Proof Complexity Lower Bounds from Graph Expansion and Combinatorial Games

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October 30, 2017

Based on joint work with Massimo Lauria and Mladen Mikša

The Satisfiability Problem (SAT)

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1. Can this problem be solved efficiently?

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Or is it always the case that some constraint must fail to hold?

1. Can this problem be solved efficiently?
2. Is there an efficiently verifiable certificate for correct answer?

SAT and Proof Complexity

SAT, NP, and coNP

- SAT NP-complete [Coo71, Lev73], hence unlikely to be solvable efficiently worst-case
- Satisfiable formulas have small certificates (assignment)
- Unsatisfiable formulas don't, unless NP = coNP
  Starting point for proof complexity [CR79]

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Proof complexity

- Prove lower bounds on certificate size for increasingly stronger formal methods of reasoning ($\approx$"separation NP $\neq$ coNP in weak computational models")
- Analyze algorithms used in practice for SAT solving
- Quantify hardness/depth of different mathematical theorems

Proof Complexity and Expansion

- **General goal:** Prove that concrete proof systems cannot efficiently certify unsatisfiability of concrete CNF formulas
- **General theme:**

CNF formula $\mathcal{F}$ "expanding"

$\Downarrow$

Large proofs needed to refute $\mathcal{F}$

- Paradigm implemented for
  - resolution: well-developed machinery
  - polynomial calculus: very much less so

  (Will define these proof systems shortly)
- What "expanding" means is usually a formula-specific hack

A General Expansion Criterion for Hardness

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$, build bipartite graph

- Left vertex set partition of clauses into $\mathcal{F} = \bigcup_{i=1}^{m} F_i$
- Right vertex set division of variables $\mathcal{V} = \bigcup_{j=1}^{n} V_j$
- Edge (F_i, V_j) if $Vars(F_i) \cap V_j \neq \emptyset$

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Lower bound on proof size if

1. Bipartite graph is an expander (very well-connected)
2. We can win the edge game on every edge (F_i, V_j)

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Edge game on (F_i, V_j)

- Adversary assigns all variables $\mathcal{V} \setminus V_j$
- We assign V_j
- We win if F_i true

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- Adversary has to start $\Rightarrow$ resolution lower bound
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Consequences

- Extends techniques in [BW01] and [AR03]
- Unifies many previous lower bounds
- And yields some new ones

Outline

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $\mathcal{F} = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  $k = \mathcal{O}(1)$ constant in this talk
- $\mathit{true} = 1$; $\mathit{false} = 0$
- M = size of formula = # literals (≈ # clauses for k-CNF)
- N = # variables $\leq M$

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\bot$ derived

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Can represent refutation as

- annotated list or
- directed acyclic graph

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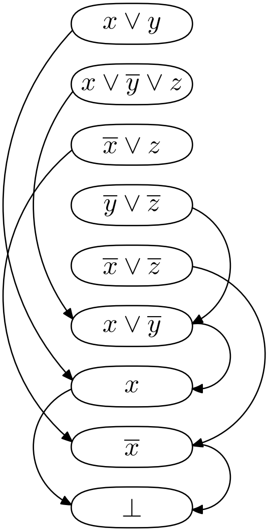

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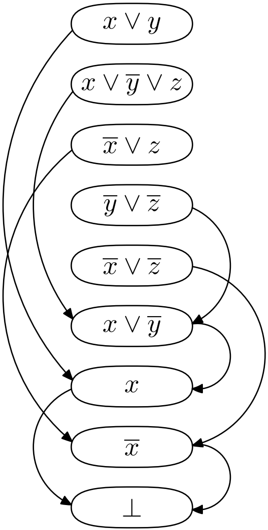
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $=$ $\#$ clauses in refutation [9 in our example]

Most fundamental measure in proof complexity

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(M))$ lower bounds known

(Recall $N = \#$ variables $\leq$ formula size $= M$)

Examples of Hard Formulas w.r.t Resolution Size (1/3)

Pigeonhole principle (PHP) [Hak85]
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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Even onto functional PHP formulas are hard for resolution
"Resolution cannot count"

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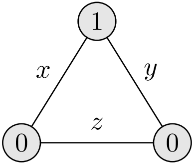
But only lower bound $\exp\left(\Omega\left(\sqrt[3]{M}\right)\right)$ in terms of formula size

Examples of Hard Formulas w.r.t Resolution Size (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



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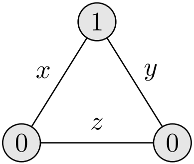

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Requires size $\exp(\Omega(M))$ on bounded-degree edge expanders
"Resolution cannot count $\bmod 2$"

Examples of Hard Formulas w.r.t Resolution Size (3/3)

Random k-CNF formulas [CS88, BKPS02]
Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(M))$

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Again lower bound $\exp(\Omega(M))$

And more...

- k-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Subset cardinality formulas [Spe10, VS10, MN14]
- Et cetera...

Resolution Width

Width = size of largest clause in refutation (always $\leq N$)

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Width upper bound $\Rightarrow$ size upper bound

Proof: at most $(2N)^{\text{width}}$ distinct clauses
(And this counting argument is essentially tight [ALN16])

Resolution Width

Width $=$ size of largest clause in refutation (always $\leq N$)

Width upper bound $\Rightarrow$ size upper bound

Proof: at most $(2N)^{\text{width}}$ distinct clauses
(And this counting argument is essentially tight [ALN16])

Width lower bound $\Rightarrow$ size lower bound

Much less obvious. . .

Width Lower Bounds Imply Size Lower Bounds

Theorem ([BW01])

For k-CNF formula over N variables

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For tree-like resolution have proof size $\geq 2^{\text{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no size lower bounds — possible to tighten analysis? **No!**

Optimality of the Size-Width Lower Bound

Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
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But initial clauses have width $\Omega(n) = \Omega\big(\sqrt{N}\big)$ — a bit more work needed to make the width lower bound meaningful. . .

Conversion to k-CNF "Graph Versions" of Formulas

- Need bounded-width CNFs to use lower bound in [BW01]
- But PHP and ordering principle formulas have wide clauses
- **Solution:** Restrict formulas to bounded-degree graphs

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- Now strong width lower bounds $\Rightarrow$ strong size lower bounds
- And size lower bounds hold for original, unrestricted formulas

Polynomial Calculus (PC)

From [CEI96]; with adjustment in [ABRW02]

Clauses interpreted as polynomial equations over field $\mathbb{F}$

Example: $x \vee y \vee \overline{z}$ gets translated to $\overline{x}\,\overline{y}z = 0$

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Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$ Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$ Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Formalizes Gröbner basis computation

Polynomial Calculus Size and Degree

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Write out all polynomials as sums of monomials

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Write out all polynomials as sums of monomials
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Size — analogue of resolution length/size
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently

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Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas (over $\mathrm{GF}(2)$ can do Gaussian elimination)
- Onto functional pigeonhole principle (over any field) [Rii93]
- Also other examples

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- **Examples of open problems:**
  - Hardness of functional PHP and onto PHP formulas?
  - Hardness of k-colouring formulas?

Lower Bounds via Graph Expansion

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Lower bounds from expansion

Simplest example is the clause-variable incidence graph (CVIG)

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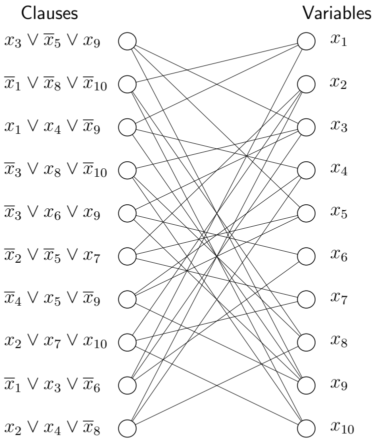

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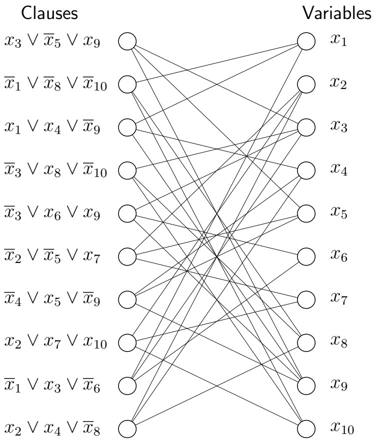
Boundary expansion:
Subsets of left vertices have many unique right neighbours



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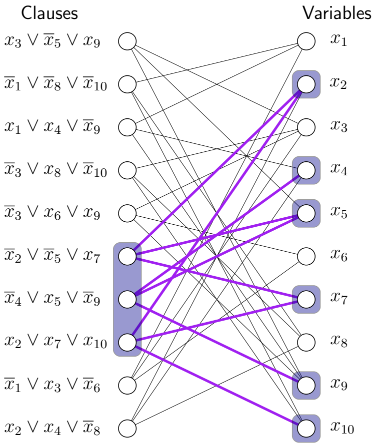

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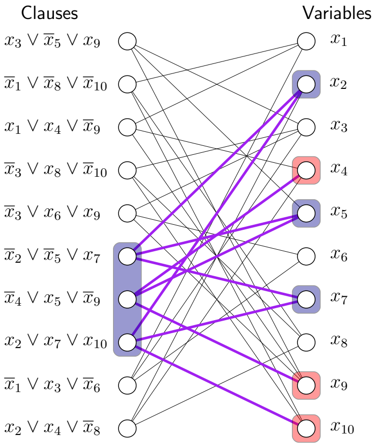

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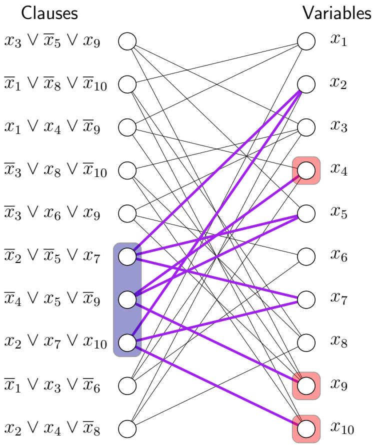

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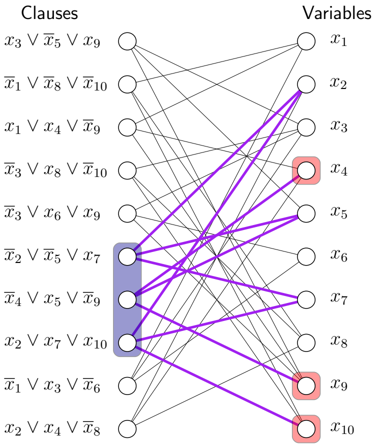

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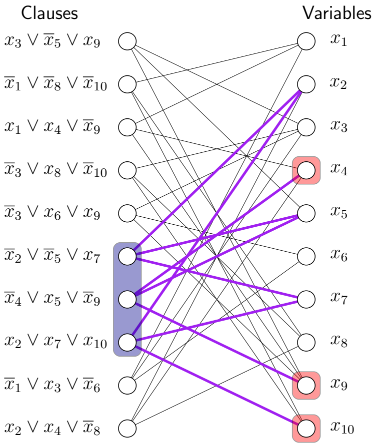
Need graph capturing combinatorial structure!



Generalized Incidence Graphs for CNF Formulas

Given CNF formula $\mathcal{F}$ over variables $\mathcal{V}$

- Partition clauses into $\mathcal{F} = E \cup \bigcup_{i=1}^{m} F_i$ (for E satisifiable)
- Divide variables into $\mathcal{V} = \bigcup_{j=1}^{n} V_j$ — **not** always partition
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Build bipartite $(\mathcal{U}, \mathcal{V})_E$-graph $\mathcal{G}$

- Left vertices $\mathcal{U} = \{F_1, \ldots, F_m\}$
- Right vertices $\mathcal{V} = \{V_1, \ldots, V_n\}$
- Edge (F_i, V_j) if $\mathit{Vars}(F_i) \cap V_j \neq \emptyset$

The Resolution Edge Game

Resolution edge game on (F_i, V_j) w.r.t. "filtering set" E

- Adversary choses any total assignment α such that $\alpha(E) = 1$
- We can modify α on V_j to get α'
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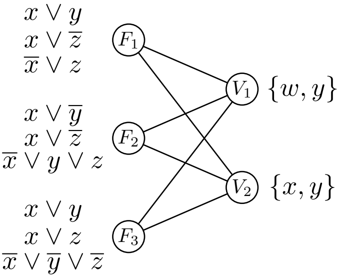

$$E = \{\overline{y} \vee z\}$$

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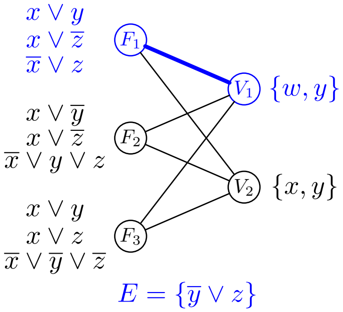

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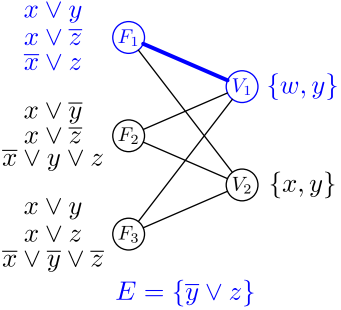

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Take $\alpha_1 = \{x \mapsto 1, y \mapsto 0, z \mapsto 0\}$

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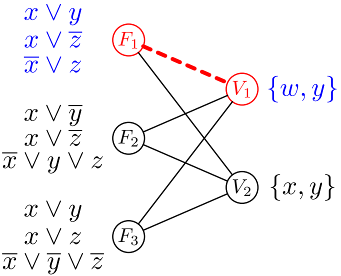

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Can't win, since

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- can't flip x or z (not in V_1)

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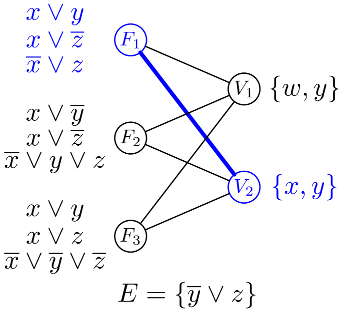

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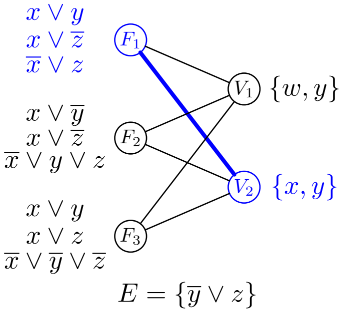

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Take (partial) $\alpha_2 = \{y \mapsto 0, z \mapsto 0\}$

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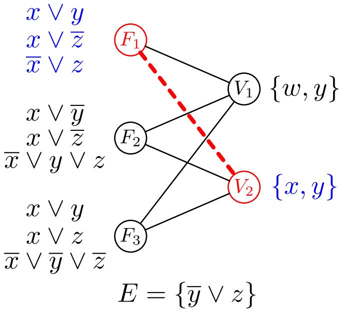

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Take (partial) $\alpha_2 = \{y \mapsto 0, z \mapsto 0\}$

Again can't win, since

- can't flip z (not in V_2)
- flipping $y \in V_2$ falsifies E
- $F_1{\restriction}_{\alpha_2} = \{x, \overline{x}\}$

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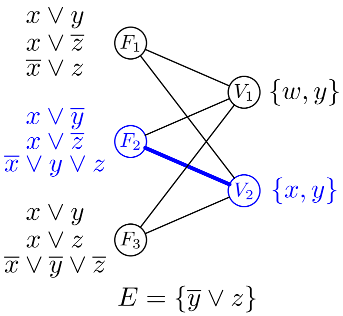

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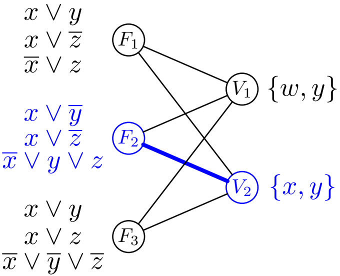

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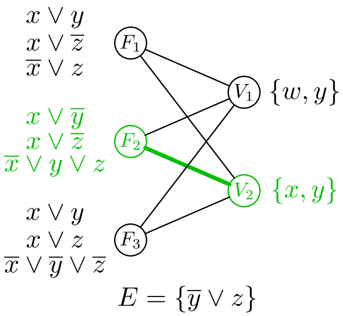

Edge game on (F_2, V_2) w.r.t. E

Now we can win!

Given any α_3 s.t. $\alpha_3(E) = 1$:

- assign $x \mapsto \alpha_3(y \vee z)$
- E still OK — didn't touch y, z
- F_2 OK — encodes $x \leftrightarrow (y \vee z)$

Edge Game, Expansion, and Width Lower Bounds

Recall boundary $\partial(\mathcal{U}') = \{V \in \mathcal{N}(\mathcal{U}') \mid \mathcal{N}(V) \cap \mathcal{U}' = \{F\} \text{ unique}\}$

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Resolution expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-resolution expander if

- For all $\mathcal{U}' \subseteq \mathcal{U}$, $|\mathcal{U}'| \leq s$ it holds that $|\partial(\mathcal{U}')| \geq \delta|\mathcal{U}'|$
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Theorem (essentially [BW01])

If the CNF formula $\mathcal{F}$ admits an (s, δ, E)-resolution expander with overlap ℓ, then

$$\textit{resolution proof width} > \frac{\delta s}{2\ell}$$

Progress Measure Approach (1/4)

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Proof overview: Define "progress measure" $\mu : \{\text{clauses}\} \to \mathbb{N}$ such that

1. $\mu(\text{axiom clause}) = \mathcal{O}(1)$
2. $\mu(C \vee D) \leq \mu(C \vee x) + \mu(D \vee \overline{x})$
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$\Rightarrow$ such clause C has width $\geq \delta\sigma/\ell$ □

Progress Measure Approach (2/4)

Given (s, δ, E)-resolution expander $(\mathcal{U}, \mathcal{V})_E$ for $\mathcal{F}$, define

$$\mu(C) := \min\left\{ \left|\mathcal{U}'\right| ; \bigwedge_{F \in \mathcal{U}'} F \wedge E \vDash C \right\}$$

Progress Measure Approach (2/4)

Given (s, δ, E)-resolution expander $(\mathcal{U}, \mathcal{V})_E$ for $\mathcal{F}$, define

$$\mu(C) := \min\{|\mathcal{U}'| \,;\, \bigwedge_{F \in \mathcal{U}'} F \wedge E \vDash C\}$$

1. $\mu(A) = \mathcal{O}(1)$ for axioms $A \in \mathcal{F} = \bigcup_{i=1}^{m} F_i \cup E$
   - $A \in E$: $\mu(A) = 0$ since $E \vDash A$
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2. $\mu(C \vee D) \leq \mu(C \vee x) + \mu(D \vee \overline{x})$
   - Fix minimal $\mathcal{U}_1$ s.t. $\bigwedge_{F \in \mathcal{U}_1} F \wedge E \vDash C \vee x$
   - Fix minimal $\mathcal{U}_2$ s.t. $\bigwedge_{F \in \mathcal{U}_2} F \wedge E \vDash D \vee \overline{x}$
   - Then it holds that

$$\bigwedge_{F \in \mathcal{U}_1 \cup \mathcal{U}_2} F \wedge E \vDash C \vee D ,$$

so $\mu(C \vee D) \leq \left|\mathcal{U}_1 \cup \mathcal{U}_2\right| \leq \left|\mathcal{U}_1\right| + \left|\mathcal{U}_2\right| = \mu(C \vee x) + \mu(D \vee \overline{x})$

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   - By "peeling argument" $\exists$ matching $F_1 \leftrightarrow V_1, \ldots, F_s \leftrightarrow V_s$ such that $V_i \in \mathcal{N}(F_i) \setminus \mathcal{N}\left(\bigcup_{j=1}^{i-1} F_j\right)$ (i.e., V_i is not a neighbour of any F_j, $j < i$)

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   - Yields α' such that $\alpha'\left(\bigwedge_{F_i \in \mathcal{U}'} F_i \wedge E\right) = 1$
   - So $\bigwedge_{F_i \in \mathcal{U}'} F_i \wedge E \nvDash \bot$ for any $\lvert\mathcal{U}'\rvert \leq s$ and hence $\mu(\bot) > s$

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Since every variable occurs in $\leq \ell$ sets V, the clause C then must have width $\geq \left|\partial(\mathcal{U}_C)\right|/\ell \geq \delta\left|\mathcal{U}_C\right|/\ell = \delta\sigma/\ell$ □

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Proof of claim: Another flipping argument using the resolution edge game:

- Fix $V \in \partial(\mathcal{U}_C)$ and unique neighbour $F_V \in \mathcal{U}_C$ of V
- By minimality, $\exists\, \alpha$ s.t. $\alpha\left(\bigwedge_{F \in \mathcal{U}_C \setminus \{F_V\}} F \wedge E\right) = 1$ but $\alpha(C) = 0$
- If $V \cap \mathit{Vars}(C) = \emptyset$, then flip α on V to satisfy $F_V \wedge E$ ↯

Applications: Tseitin and Onto-FPHP

Tseitin formulas

- F_i = clauses encoding parity constraint for ith vertex
- V_j = singleton set with jth edge (so overlap $\ell = 1$)
- $E = \emptyset$
- If underlying graph edge expander, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

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Onto functional PHP formulas

- F_i = singleton set with pigeon axiom for pigeon i
- V_j = all variables $p_{i,j}$ mentioning hole j (again overlap $\ell = 1$)
- E = all hole, functional, and onto axioms
- If onto FPHP restricted to bipartite graph, then $(\mathcal{U}, \mathcal{V})_E$-graph is resolution expander with same parameters

From Resolution to Polynomial Calculus

So far: Obtain resolution width lower bounds from expander graphs where we can win following game on all edges

Resolution edge game on (F, V) with respect to E

1. Adversary provides total assignment α such that $\alpha(E) = 1$
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Polynomial calculus degree lower bounds require harder game

Polynomial calculus edge game on (F, V) with respect to E

1. Commit to partial assignment $\alpha_V : V \to \{0, 1\}$
2. Adversary provides total assignment α such that $\alpha(E) = 1$
3. Substituting α_V for V should yield $\alpha[\alpha_V/V](F \wedge E) = 1$

The Polynomial Calculus Edge Game

To win PC edge game on (F, V), need to find $\alpha_V : V \rightarrow \{0, 1\}$ s.t.

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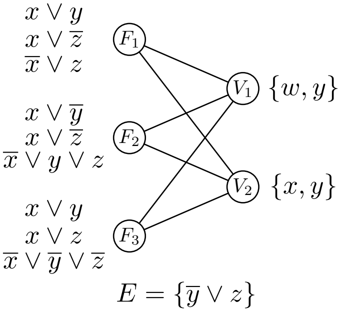

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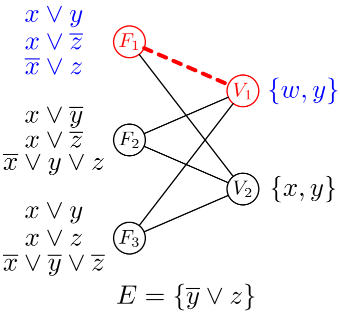

Recall that for resolution edge game we:

- Lose on (F_1, V_1)

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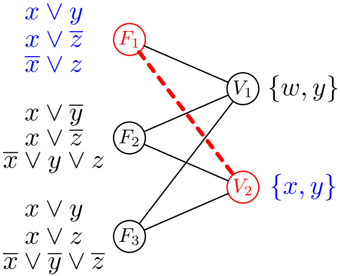

$E = \{\overline{y} \vee z\}$

Recall that for resolution edge game we:

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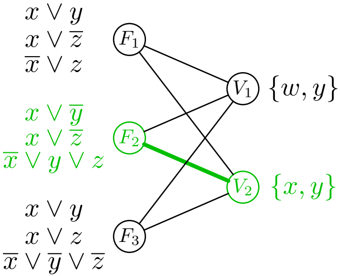

$E = \{\overline{y} \vee z\}$

Recall that for resolution edge game we:

- Lose on (F_1, V_1)
- Lose on (F_1, V_2)
- Win on (F_2, V_2)

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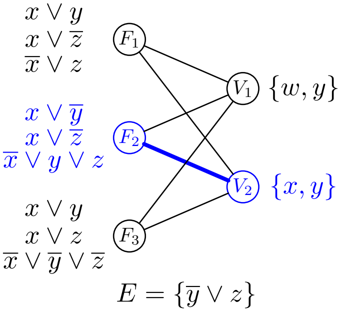

PC edge game on (F_2, V_2) w.r.t. E

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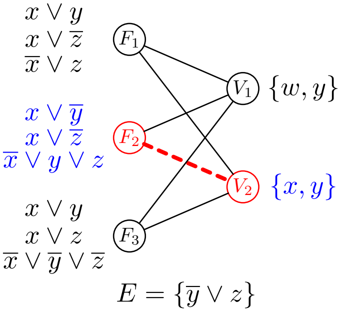

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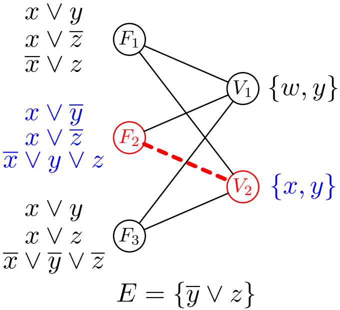

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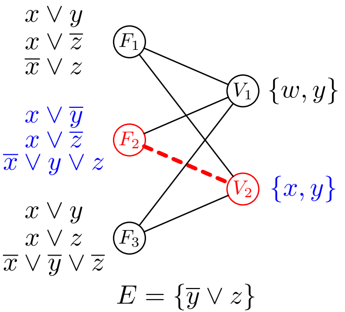

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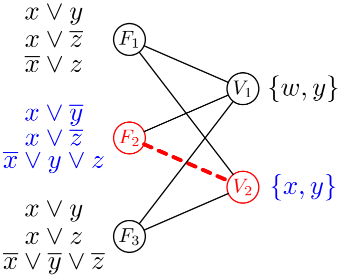

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- Adversary sets $z \mapsto 1 - \alpha_V(x)$

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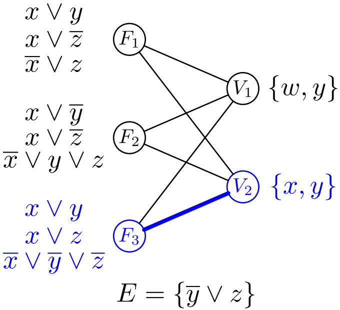

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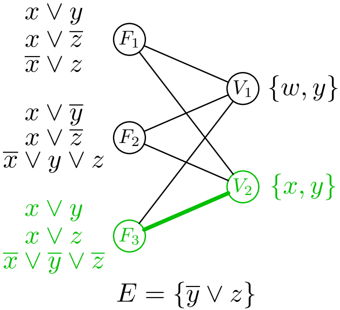

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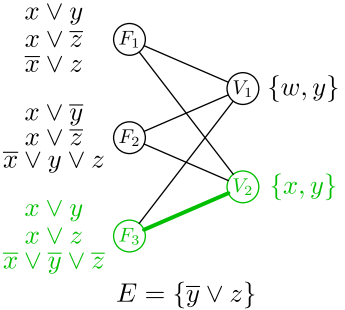

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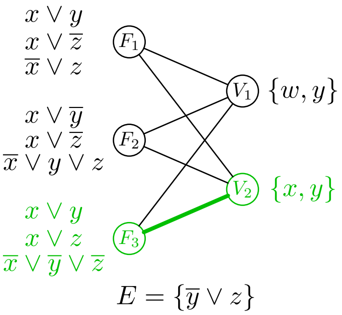

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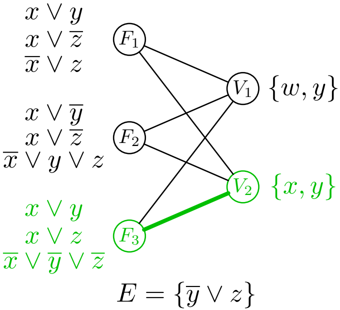

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A Generalized Method for PC Degree Lower Bounds

Polynomial calculus expander

Say that an $(\mathcal{U}, \mathcal{V})_E$-graph is an (s, δ, E)-PC expander if

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Theorem ([MN15] building on [AR03])

If $\mathcal{F}$ admits an (s, δ, E)-PC expander with overlap ℓ, then

$$\textit{PC proof degree} > \frac{\delta s}{2\ell}$$

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Also holds for sets of polynomials not obtained from CNFs
Proof by carefully adapting [AR03] (fairly involved — can't say much)

Consequences

Common framework for previous lower bounds

- Random k-CNF formulas [BI10, AR03]
- CNF formulas with expanding CVIGs [AR03]
- "Vanilla" PHP formulas [AR03]
- Ordering principle formulas [GL10]
- Subset cardinality formulas [MN14]

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New lower bounds

- Functional pigeonhole principle [MN15]
- Graph colouring [LN17]

Hardness of Different Flavours of PHP

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- Observe that [AR03] proves hardness of Onto-PHP
- Prove that functional PHP is hard for polynomial calculus (answering open question in [Raz02, Raz14])

Degree Lower Bound for Functional PHP

Theorem ([MN15])

If G is a (standard) bipartite (s, δ)-boundary expander with left degree $\leq d$, then $FPHP_G$ requires PC degree $> \delta s/(2d)$

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- So get same expansion parameters, and theorem follows □

Graph Colouring

Graph k-colouring formulas
"$G = (V, E)$ is k-colourable"

Variables $x_{v,c}$ = "vertex v gets colour c"

$x_{v,1} \lor x_{v,2} \lor \cdots \lor x_{v,k}$	every vertex v gets a colour
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On the contrary, [DLMM08, DLMO09, DLMM11, DMP^{+}15] claim very efficient algorithms based on Nullstellensatz ("static PC") for slightly different encoding using primitive kth roots of unity

Polynomial Calculus Lower Bound for Colouring

Joint work with Massimo Lauria [LN17]:

Theorem ([LN17])

For any $k \geq 3$ $\exists$ constant-degree graphs which require linear PC degree, and hence exponential size, to be proven non-k-colourable

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Lower bound applies also to kth-root-of-unity encoding

Answers open question raised in [DLMO09, LLO16]

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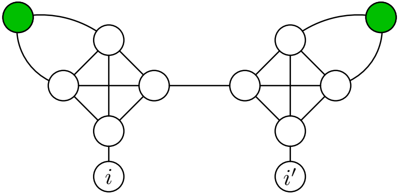

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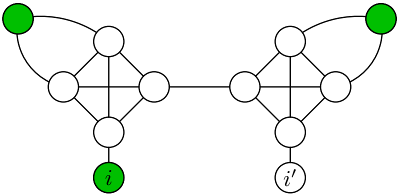

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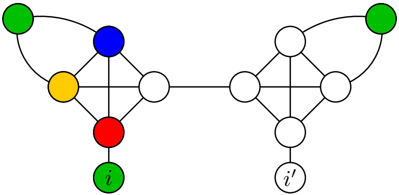

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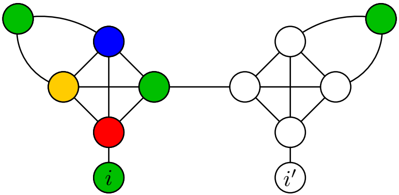

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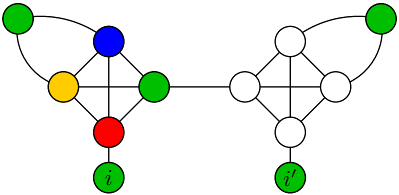

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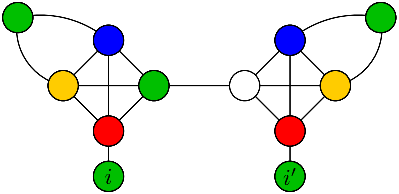

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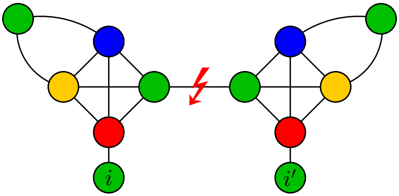

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Symmetric argument in right subgadget $\Rightarrow$ **contradiction**

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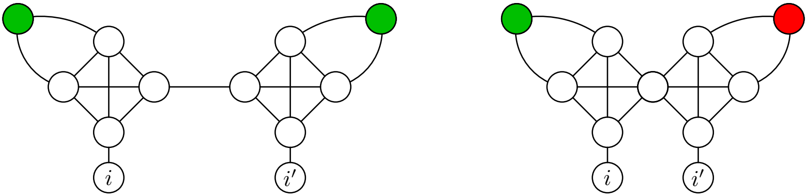

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not i green and i' red

Open Problems

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- Go beyond polynomial calculus (e.g. to Positivstellensatz, a.k.a. Lasserre/sums-of-squares)

Take-away Message

Generalized method for width and degree lower bounds

- Unified framework for most previous lower bounds
- Highlights similarities and differences between resolution and polynomial calculus
- Exponential polynomial calculus size lower bound for
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Future directions

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Thank you for your attention!

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