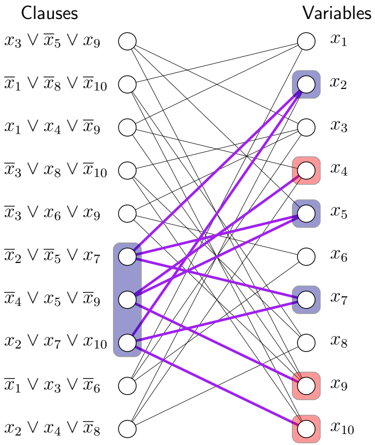

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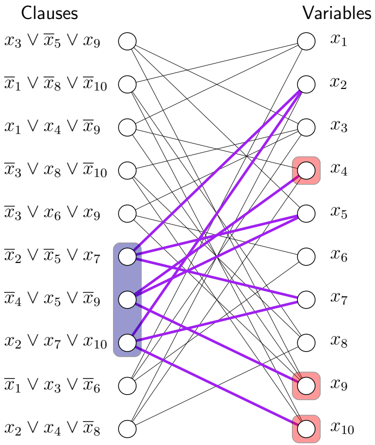

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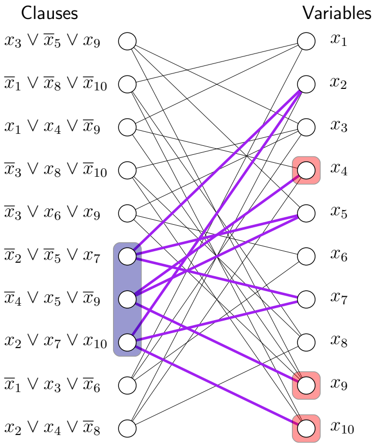

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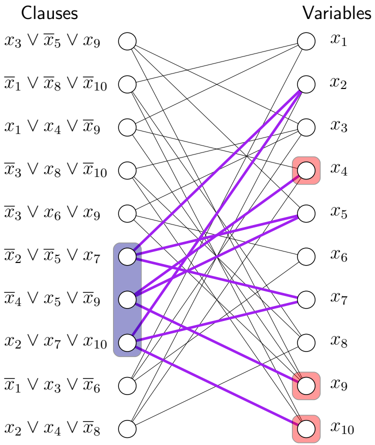
Need graph capturing combinatorial structure!



Generalized Incidence Graphs for CNF Formulas

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$

- Partition clauses into $\mathcal{F} = E \cup \bigcup_{i=1}^{m} F_i$ (for E satisifiable)
- Divide variables into $\mathcal{V} = \bigcup_{j=1}^{n} V_j$ — **not** always partition
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Build bipartite $(\mathcal{U}, \mathcal{V})_E$-graph $\mathcal{G}$

- Left vertices $\mathcal{U} = \{F_1, \ldots, F_m\}$
- Right vertices $\mathcal{V} = \{V_1, \ldots, V_n\}$
- Edge (F_i, V_j) if $Vars(F_i) \cap V_j \neq \emptyset$

The Resolution Edge Game

Resolution edge game on (F_i, V_j) w.r.t. “filtering set” E

- Adversary choses any total assignment α such that $\alpha(E) = 1$
- We can modify α on V_j to get α'
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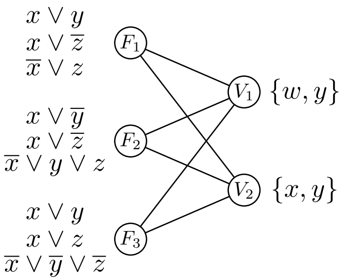

$E = \{\overline{y} \vee z\}$

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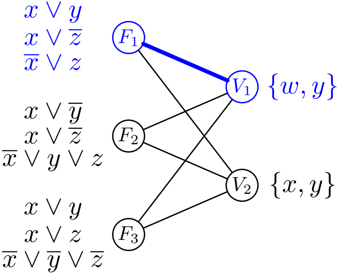

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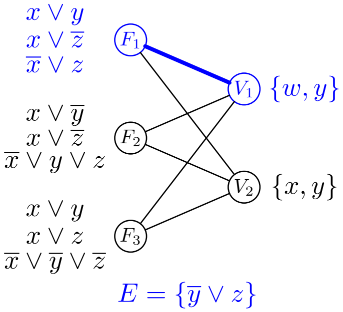

Edge game on (F_1, V_1) w.r.t. E

Take $\alpha_1 = \{x \mapsto 1, y \mapsto 0, z \mapsto 0\}$

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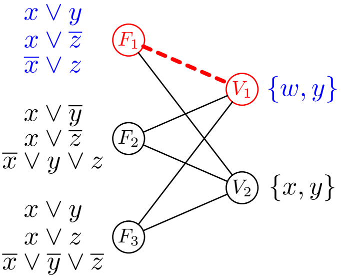

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Take $\alpha_1 = \{x \mapsto 1, y \mapsto 0, z \mapsto 0\}$

Can't win, since

- $\alpha_1(\overline{x} \vee z) = 0$
- can't flip x or z (not in V_1)

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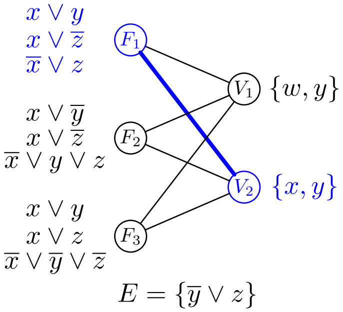

Edge game on (F_1, V_2) w.r.t. E

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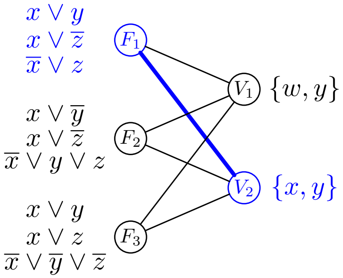

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Edge game on (F_1, V_2) w.r.t. E

Take (partial) $\alpha_2 = \{y \mapsto 0, z \mapsto 0\}$

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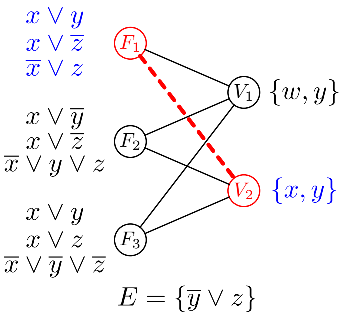

Edge game on (F_1, V_2) w.r.t. E

Take (partial) $\alpha_2 = \{y \mapsto 0, z \mapsto 0\}$

Again can't win, since

- can't flip z (not in V_2)
- flipping $y \in V_2$ falsifies E
- $F_1\restriction_{\alpha_2} = \{x, \overline{x}\}$

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$x \vee y$
$x \vee \overline{z}$
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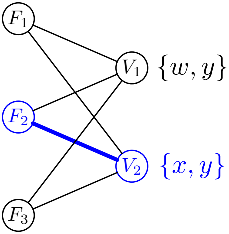

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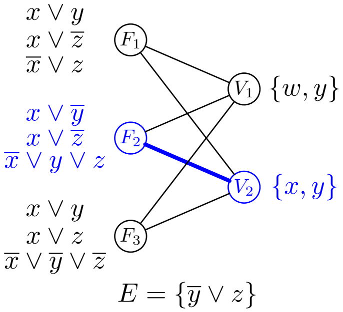

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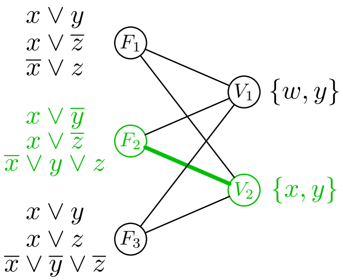

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Edge game on (F_2, V_2) w.r.t. E

Now we can win!

Given any α_3 s.t. $\alpha_3(E) = 1$:

- assign $x \mapsto \alpha_3(y \vee z)$
- E still OK — didn't touch y, z
- F_2 OK — encodes $x \leftrightarrow (y \vee z)$

Edge Game, Expansion, and Width Lower Bounds

Recall boundary $\partial(\mathcal{U}') = \{V \in \mathcal{N}(\mathcal{U}') \mid \mathcal{N}(V) \cap \mathcal{U}' = \{F\} \text{ unique}\}$

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Resolution expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-resolution expander if

- For all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'|$
- For all edges (F_i, V_j) we can win the resolution edge game with respect to E

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Theorem (essentially [BW01])

If the CNF formula $\mathcal{F}$ admits an (s, δ, E)-resolution expander with overlap ℓ, then

$$\textit{resolution proof width} > \frac{\delta s}{2\ell}$$

Progress Measure Approach (1/4)

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Proof overview: Define “progress measure” $\mu : \{\text{clauses}\} \to \mathbb{N}$ such that

1. $\mu(\text{axiom clause}) = \mathcal{O}(1)$
2. $\mu(C \lor D) \leq \mu(C \lor x) + \mu(D \lor \overline{x})$
3. $\mu(\bot) > s$

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$\Rightarrow$ in any resolution proof $\exists\, C$ with $\mu(C) = \sigma$ for $s/2 < \sigma \leq s$

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$\Rightarrow$ such clause C has width $\geq \delta\sigma/\ell$ $\square$

Progress Measure Approach (2/4)

Given (s, δ, E)-resolution expander $(\mathcal{U}, \mathcal{V})_E$ for $\mathcal{F}$, define

$$\mu(C) := \min\{|\mathcal{U}'| \, ; \, \bigwedge_{F \in \mathcal{U}'} F \wedge E \vDash C\}$$

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1. $\mu(A) = \mathcal{O}(1)$ for axioms $A \in \mathcal{F} = \bigcup_{i=1}^{m} F_i \cup E$
   - $A \in E$: $\mu(A) = 0$ since $E \vDash A$
   - $A \in F_i$: $\mu(A) = 1$ since $F_i \wedge E \vDash A$

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2. $\mu(C \vee D) \leq \mu(C \vee x) + \mu(D \vee \overline{x})$
    - Fix minimal $\mathcal{U}_1$ s.t. $\bigwedge_{F \in \mathcal{U}_1} F \wedge E \vDash C \vee x$
    - Fix minimal $\mathcal{U}_2$ s.t. $\bigwedge_{F \in \mathcal{U}_2} F \wedge E \vDash D \vee \overline{x}$
    - Then it holds that

$$\bigwedge_{F \in \mathcal{U}_1 \cup \mathcal{U}_2} F \wedge E \vDash C \vee D \ ,$$

so $\mu(C \vee D) \leq |\mathcal{U}_1 \cup \mathcal{U}_2| \leq |\mathcal{U}_1| + |\mathcal{U}_2| = \mu(C \vee x) + \mu(D \vee \overline{x})$

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   - By expansion $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'| > 0$

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   - Yields α' such that $\alpha'\left(\bigwedge_{F_i \in \mathcal{U}'} F_i \land E\right) = 1$
   - So $\bigwedge_{F_i \in \mathcal{U}'} F_i \land E \nvDash \bot$ for any $|\mathcal{U}'| \leq s$ and hence $\mu(\bot) > s$

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Given (s, δ, E)-resolution expander $(\mathcal{U}, \mathcal{V})_E$ with overlap ℓ

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Proof of claim: Another flipping argument using the resolution edge game:

- Fix $V \in \partial(\mathcal{U}_C)$ and unique neighbour $F_V \in \mathcal{U}_C$ of V
- By minimality, $\exists\, \alpha$ s.t. $\alpha(\bigwedge_{F \in \mathcal{U}_C \setminus \{F_V\}} F \wedge E) = 1$ but $\alpha(C) = 0$
- If $V \cap \mathit{Vars}(C) = \emptyset$, then flip α on V to satisfy $F_V \wedge E$ $\lightning$

Applications: Tseitin and Onto-FPHP

Tseitin formulas

- $F_i =$ clauses encoding parity constraint for ith vertex
- $V_j =$ singleton set with jth edge (so overlap $\ell = 1$)
- $E = \emptyset$
- If underlying graph edge expander, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

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Onto functional PHP formulas

- F_i = singleton set with pigeon axiom for pigeon i
- V_j = all variables $p_{i,j}$ mentioning hole j (again overlap $\ell = 1$)
- E = all hole, functional, and onto axioms
- If onto FPHP restricted to bipartite graph, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

From Resolution to Polynomial Calculus

So far: Obtain resolution width lower bounds from expander graphs where we can win following game on all edges

Resolution edge game on (F, V) with respect to E

1. Adversary provides total assignment α such that $\alpha(E) = 1$
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Polynomial calculus degree lower bounds require harder game

Polynomial calculus edge game on (F, V) with respect to E

1. Commit to partial assignment $\alpha_V : V \to \{0, 1\}$
2. Adversary provides total assignment α such that $\alpha(E) = 1$
3. Substituting α_V for V should yield $\alpha[\alpha_V/V](F \wedge E) = 1$

The Polynomial Calculus Edge Game

To win PC edge game on (F, V), need to find $\alpha_V : V \to \{0, 1\}$ s.t.

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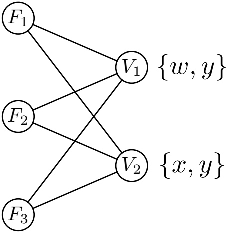

$E = \{\overline{y} \lor z\}$

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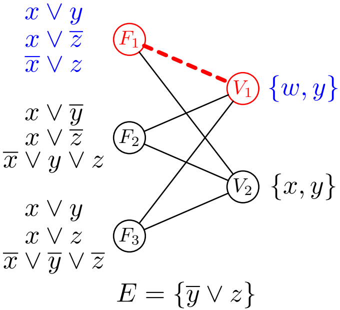

Recall that for resolution edge game we:

- Lose on (F_1, V_1)

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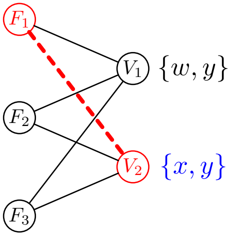

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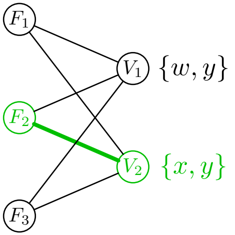

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Recall that for resolution edge game we:

- Lose on (F_1, V_1)
- Lose on (F_1, V_2)
- Win on (F_2, V_2)

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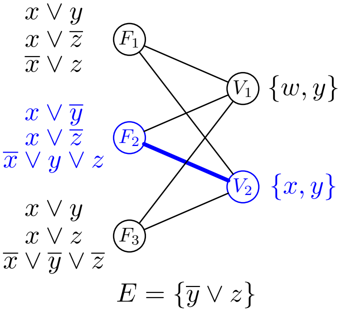

PC edge game on (F_2, V_2) w.r.t. E

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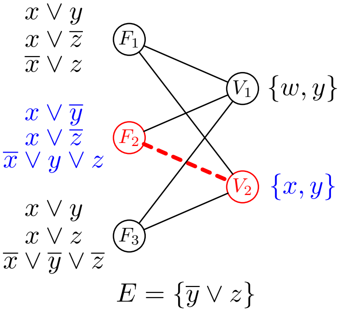

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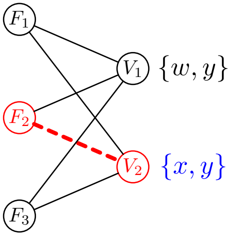

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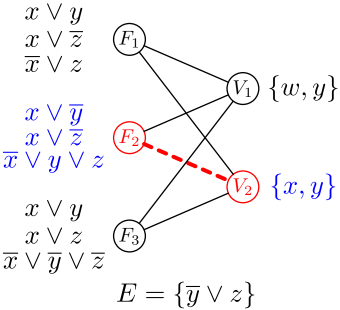

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- $E = \{\overline{y} \vee z\}$ needs $y \mapsto 0$
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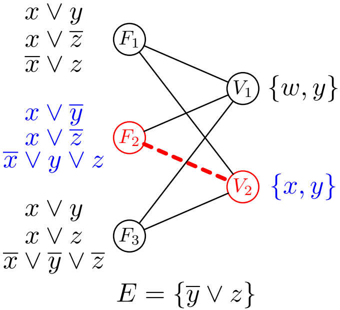

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- Adversary sets $z \mapsto 1 - \alpha_V(x)$

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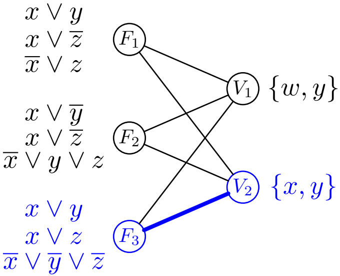

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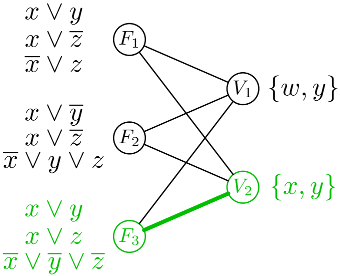

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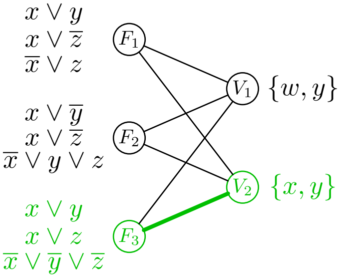

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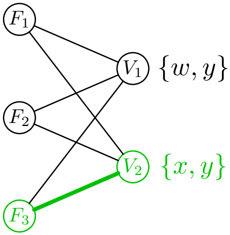

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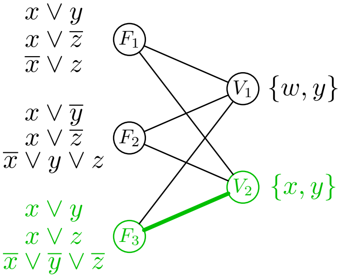

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On this edge we can win!

- Choose $\alpha_V = \{x \mapsto 1, y \mapsto 0\}$
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- $V_2 \cap Vars(\overline{y} \vee z) \neq \emptyset$ but $\alpha_V(\overline{y} \vee z) = 1$

A Generalized Method for PC Degree Lower Bounds

Polynomial calculus expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-PC expander if

- For all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'|$
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Theorem ([MN15] building on [AR03])

If $\mathcal{F}$ admits an (s, δ, E)-PC expander with overlap ℓ, then

$$\textit{PC proof degree} > \frac{\delta s}{2\ell}$$

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Also holds for sets of polynomials not obtained from CNFs
Proof by carefully adapting [AR03] (fairly involved — can't say much)

Consequences

Common framework for previous lower bounds

- Random k-CNF formulas [BI10, AR03]
- CNF formulas with expanding CVIGs [AR03]
- "Vanilla" PHP formulas [AR03]
- Ordering principle formulas [GL10]
- Subset cardinality formulas [MN14]

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New lower bounds

- Functional pigeonhole principle [MN15]
- Graph colouring [LN17]

Hardness of Different Flavours of PHP

Variant	Resolution	Polynomial calculus
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FPHP		
Onto-PHP		
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Hardness of Different Flavours of PHP

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Hardness of Different Flavours of PHP

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- Observe that [AR03] proves hardness of Onto-PHP
- Prove that functional PHP is hard for polynomial calculus (answering open question in [Raz02, Raz14])

Degree Lower Bound for Functional PHP

Theorem ([MN15])

If G is a (standard) bipartite (s, δ)-boundary expander with left degree $\leq d$, then $FPHP_G$ requires PC degree $> \delta s/(2d)$

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- So get same expansion parameters, and theorem follows □

Graph Colouring

Graph k-colouring formulas

"$G = (V, E)$ is k-colourable"

Variables $x_{v,c}$ = "vertex v gets colour c"

$x_{v,1} \lor x_{v,2} \lor \cdots \lor x_{v,k}$	every vertex v gets a colour
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On the contrary, [DLMM08, DLMO09, DLMM11, DMP$^+$15] claim very efficient algorithms based on Nullstellensatz ("static PC") for slightly different encoding using primitive kth roots of unity

Polynomial Calculus Lower Bound for Colouring

Joint work with Massimo Lauria [LN17]:

Theorem ([LN17])

For any $k \geq 3$ $\exists$ constant-degree graphs which require linear PC degree, and hence exponential size, to be proven non-k-colourable

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Lower bound applies also to kth-root-of-unity encoding

Answers open question raised in [DLMO09, LLO16]

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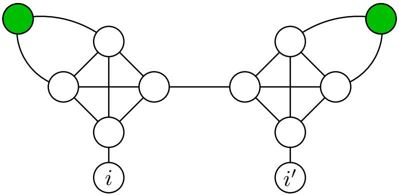

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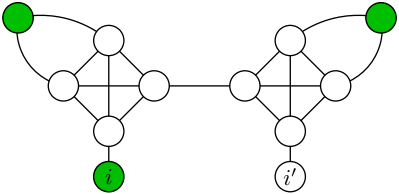

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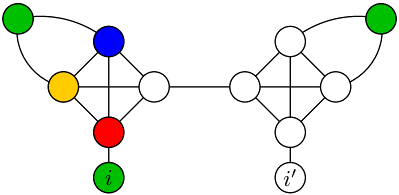

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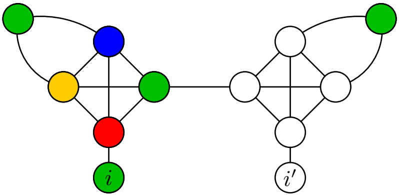

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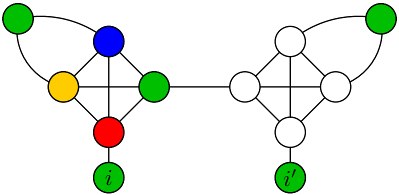

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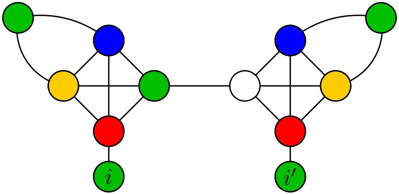

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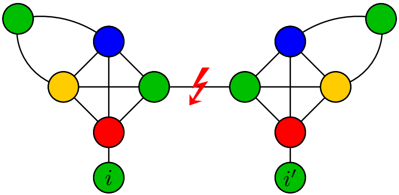

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Symmetric argument in right subgadget $\Rightarrow$ **contradiction**

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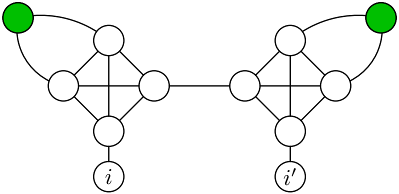

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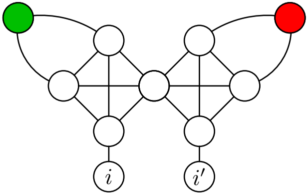

not i green and i' red

Open Problems

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- Go beyond polynomial calculus (e.g. to Positivstellensatz, a.k.a. Lasserre/sums-of-squares)

Take-away Message

Generalized method for width and degree lower bounds

- Unified framework for most previous lower bounds
- Highlights similarities and differences between resolution and polynomial calculus
- Exponential polynomial calculus size lower bound for
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Thank you for your attention!

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