# On the Interplay Between Proof Complexity and SAT Solving

Jakob Nordström

KTH Royal Institute of Technology
Stockholm, Sweden

National Research University Higher School of Economics
Moscow, Russia
April 8, 2016

# The Satisfiability Problem (SAT)

$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

# The Satisfiability Problem (SAT)

$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

- Variables should be set to true or false

# The Satisfiability Problem (SAT)

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

- Variables should be set to true or false
- Constraint $(x \lor \overline{y} \lor z)$: means $x$ or $z$ should be true or $y$ false

# The Satisfiability Problem (SAT)

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

- Variables should be set to true or false
- Constraint $(x \lor \overline{y} \lor z)$: means $x$ or $z$ should be true or $y$ false
- $\land$ means all constraints should hold simultaneously

# The Satisfiability Problem (SAT)

$$(x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z})$$

- Variables should be set to true or false
- Constraint $(x \vee \overline{y} \vee z)$: means $x$ or $z$ should be true or $y$ false
- $\wedge$ means all constraints should hold simultaneously

Is there a truth value assignment satisfying all these conditions?
Or is it always the case that some constraint must fail to hold?

# The Satisfiability Problem (SAT)

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

- Variables should be set to true or false
- Constraint $(x \lor \overline{y} \lor z)$: means $x$ or $z$ should be true or $y$ false
- $\land$ means all constraints should hold simultaneously

Is there a truth value assignment satisfying all these conditions?
Or is it always the case that some constraint must fail to hold?

Can we use computers to solve the SAT problem efficiently?

# Computational Complexity Theory and SAT Solving

**Complexity theory**

- Satisfiability of formulas in propositional logic (SAT) foundational problem
- SAT proven NP-complete by Stephen Cook in 1971
- Hence most likely totally intractable
- Just remains to prove this — one of the million-dollar "Millennium Problems"

# Computational Complexity Theory and SAT Solving

**Complexity theory**

- Satisfiability of formulas in propositional logic (SAT) foundational problem
- SAT proven NP-complete by Stephen Cook in 1971
- Hence most likely totally intractable
- Just remains to prove this — one of the million-dollar "Millennium Problems"

**Applied SAT solving**

- Dramatic performance increase last 15–20 years
- State-of-the-art SAT solvers can deal with real-world formulas containing millions of variables
- But best solvers still based on methods from early 1960s
- Also, tiny formulas known that are totally beyond reach

# SAT Solving and Proof Complexity

- How can state-of-the-art SAT solvers decide satisfiability of such huge formulas?
- Why do they work so well? And why do they sometimes miserably fail?
- Best current SAT solvers
  - Based on so-called conflict-driven clause learning (CDCL)
  - Sometimes algebraic reasoning (e.g., Gaussian elimination)
  - Sometimes geometric reasoning (e.g., cardinality constraints)
  - And even perhaps some so-called extended resolution
- How can we analyze the power of these methods?
  Question addressed by research area of **proof complexity**

# Outline of This Presentation

This talk: overview of (or crash course in) proof complexity

Focus on connections with current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Algebraic Gröbner basis computations — polynomial calculus
- Geometric pseudo-Boolean solvers — cutting planes
- Might also mention extended resolution, but if so very briefly

Survey (some of) what is known about these proof systems

Show theoretical "benchmark formulas" used to understand potential and limitations of methods of reasoning

# Outline of This Presentation

This talk: overview of (or crash course in) proof complexity

Focus on connections with current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Algebraic Gröbner basis computations — polynomial calculus
- Geometric pseudo-Boolean solvers — cutting planes
- Might also mention extended resolution, but if so very briefly

Survey (some of) what is known about these proof systems

Show theoretical "benchmark formulas" used to understand potential and limitations of methods of reasoning

**Caveats:**

- By necessity, selective and somewhat subjective coverage
- Won't do too much name-dropping — full references at end of slides

# Outline


1. Resolution
   - Preliminaries
   - Length, Width and Space
   - Resolution Trade-offs
2. Connections Between Resolution and CDCL
   - Resolution and SAT Solving
   - Complexity Measures and CDCL
   - Research Questions and Future Directions
3. Stronger Proof Systems than Resolution
   - Polynomial Calculus
   - Cutting Planes
   - And Beyond...

# Some Notation and Terminology

- Literal $a$: variable $x$ or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- $k$-CNF formula: CNF formula with clauses of size $\leq k$
  (where $k$ is some constant)
- Mostly assume formulas $k$-CNFs (for simplicity of exposition)
  Conversion to $3$-CNF (most often) doesn't change much
- $N$ denotes size of formula ($\#$ literals, which is $\approx \#$ clauses)

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\bot$ derived

1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
5. $\overline{x} \lor \overline{z}$

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| **2.** | $\boldsymbol{x \lor \overline{y} \lor z}$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| **4.** | $\boldsymbol{\overline{y} \lor \overline{z}}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| **2.** | $\boldsymbol{x \lor \overline{y} \lor z}$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| **4.** | $\boldsymbol{\overline{y} \lor \overline{z}}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| **6.** | $\boldsymbol{x \lor \overline{y}}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\bot$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| **6.** | $\boldsymbol{x \lor \overline{y}}$ | Res$(2,4)$ |
| 7. | $x$ | Res$(1,6)$ |
| 8. | $\overline{x}$ | Res$(3,5)$ |
| 9. | $\bot$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| **1.** | $\boldsymbol{x \lor y}$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| **6.** | $\boldsymbol{x \lor \overline{y}}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| **1.** | $\boldsymbol{x \lor y}$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| **6.** | $\boldsymbol{x \lor \overline{y}}$ | Res$(2,4)$ |
| **7.** | $\boldsymbol{x}$ | Res$(1,6)$ |
| 8. | $\overline{x}$ | Res$(3,5)$ |
| 9. | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \vee y$ | Axiom |
| 2. | $x \vee \overline{y} \vee z$ | Axiom |
| 3. | $\overline{x} \vee z$ | Axiom |
| 4. | $\overline{y} \vee \overline{z}$ | Axiom |
| 5. | $\overline{x} \vee \overline{z}$ | Axiom |
| 6. | $x \vee \overline{y}$ | Res$(2,4)$ |
| **7.** | $\boldsymbol{x}$ | Res$(1,6)$ |
| 8. | $\overline{x}$ | Res$(3,5)$ |
| 9. | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| **3.** | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| **5.** | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2,4)$ |
| 7. | $x$ | Res$(1,6)$ |
| 8. | $\overline{x}$ | Res$(3,5)$ |
| 9. | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \vee y$ | Axiom |
| 2. | $x \vee \overline{y} \vee z$ | Axiom |
| **3.** | $\boldsymbol{\overline{x} \vee z}$ | **Axiom** |
| 4. | $\overline{y} \vee \overline{z}$ | Axiom |
| **5.** | $\boldsymbol{\overline{x} \vee \overline{z}}$ | **Axiom** |
| 6. | $x \vee \overline{y}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| **8.** | $\boldsymbol{\overline{x}}$ | **Res$(3, 5)$** |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2,4)$ |
| 7. | $x$ | Res$(1,6)$ |
| **8.** | $\boldsymbol{\overline{x}}$ | Res$(3,5)$ |
| 9. | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2, 4)$ |
| **7.** | $\boldsymbol{x}$ | Res$(1, 6)$ |
| **8.** | $\boldsymbol{\overline{x}}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2,4)$ |
| **7.** | $\boldsymbol{x}$ | Res$(1,6)$ |
| **8.** | $\boldsymbol{\overline{x}}$ | Res$(3,5)$ |
| **9.** | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

| | | |
|---|---|---|
| 1. | $x \vee y$ | Axiom |
| 2. | $x \vee \overline{y} \vee z$ | Axiom |
| 3. | $\overline{x} \vee z$ | Axiom |
| 4. | $\overline{y} \vee \overline{z}$ | Axiom |
| 5. | $\overline{x} \vee \overline{z}$ | Axiom |
| 6. | $x \vee \overline{y}$ | Res$(2,4)$ |
| 7. | $x$ | Res$(1,6)$ |
| 8. | $\overline{x}$ | Res$(3,5)$ |
| **9.** | $\perp$ | Res$(7,8)$ |

# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

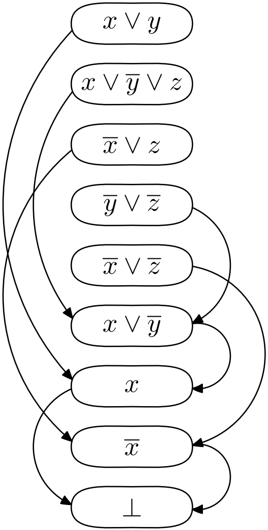
Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\bot$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph



# The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

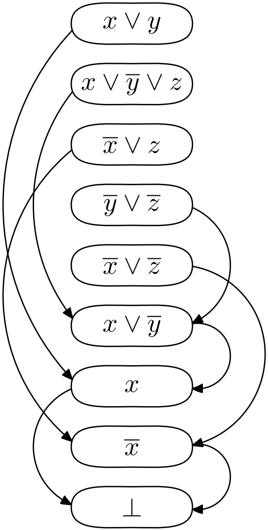
$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
- directed acyclic graph

Tree-like resolution if DAG is tree



# Resolution Size/Length

**Size/length** $= \#$ clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

# Examples of Hard Formulas w.r.t Resolution Length (1/3)

**Pigeonhole principle (PHP)** [Hak85]*

"$n+1$ pigeons don't fit into $n$ holes"

Variables $p_{i,j}$ = "pigeon $i$ goes into hole $j$"

| | |
|---|---|
| $p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$ | every pigeon $i$ gets a hole |
| $\overline{p}_{i,j} \lor \overline{p}_{i',j}$ | no hole $j$ gets two pigeons $i \neq i'$ |

Can also add "functionality" and "onto" axioms

| | |
|---|---|
| $\overline{p}_{i,j} \lor \overline{p}_{i,j'}$ | no pigeon $i$ gets two holes $j \neq j'$ |
| $p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$ | every hole $j$ gets a pigeon |

# Examples of Hard Formulas w.r.t Resolution Length (1/3)

**Pigeonhole principle (PHP)** [Hak85]*
"$n+1$ pigeons don't fit into $n$ holes"

Variables $p_{i,j}$ = "pigeon $i$ goes into hole $j$"

| | |
|---|---|
| $p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$ | every pigeon $i$ gets a hole |
| $\overline{p}_{i,j} \lor \overline{p}_{i',j}$ | no hole $j$ gets two pigeons $i \neq i'$ |

Can also add "functionality" and "onto" axioms

| | |
|---|---|
| $\overline{p}_{i,j} \lor \overline{p}_{i,j'}$ | no pigeon $i$ gets two holes $j \neq j'$ |
| $p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$ | every hole $j$ gets a pigeon |

Even onto functional PHP formula is hard for resolution
**"Resolution cannot count"**

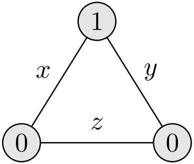
(*) List of full references given at the end of the slides (also available online)

# Examples of Hard Formulas w.r.t Resolution Length (2/3)

**Tseitin formulas** [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label




$$
\begin{array}{rr}
(x \lor y) & \land\ (\overline{x} \lor z) \\
\land\ (\overline{x} \lor \overline{y}) & \land\ (y \lor \overline{z}) \\
\land\ (x \lor \overline{z}) & \land\ (\overline{y} \lor z)
\end{array}
$$

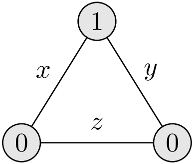
# Examples of Hard Formulas w.r.t Resolution Length (2/3)

**Tseitin formulas** [Urq87]

"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label




$$
\begin{array}{ll}
(x \lor y) & \land\ (\overline{x} \lor z) \\
\land\ (\overline{x} \lor \overline{y}) & \land\ (y \lor \overline{z}) \\
\land\ (x \lor \overline{z}) & \land\ (\overline{y} \lor z)
\end{array}
$$

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders

**"Resolution cannot count** $\mathrm{mod}\ 2$**"**

# Examples of Hard Formulas w.r.t Resolution Length (3/3)

**Random $k$-CNF formulas** [CS88]

$\Delta n$ randomly sampled $k$-clauses over $n$ variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable $3$-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

# Examples of Hard Formulas w.r.t Resolution Length (3/3)

**Random $k$-CNF formulas** [CS88]
$\Delta n$ randomly sampled $k$-clauses over $n$ variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable $3$-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

**And more...**

- $k$-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera...

# Resolution Width

**Width** $=$ size of largest clause in refutation (always $\leq N$)

# Resolution Width

**Width** = size of largest clause in refutation (always $\leq N$)

Width upper bound $\Rightarrow$ length upper bound

**Proof:** at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [ALN14])

# Resolution Width

**Width** = size of largest clause in refutation (always $\leq N$)

Width upper bound $\Rightarrow$ length upper bound

**Proof:** at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [ALN14])

Width lower bound $\Rightarrow$ length lower bound

Much less obvious. . .

# Width Lower Bounds Imply Length Lower Bounds

**Theorem ([BW01])**

$$\textit{length} \geq \exp\left(\Omega\left(\frac{(\textit{width})^2}{(\textit{formula size } N)}\right)\right)$$

# Width Lower Bounds Imply Length Lower Bounds

**Theorem ([BW01])**

$$\textit{length} \geq \exp\left(\Omega\left(\frac{(\textit{width})^2}{(\textit{formula size } N)}\right)\right)$$

Yields superpolynomial length bounds for width $\omega\left(\sqrt{N \log N}\right)$
Almost all known lower bounds on length derivable via width

# Width Lower Bounds Imply Length Lower Bounds

**Theorem ([BW01])**

$$\mathit{length} \geq \exp\left(\Omega\left(\frac{(\mathit{width})^2}{(\mathit{formula\ size}\ N)}\right)\right)$$

Yields superpolynomial length bounds for width $\omega\left(\sqrt{N \log N}\right)$
Almost all known lower bounds on length derivable via width

For tree-like resolution have length $\geq 2^{\mathit{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no length lower bounds — possible to tighten analysis? **No!**

# Optimality of the Length-Width Lower Bound

**Ordering principles** [Stå96, BG01]

"Every finite ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

| | |
|---|---|
| $\overline{x}_{i,j} \lor \overline{x}_{j,i}$ | anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$ |
| $\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$ | transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$ |
| $\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$ | $e_j$ is not a minimal element |

Can also add "total order" axioms

| | |
|---|---|
| $x_{i,j} \lor x_{j,i}$ | totality; either $e_i < e_j$ or $e_j < e_i$ |

# Optimality of the Length-Width Lower Bound

**Ordering principles** [Stå96, BG01]
"Every finite ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

| | |
|---|---|
| $\overline{x}_{i,j} \lor \overline{x}_{j,i}$ | anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$ |
| $\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$ | transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$ |
| $\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$ | $e_j$ is not a minimal element |

Can also add "total order" axioms

| | |
|---|---|
| $x_{i,j} \lor x_{j,i}$ | totality; either $e_i < e_j$ or $e_j < e_i$ |

Refutable in resolution in length $\mathcal{O}(N)$
Requires resolution width $\Omega\big(\sqrt[3]{N}\big)$ (for $3$-CNF version)

# Resolution Space

**Space** = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step $t$: # clauses at steps $\leq t$ used at steps $\geq t$

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# Resolution Space

**Space** = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step $t$: # clauses at steps $\leq t$ used at steps $\geq t$

**Example:** Space at step 7 ...

| | | |
|---|---|---|
| 1. | $x \lor y$ | Axiom |
| 2. | $x \lor \overline{y} \lor z$ | Axiom |
| 3. | $\overline{x} \lor z$ | Axiom |
| 4. | $\overline{y} \lor \overline{z}$ | Axiom |
| 5. | $\overline{x} \lor \overline{z}$ | Axiom |
| 6. | $x \lor \overline{y}$ | Res$(2, 4)$ |
| 7. | $x$ | Res$(1, 6)$ |
| 8. | $\overline{x}$ | Res$(3, 5)$ |
| 9. | $\perp$ | Res$(7, 8)$ |

# Resolution Space

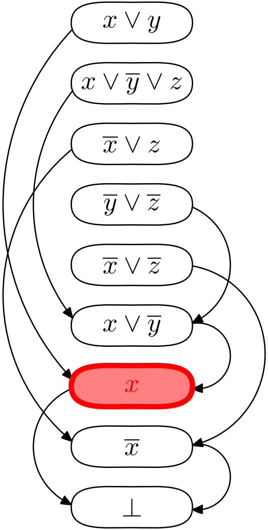
**Space** = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step $t$: # clauses at steps $\leq t$ used at steps $\geq t$

**Example:** Space at step 7 ...



# Resolution Space

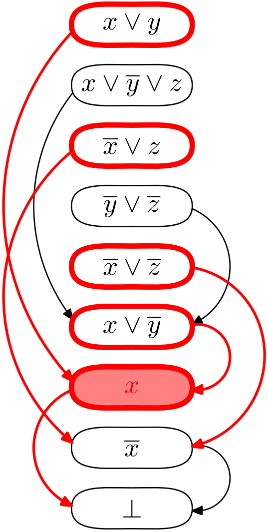
**Space** = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step $t$: # clauses at steps $\leq t$ used at steps $\geq t$

**Example:** Space at step $7$ is $5$



# Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Lower bounds subsequently proven for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random $k$-CNFs [BG03]

# Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Lower bounds subsequently proven for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random $k$-CNFs [BG03]

Results always exactly matching width lower bounds
And proofs of very similar flavour...
Just a coincidence?

# Space vs. Width

**Theorem ([AD08])**

$$\textit{space} \geq \textit{width} + \mathcal{O}(1)$$

# Space vs. Width

**Theorem ([AD08])**

$$\textit{space} \geq \textit{width} + \mathcal{O}(1)$$

Width lower bound $\Rightarrow$ length **and space** lower bounds!

(Not a trivial claim, since space counts clauses)

# Space vs. Width

**Theorem ([AD08])**

$$\mathit{space} \geq \mathit{width} + \mathcal{O}(1)$$

Width lower bound $\Rightarrow$ length **and space** lower bounds!
(Not a trivial claim, since space counts clauses)

Are space and width asymptotically always the same? **No!**

# Space vs. Width

**Theorem ([AD08])**

$$\textit{space} \geq \textit{width} + \mathcal{O}(1)$$

Width lower bound $\Rightarrow$ length **and space** lower bounds!
(Not a trivial claim, since space counts clauses)

Are space and width asymptotically always the same? **No!**

**Pebbling formulas** [BN08]

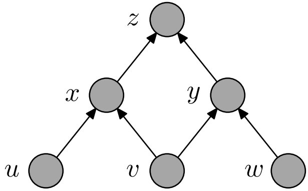
- Can be refuted in width $\mathcal{O}(1)$
- May require space $\Omega(N/\log N)$

A bit more involved to describe than previous benchmarks. . .

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

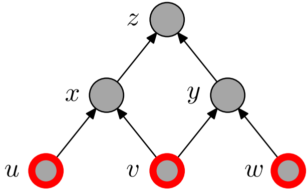



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

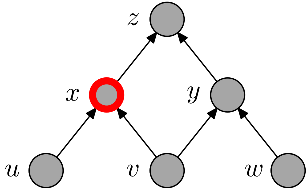



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

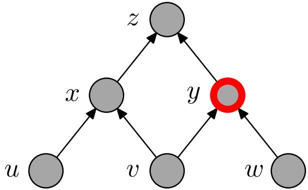



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

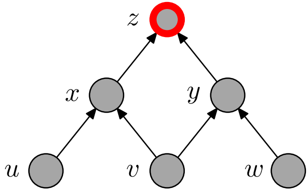



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

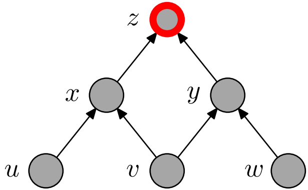



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

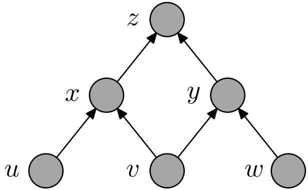



- sources are true
- truth propagates upwards
- but sink is false

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$




- sources are true
- truth propagates upwards
- but sink is false

Extensive literature on pebbling space and time-space trade-offs from 1970s and 80s

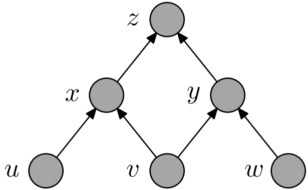
Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas.

# Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. $u$
2. $v$
3. $w$
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$




- sources are true
- truth propagates upwards
- but sink is false

Extensive literature on pebbling space and time-space trade-offs from 1970s and 80s

Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas. **Except...**

# Substituted Pebbling Formulas

Won't work — formulas are supereasy (solved by unit propagation)

Make formula harder by substituting $x_1 \oplus x_2$ for every variable $x$ (also works for other Boolean functions with "right" properties):

$$\overline{x} \lor y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \lor \overline{x}_2 \lor y_1 \lor y_2) \\ \land\ & (x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor y_1 \lor y_2) \\ \land\ & (\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2) \end{aligned}$$

Now CNF formula inherits pebbling graph properties!

# Trade-offs Between Complexity Measures?

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures simultaneously?

# Trade-offs Between Complexity Measures?

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures simultaneously?

- Space vs. width: **No!** [Ben09]

# Trade-offs Between Complexity Measures?

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures simultaneously?

- Space vs. width: **No!** [Ben09]
- Length vs. width: **No!** [Tha14]

# Trade-offs Between Complexity Measures?

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures simultaneously?

- Space vs. width: **No!** [Ben09]
- Length vs. width: **No!** [Tha14]
- Length vs. space: Arguably most interesting case
  Length $\approx$ running time
  Space $\approx$ memory consumption
  SAT solvers aggressively try to minimize both

# Length-Space Trade-offs

**Theorem ([BN11, BBI12, BNT13])**

*There are formulas for which*

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

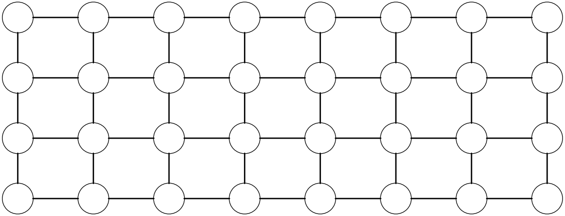
Holds for

- Substituted pebbling formulas on the right graphs
- Tseitin formulas on long, narrow rectangular grids

So no meaningful simultaneous optimization possible in worst case

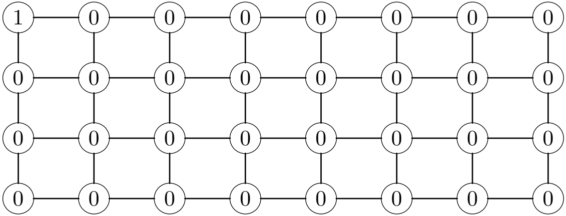
# A Closer Look at Tseitin Formulas on Long, Narrow Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$



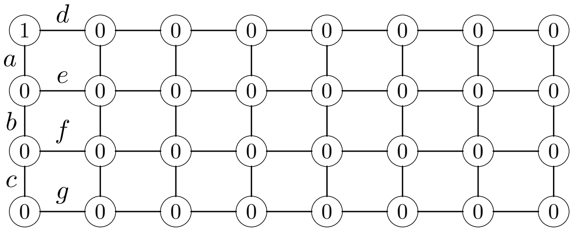
# A Closer Look at Tseitin Formulas on Long, Narrow Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd



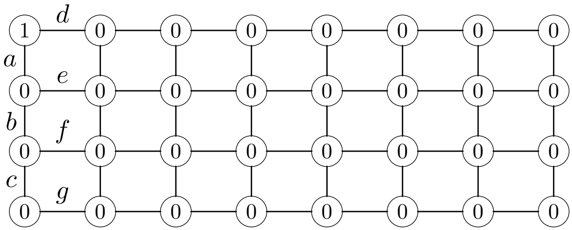
# A Closer Look at Tseitin Formulas on Long, Narrow Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges



# A Closer Look at Tseitin Formulas on Long, Narrow Grids

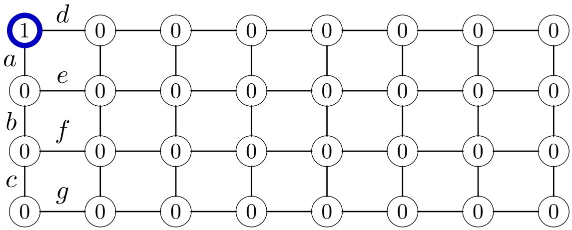
- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges
- We have clauses encoding constraints
  "vertex label = parity of incident edges"



# A Closer Look at Tseitin Formulas on Long, Narrow Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges
- We have clauses encoding constraints
  "vertex label = parity of incident edges"

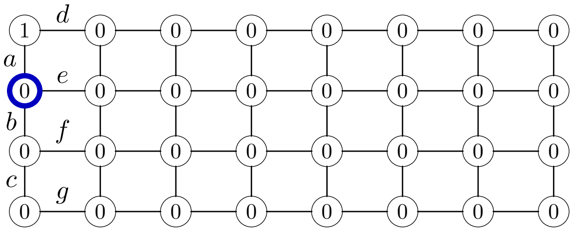
$$(a \lor d) \land (\overline{a} \lor \overline{d})$$



# A Closer Look at Tseitin Formulas on Long, Narrow Grids

- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges
- We have clauses encoding constraints
  "vertex label = parity of incident edges"

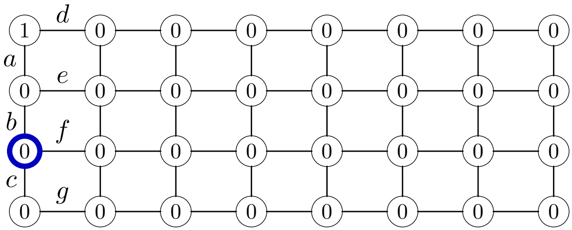
$$
\begin{aligned}
&(a \lor d) \\
\land\ &(\overline{a} \lor \overline{d}) \\
\land\ &(a \lor b \lor \overline{e}) \\
\land\ &(a \lor \overline{b} \lor e) \\
\land\ &(\overline{a} \lor b \lor e) \\
\land\ &(\overline{a} \lor \overline{b} \lor \overline{e})
\end{aligned}
$$



# A Closer Look at Tseitin Formulas on Long, Narrow Grids

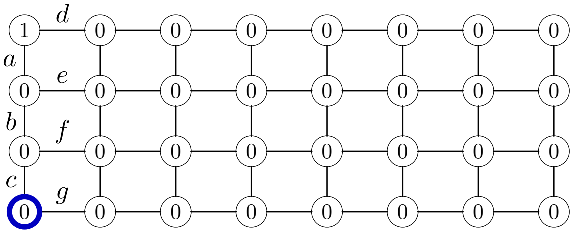
- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges
- We have clauses encoding constraints
  "vertex label = parity of incident edges"

$$
\begin{aligned}
& (a \lor d) \\
\land\ & (\overline{a} \lor \overline{d}) \\
\land\ & (a \lor b \lor \overline{e}) \\
\land\ & (a \lor \overline{b} \lor e) \\
\land\ & (\overline{a} \lor b \lor e) \\
\land\ & (\overline{a} \lor \overline{b} \lor \overline{e}) \\
\land\ & (b \lor c \lor \overline{f}) \\
\land\ & (b \lor \overline{c} \lor f) \\
\land\ & (\overline{b} \lor c \lor f) \\
\land\ & (\overline{b} \lor \overline{c} \lor \overline{f})
\end{aligned}
$$



# A Closer Look at Tseitin Formulas on Long, Narrow Grids

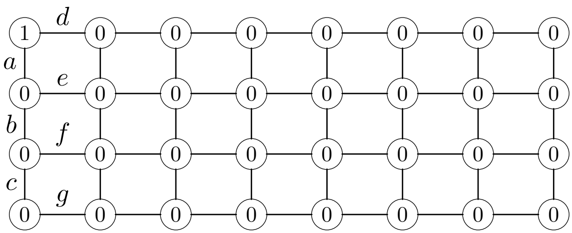
- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables $=$ edges
- We have clauses encoding constraints
  "vertex label $=$ parity of incident edges"




$$
\begin{aligned}
& (a \lor d) \\
\land\ & (\overline{a} \lor \overline{d}) \\
\land\ & (a \lor b \lor \overline{e}) \\
\land\ & (a \lor \overline{b} \lor e) \\
\land\ & (\overline{a} \lor b \lor e) \\
\land\ & (\overline{a} \lor \overline{b} \lor \overline{e}) \\
\land\ & (b \lor c \lor \overline{f}) \\
\land\ & (b \lor \overline{c} \lor f) \\
\land\ & (\overline{b} \lor c \lor f) \\
\land\ & (\overline{b} \lor \overline{c} \lor \overline{f}) \\
\land\ & (c \lor \overline{g}) \\
\land\ & (\overline{c} \lor g)
\end{aligned}
$$

# A Closer Look at Tseitin Formulas on Long, Narrow Grids

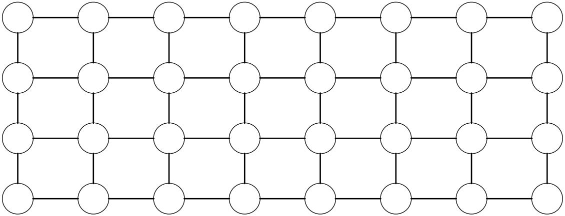
- Take $w \times m$ grid, $w = \mathcal{O}(\log m)$
- Label vertices $0/1$ with total charge odd
- Recall that variables = edges
- We have clauses encoding constraints "vertex label = parity of incident edges"
- Unsatisfiable — every edge counted twice, so total sum can't be odd




$$
\begin{aligned}
& (a \lor d) \\
\land\ & (\overline{a} \lor \overline{d}) \\
\land\ & (a \lor b \lor \overline{e}) \\
\land\ & (a \lor \overline{b} \lor e) \\
\land\ & (\overline{a} \lor b \lor e) \\
\land\ & (\overline{a} \lor \overline{b} \lor \overline{e}) \\
\land\ & (b \lor c \lor \overline{f}) \\
\land\ & (b \lor \overline{c} \lor f) \\
\land\ & (\overline{b} \lor c \lor f) \\
\land\ & (\overline{b} \lor \overline{c} \lor \overline{f}) \\
\land\ & (c \lor \overline{g}) \\
\land\ & (\overline{c} \lor g) \\
& \vdots
\end{aligned}
$$

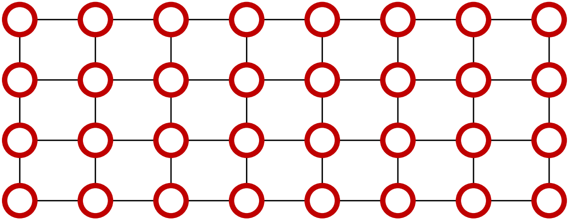
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges



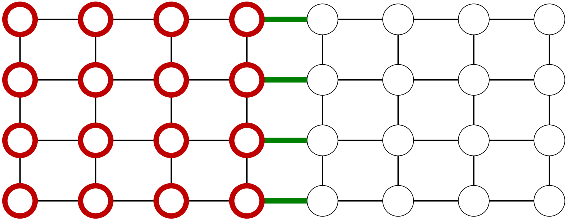
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component



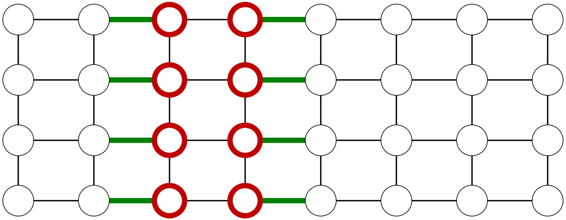
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse



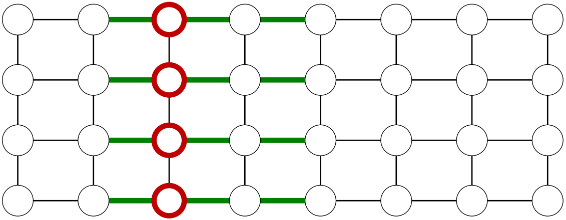
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse



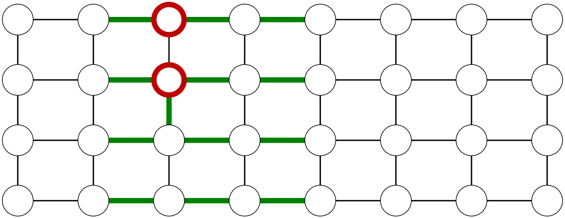
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse



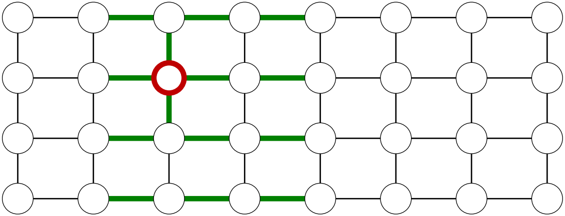
# Small-Space “Divide-and-Conquer” Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse



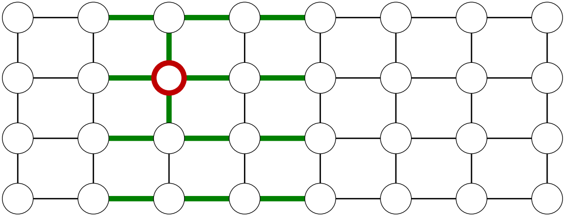
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse
- Violated vertex found after $w \log m$ queries



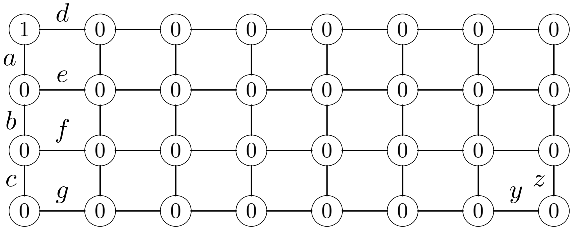
# Small-Space "Divide-and-Conquer" Proof

- Build DPLL search tree querying edges
- Identify odd-charge component
- Disconnect into two pieces by querying edges; then recurse
- Violated vertex found after $w \log m$ queries
- Height of tree = proof space = $w \log m$
  (very space-efficient, but proof size exponential in space)



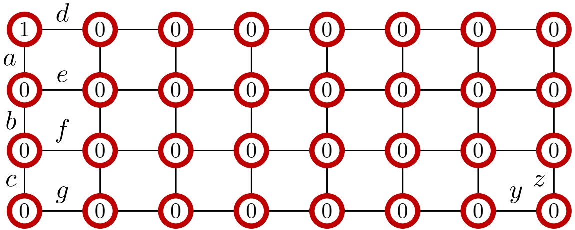
# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$



# Small-Size "Dynamic Programming" Proof

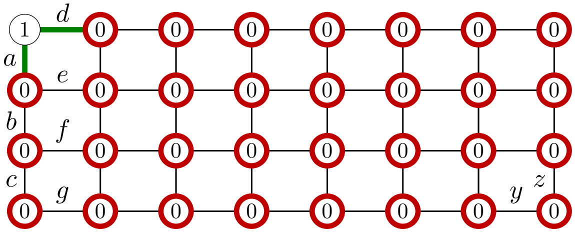
- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

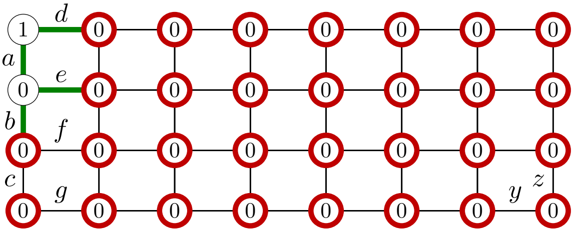
$a + d = 1$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

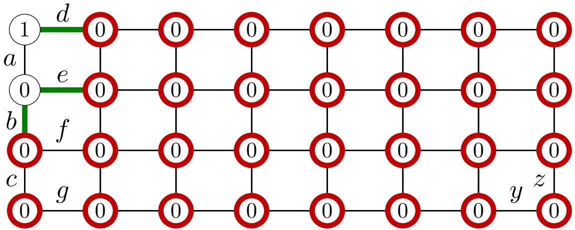
$$a + d = 1$$
$$a + b + e = 0$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

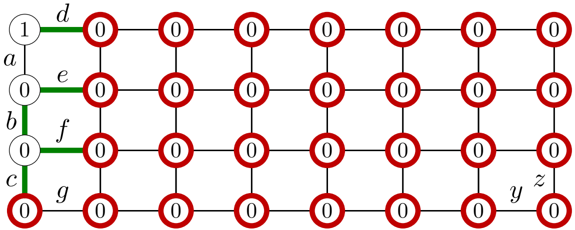
$$a + d = 1$$
$$a + b + e = 0$$
$$b + d + e = 1$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

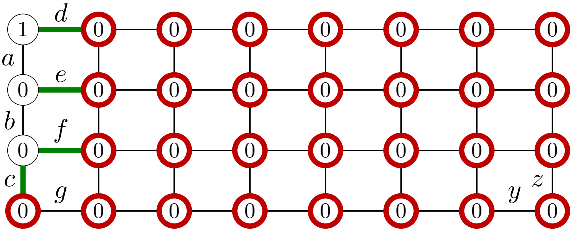
$$a + d = 1$$
$$a + b + e = 0$$
$$b + d + e = 1$$
$$b + c + f = 0$$



# Small-Size “Dynamic Programming” Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

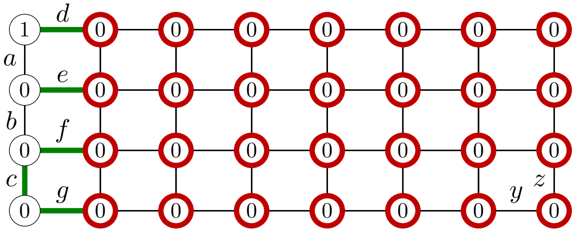
$$a+d=1$$
$$a+b+e=0$$
$$b+d+e=1$$
$$b+c+f=0$$
$$c+d+e+f=1$$



# Small-Size “Dynamic Programming” Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

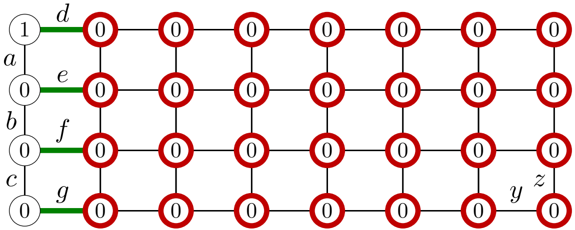
$$a+d=1$$
$$a+b+e=0$$
$$b+d+e=1$$
$$b+c+f=0$$
$$c+d+e+f=1$$
$$c+g=0$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

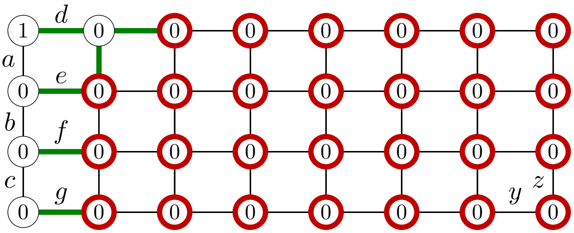
$$a + d = 1$$
$$a + b + e = 0$$
$$b + d + e = 1$$
$$b + c + f = 0$$
$$c + d + e + f = 1$$
$$c + g = 0$$
$$d + e + f + g = 1$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

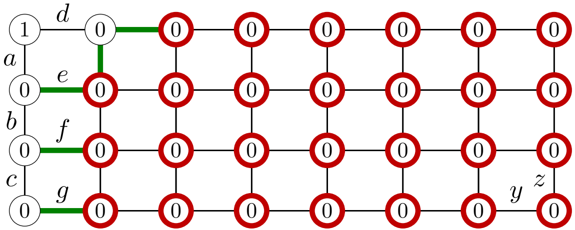
$$a + d = 1$$
$$a + b + e = 0$$
$$b + d + e = 1$$
$$b + c + f = 0$$
$$c + d + e + f = 1$$
$$c + g = 0$$
$$d + e + f + g = 1$$
$$\vdots$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow$ $2^w$ clauses

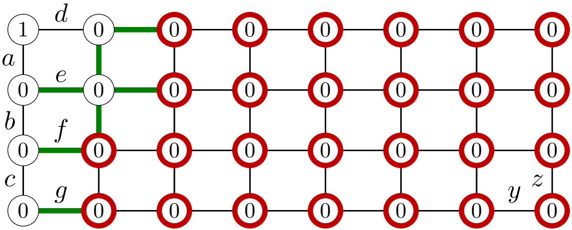
$$a + d = 1$$
$$a + b + e = 0$$
$$b + d + e = 1$$
$$b + c + f = 0$$
$$c + d + e + f = 1$$
$$c + g = 0$$
$$d + e + f + g = 1$$
$$\vdots$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

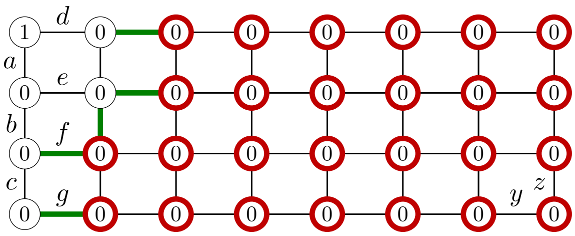
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

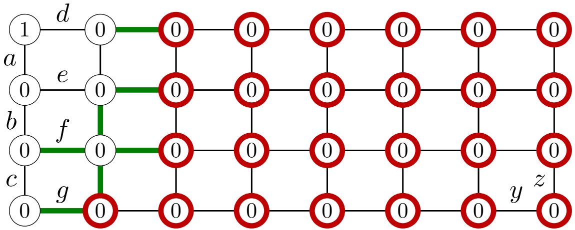
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

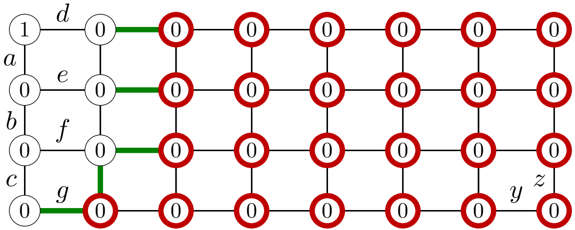
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

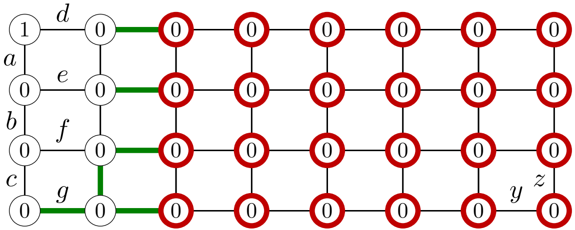
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

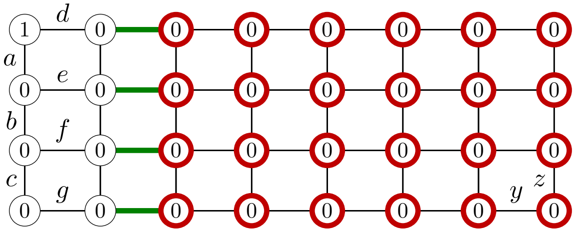
$$\begin{aligned} a+d &= 1 \\ a+b+e &= 0 \\ b+d+e &= 1 \\ b+c+f &= 0 \\ c+d+e+f &= 1 \\ c+g &= 0 \\ d+e+f+g &= 1 \\ &\vdots \end{aligned}$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow$ $2^w$ clauses

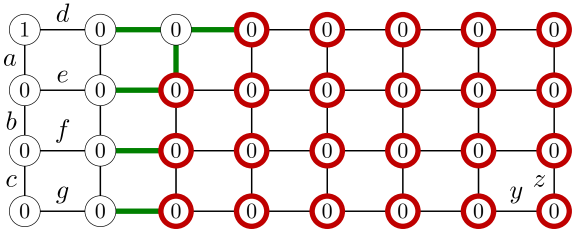
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

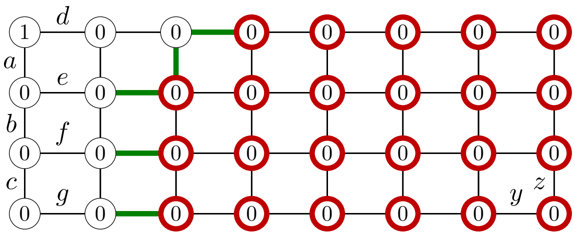
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

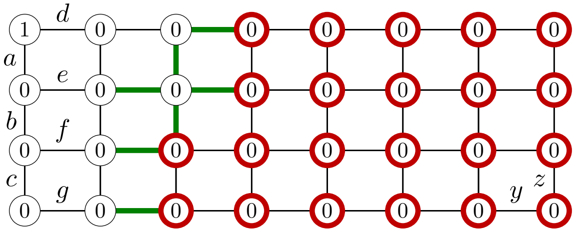
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

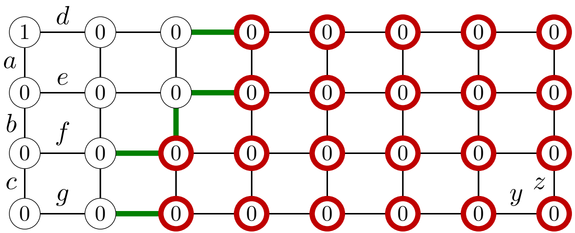
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow$ $2^w$ clauses

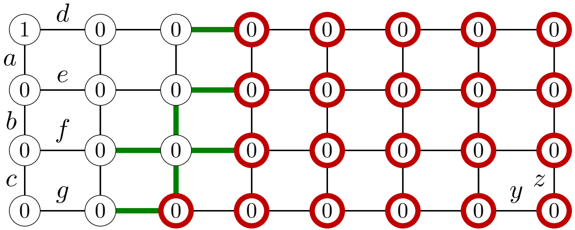
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

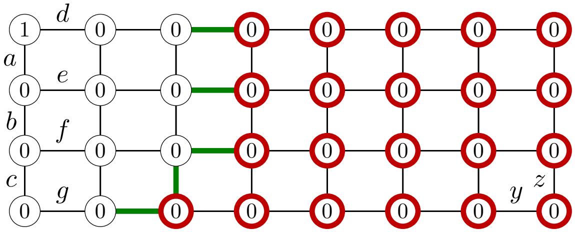
$$
\begin{aligned}
a + d &= 1\\
a + b + e &= 0\\
b + d + e &= 1\\
b + c + f &= 0\\
c + d + e + f &= 1\\
c + g &= 0\\
d + e + f + g &= 1\\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

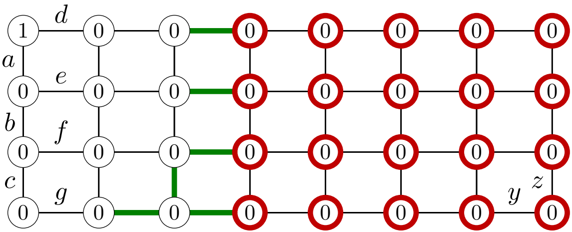
$$
\begin{aligned}
a+d&=1\\
a+b+e&=0\\
b+d+e&=1\\
b+c+f&=0\\
c+d+e+f&=1\\
c+g&=0\\
d+e+f+g&=1\\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses

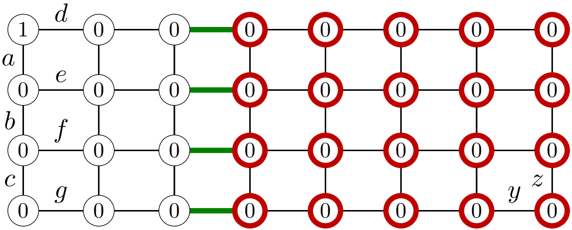
$$
\begin{aligned}
a+d&=1\\
a+b+e&=0\\
b+d+e&=1\\
b+c+f&=0\\
c+d+e+f&=1\\
c+g&=0\\
d+e+f+g&=1\\
&\vdots
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w + 1$ variables $\Rightarrow 2^w$ clauses

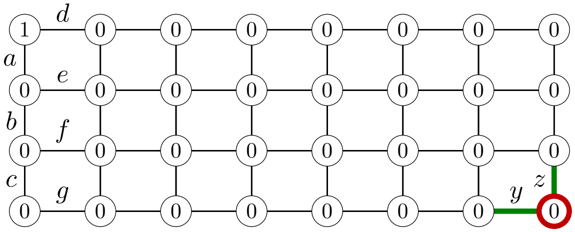
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots
\end{aligned}
$$



# Small-Size “Dynamic Programming” Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow$ $2^w$ clauses

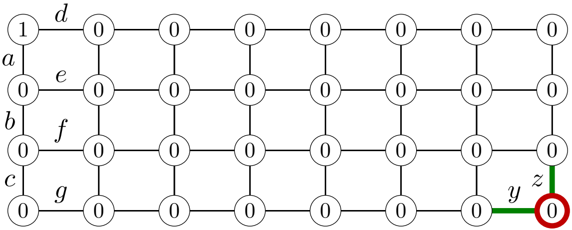
$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots \\
y + z &= 1
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\bmod 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow$ $2^w$ clauses

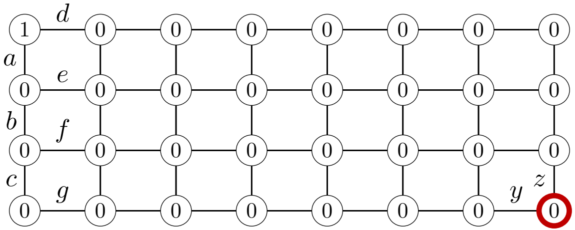
$$a+d=1$$
$$a+b+e=0$$
$$b+d+e=1$$
$$b+c+f=0$$
$$c+d+e+f=1$$
$$c+g=0$$
$$d+e+f+g=1$$
$$\vdots$$
$$y+z=1$$
$$y+z=0$$



# Small-Size “Dynamic Programming” Proof

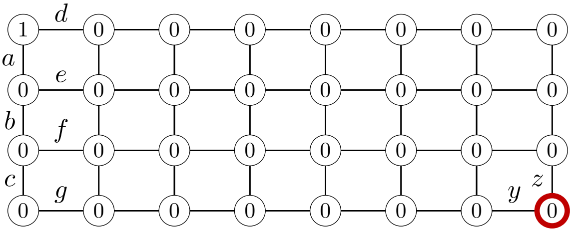
- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses
- Total of $mw$ summations

$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots \\
y + z &= 1 \\
y + z &= 0 \\
0 &= 1
\end{aligned}
$$



# Small-Size "Dynamic Programming" Proof

- View constraints as linear equations $\mathrm{mod}\ 2$
- Sum constraints vertex by vertex
- Can be done in resolution by completeness
  But parity of $w+1$ variables $\Rightarrow 2^w$ clauses
- Total of $mw$ summations
- Small proof size $\mathcal{O}\big(mw2^w\big) = poly(m)$
  However, space $\approx$ size — superlinear!




$$
\begin{aligned}
a + d &= 1 \\
a + b + e &= 0 \\
b + d + e &= 1 \\
b + c + f &= 0 \\
c + d + e + f &= 1 \\
c + g &= 0 \\
d + e + f + g &= 1 \\
&\vdots \\
y + z &= 1 \\
y + z &= 0 \\
0 &= 1
\end{aligned}
$$

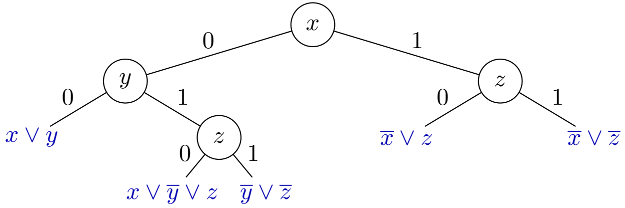
# Resolution-Based SAT Solvers

- Resolution used for SAT algorithms already in 1960s
- Basis of best modern SAT solvers still DPLL method [DP60, DLL62]
- Addition of conflict-driven clause learning (CDCL) [BS97, MS99] exponential increase in reasoning power
- Plus lots of smart engineering and heuristics to make it fly in practice [MMZ$^+$01]
- Today there are highly successful CDCL SAT solvers such as, e.g., MiniSat [ES04], Glucose [AS09], and Lingeling [Bie10]

# A Very Simplified Description of DPLL

Visualize execution of DPLL algorithm as search tree

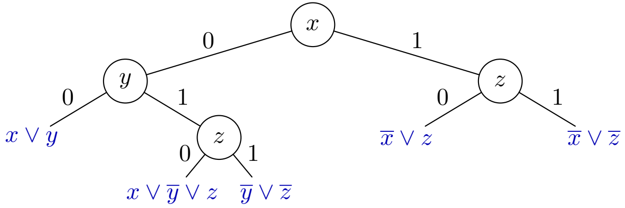
- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



# A Very Simplified Description of DPLL

Visualize execution of DPLL algorithm as search tree

- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



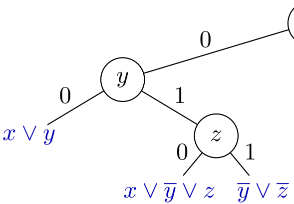

Many more ingredients in modern SAT solvers, for instance:

- Choice of branching variables crucial
- In leaf, compute & add reason for failure (clause learning)
- Restart every once in a while (but save computed info)

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

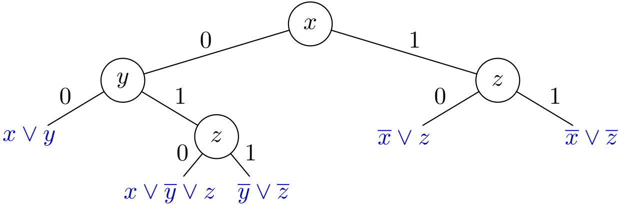
Look at our example again:



# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:

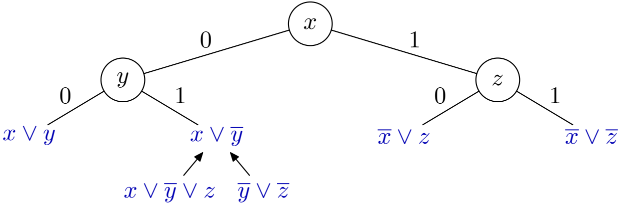



and apply resolution rule bottom-up

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:

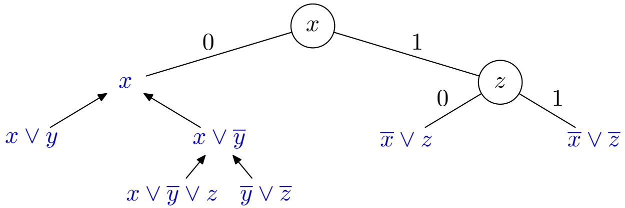



and apply resolution rule bottom-up

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:

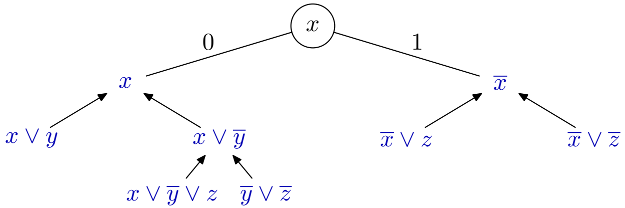



and apply resolution rule bottom-up

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:

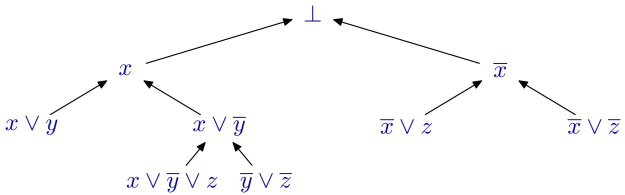



and apply resolution rule bottom-up

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:

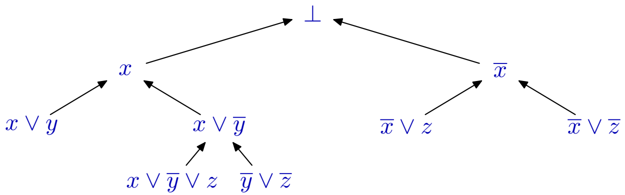



and apply resolution rule bottom-up

# DPLL and Resolution

A DPLL execution is essentially a resolution proof

Look at our example again:




and apply resolution rule bottom-up

Holds also for clause learning — makes tree into a DAG

# Complexity Measures for Resolution: Summary

Recall that $N =$ size of formula

**Length**

$\#$ clauses in refutation — at most $\exp(N)$

**Width**

Size of largest clause in refutation — at most $N$

**Space**

Max $\#$ clauses one needs to remember when "verifying correctness of refutation" — at most $N$ (!)

# Proof Complexity Measures and CDCL Proof Search

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

# Proof Complexity Measures and CDCL Proof Search

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

## Length

- Lower bound on running time for CDCL
- CDCL polynomially simulates resolution [PD11]
- But short proofs may be worst-case intractable to find [AR08]

# Proof Complexity Measures and CDCL Proof Search

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

**Length**

- Lower bound on running time for CDCL
- CDCL polynomially simulates resolution [PD11]
- But short proofs may be worst-case intractable to find [AR08]

**Width**

- Searching in small width known heuristic in AI community
- Small width $\Rightarrow$ CDCL solver will run fast [AFT11]

# Proof Complexity Measures and CDCL Proof Search

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

## Length

- Lower bound on running time for CDCL
- CDCL polynomially simulates resolution [PD11]
- But short proofs may be worst-case intractable to find [AR08]

## Width

- Searching in small width known heuristic in AI community
- Small width $\Rightarrow$ CDCL solver will run fast [AFT11]

## Space

- In practice, memory consumption important bottleneck
- Space complexity gives lower bound on clause database size
- Plus assumes solver knows **exactly** which clauses to keep $\Rightarrow$ in reality, probably (much) more memory needed

# Bridging the Gap Between Theory and Practice?

- CDCL hardness related to width and/or space?
  Preliminary work in [JMNŽ12] — no clear-cut answers
- Or is CDCL as good as general resolution?
  Are [PD11] and [AFT11] results "true in practice"? Doubt it
- CDCL explores only small part of resolution search space —
  Can time-space trade-offs in this talk occur in principle? Yes
- Do such time-space trade-offs occur in practice?
  Great question — on our to-do list

Not all mathematically well-defined questions. . .

Still possible to do experiments and draw interesting conclusions?

# Using Theoretical Benchmarks to Shed Light on CDCL?

CDCL performance on theory benchmarks can be surprising:

- Sometimes worse behaviour with heuristics than without
  Pigeonhole principle formulas [Hak85]
- Sometimes "easy" formulas harder than "hard" ones?!
  Zero-one designs [VS10, MN14]
- Sometimes minor changes in internals makes all the difference between supereasy and totally impossible
  Ordering principle formulas [Stå96, BG01]

# Using Theoretical Benchmarks to Shed Light on CDCL?

CDCL performance on theory benchmarks can be surprising:

- Sometimes worse behaviour with heuristics than without
  Pigeonhole principle formulas [Hak85]
- Sometimes "easy" formulas harder than "hard" ones?!
  Zero-one designs [VS10, MN14]
- Sometimes minor changes in internals makes all the difference between supereasy and totally impossible
  Ordering principle formulas [Stå96, BG01]

## Open Problems

- *Could explanations of above phenomena help us understand CDCL better?*
- *Could experiments on easily scalable theoretical benchmarks yield other interesting insights?*

# Polynomial Calculus

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
**Example:** $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

# Polynomial Calculus

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
**Example:** $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

## Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

**Goal:** Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

# Size, Degree and Space

Clauses turn into monomials

Write out all polynomials as sums of monomials

W.l.o.g. all polynomials multilinear (because of Boolean axioms)

# Size, Degree and Space

Clauses turn into monomials
Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

**Size** — analogue of resolution length
total # monomials in refutation counted with repetitions

(Also possible to define length measure — but can be much smaller since polynomials might be of exponential size)

**Degree** — analogue of resolution width
largest degree of monomial in refutation

**(Monomial) space** — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

# Polynomial Calculus Simulates Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

# Polynomial Calculus Simulates Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

**Example:** Resolution step:

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

# Polynomial Calculus Simulates Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

**Example:** Resolution step:

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation:

$$\cfrac{x\overline{y}z = 0 \qquad \cfrac{\cfrac{\overline{y}\,\overline{z}=0}{x\overline{y}\,\overline{z}=0} \qquad \cfrac{\cfrac{z+\overline{z}-1=0}{\overline{y}z+\overline{y}\,\overline{z}-\overline{y}=0}}{x\overline{y}z+x\overline{y}\,\overline{z}-x\overline{y}=0}}{-x\overline{y}z+x\overline{y}=0}}{x\overline{y}=0}$$

# Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas on expanders (just do Gaussian elimination)
- Onto functional pigeonhole principle [Rii93]

# Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas on expanders (just do Gaussian elimination)
- Onto functional pigeonhole principle [Rii93]

**Open Problem**

*Show that polynomial calculus is strictly stronger than resolution w.r.t. space*

# Size vs. Degree

- Degree upper bound $\Rightarrow$ size upper bound [CEI96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [ALN14]

- Degree lower bound $\Rightarrow$ size lower bound [IPS99]
  Precursor of [BW01] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [GL10]
  Example: same ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery much less developed)

# Examples of Hard Formulas w.r.t. Size (and Degree)

**Pigeonhole principle formulas**

Follows from [AR03]

Earlier work on other encodings in [Raz98, IPS99]

Hard even with functionality axioms added [MN15]

**Tseitin formulas with "wrong modulus"**

Can define Tseitin-like formulas counting $\bmod p$ for $p \neq 2$

Hard if $p \neq$ characteristic of field [BGIP01]

**Random $k$-CNF formulas**

Hard in all characteristics except $2$ [BI99]

Lower bound for all characteristics in [AR03]

# Bounds on Polynomial Calculus Space

Lower bound for PHP with wide clauses [ABRW02]

$k$-CNFs much trickier — sequence of lower bounds for

- Obfuscated $4$-CNF versions of PHP [FLN$^{+}$12]
- Random $4$-CNFs [BG13]
- Tseitin formulas on (some) $4$-regular expanders [FLM$^{+}$13]
- Random $3$-CNFs [BBG$^{+}$15]

# Bounds on Polynomial Calculus Space

Lower bound for PHP with wide clauses [ABRW02]

$k$-CNFs much trickier — sequence of lower bounds for

- Obfuscated $4$-CNF versions of PHP [FLN$^+$12]
- Random $4$-CNFs [BG13]
- Tseitin formulas on (some) $4$-regular expanders [FLM$^+$13]
- Random $3$-CNFs [BBG$^+$15]

## Open Problems

*Prove polynomial calculus space lower bounds on*

- *Tseitin formulas on arbitrary $d$-regular expanders for $d \geq 3$*
- *$3$-CNF version of PHP formulas*

# Space vs. Degree

**Open Problem (analogue of [AD08])**

*Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?*

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM$^+$13]

# Space vs. Degree

## Open Problem (analogue of [AD08])

*Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?*

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM$^+$13]

Optimal separation of space and degree in [FLM$^+$13] using flavour of Tseitin formulas which

- can be refuted in degree $\mathcal{O}(1)$
- require space $\Omega(N)$
- but separating formulas depend on characteristic of field

## Open Problem

*Prove space lower bounds for substituted pebbling formulas (would give space-degree separation independent of characteristic)*

# Trade-offs for Polynomial Calculus

- Strong, essentially optimal space-degree trade-offs [BNT13]
  Same formulas as for resolution — same parameters

- Strong size-space trade-offs [BNT13]
  Same formulas as for resolution — some loss in parameters

**Open Problem**

*Are there size-degree trade-offs in polynomial calculus?*

[Tha14] works only for resolution (so far)

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Fail to solve even FPHP formulas (though this can be fixed)

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Fail to solve even FPHP formulas (though this can be fixed)
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?

# Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Fail to solve even FPHP formulas (though this can be fixed)
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work

# Cutting Planes

Introduced in [CCT87] based on integer LP in [Gom63, Chv73]

Clauses interpreted as linear inequalities over the reals with integer coefficients

**Example:** $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$
(Now $1 \equiv \mathit{true}$ and $0 \equiv \mathit{false}$ again)

# Cutting Planes

Introduced in [CCT87] based on integer LP in [Gom63, Chv73]

Clauses interpreted as linear inequalities over the reals with integer coefficients

**Example:** $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$
(Now $1 \equiv \mathit{true}$ and $0 \equiv \mathit{false}$ again)

## Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

**Goal:** Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

# Size, Length and Space

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Space** = max # lines in memory during refutation

No (useful) analogue of width/degree

# Size, Length and Space

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Space** = max # lines in memory during refutation

No (useful) analogue of width/degree

Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously

# Size, Length and Space

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Space** = max # lines in memory during refutation

No (useful) analogue of width/degree

Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]

# Size, Length and Space

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Space** = max # lines in memory during refutation

No (useful) analogue of width/degree

Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]
- is strictly stronger w.r.t. space — can refute any CNF in constant space $5$ (!) [GPT15]

# Size, Length and Space

**Length** = total # lines/inequalities in refutation

**Size** = sum also size of coefficients

**Space** = max # lines in memory during refutation

No (useful) analogue of width/degree

Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]
- is strictly stronger w.r.t. space — can refute any CNF in constant space $5$ (!) [GPT15] (But coefficients will be exponentially large — what if also coefficient size counted?)

# Hard Formulas w.r.t Cutting Planes Length

**Clique-coclique formulas** [Pud97]
"A graph with an $m$-clique is not $(m-1)$-colourable"

$p_{i,j}$ = indicator variables for edges in an $n$-vertex graph
$q_{k,i}$ = identifiers for members of $m$-clique in graph
$r_{i,\ell}$ = encoding of legal $(m-1)$-colouring of vertices

| | |
|---|---|
| $q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$ | some vertex is $k$th member of clique |
| $\overline{q}_{k,i} \lor \overline{q}_{k,j}$ | $k$th clique member is uniquely defined |
| $p_{i,j} \lor \overline{q}_{k,i} \lor \overline{q}_{k',j}$ | clique members are connected by edges |
| $r_{i,1} \lor r_{i,2} \lor \cdots \lor r_{i,m-1}$ | every vertex $i$ has a colour |
| $\overline{p}_{i,j} \lor \overline{r}_{i,\ell} \lor \overline{r}_{j,\ell}$ | neighbours have distinct colours |

Exponential lower bound via interpolation and circuit complexity
Technique very specifically tied to structure of formula

# Open Problems for Cutting Planes Length and Space

## Open Problems

*Prove length lower bounds for cutting planes*

- *for Tseitin formulas*
- *for random $k$-CNFs*
- *for any formula using other technique than interpolation*

# Open Problems for Cutting Planes Length and Space

## Open Problems

*Prove length lower bounds for cutting planes*

- *for Tseitin formulas*
- *for random $k$-CNFs*
- *for any formula using other technique than interpolation*

## Open Problems

*Prove space lower bounds for cutting planes*

- *with constant-size coefficients (very weak bounds in [GPT15])*
- *with polynomial-size coefficients (nothing known)*

# Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of (lifted) Tseitin formulas on expanders need large space [GP14] (but probably don't exist)
- Short cutting planes refutations of (some) pebbling formulas need large space [HN12, GP14] (and such refutations exist)

Results obtained via communication complexity

# Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of (lifted) Tseitin formulas on expanders need large space [GP14] (but probably don't exist)
- Short cutting planes refutations of (some) pebbling formulas need large space [HN12, GP14] (and such refutations exist)

Results obtained via communication complexity

By [GPT15] get trade-offs with "constant space" upper bounds (but with coefficients of exponential size)

Doesn't seem like a too relevant a trade-off — exponential size coefficients doesn't feel like "small space"

# Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of (lifted) Tseitin formulas on expanders need large space [GP14] (but probably don't exist)
- Short cutting planes refutations of (some) pebbling formulas need large space [HN12, GP14] (and such refutations exist)

Results obtained via communication complexity

By [GPT15] get trade-offs with "constant space" upper bounds (but with coefficients of exponential size)

Doesn't seem like a too relevant a trade-off — exponential size coefficients doesn't feel like "small space"

### Open Problem

*Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)*

# Size-Space Trade-offs for Cutting Planes!

**Breaking news:** Yes, there are such trade-offs!

### Theorem ([dRNV16])

*There exist flavours of pebbling formulas such that*

- $\exists$ *small-size refutations with constant-size coefficients*
- $\exists$ *small-space refutations with constant-size coefficients*
- *Decreasing the space even for refutations with exponentially large coefficients causes exponential blow-up of length*

# Size-Space Trade-offs for Cutting Planes!

**Breaking news:** Yes, there are such trade-offs!

**Theorem ([dRNV16])**

*There exist flavours of pebbling formulas such that*

- $\exists$ *small-size refutations with constant-size coefficients*
- $\exists$ *small-space refutations with constant-size coefficients*
- *Decreasing the space even for refutations with exponentially large coefficients causes exponential blow-up of length*

- Results hold uniformly for resolution, polynomial calculus (regardless of field) and cutting planes
- Again uses communication complexity (+ several other twists)
- Downside: Parameters worse than in previous results

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning
- Seems hard to make competitive with CDCL

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning
- Seems hard to make competitive with CDCL
- **Roadblock 1:** Given CNF input, solvers cannot discover and use cardinality constraints (too limited form of addition)

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning
- Seems hard to make competitive with CDCL
- **Roadblock 1:** Given CNF input, solvers cannot discover and use cardinality constraints (too limited form of addition)
- But given helpful encoding, solvers can do really well (e.g., PHP formulas and zero-one designs) [BLLM14]

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning
- Seems hard to make competitive with CDCL
- **Roadblock 1:** Given CNF input, solvers cannot discover and use cardinality constraints (too limited form of addition)
- But given helpful encoding, solvers can do really well (e.g., PHP formulas and zero-one designs) [BLLM14]
- **Roadblock 2(?):** Solvers seem inefficient for systems of inequalities that have rational but not integral solutions (too limited form of division?)

# Geometric SAT Solvers?

- There are some so-called pseudo-Boolean solvers using (subset of) cutting planes reasoning
- Seems hard to make competitive with CDCL
- **Roadblock 1:** Given CNF input, solvers cannot discover and use cardinality constraints (too limited form of addition)
- But given helpful encoding, solvers can do really well (e.g., PHP formulas and zero-one designs) [BLLM14]
- **Roadblock 2(?):** Solvers seem inefficient for systems of inequalities that have rational but not integral solutions (too limited form of division?)
- Not well understood at all — work in progress

# Building SAT Solvers on Extended Resolution?

- Resolution + introduce new variables to name subformulas
- Without restrictions, corresponds to extended Frege system
- Extremely strong — pretty much no lower bounds known
- In order to study extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

# Summing up This Presentation

Overview of resolution, polynomial calculus and cutting planes
(More details in survey papers [Nor13, Nor15])

- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all

## Summing up This Presentation

Overview of resolution, polynomial calculus and cutting planes
(More details in survey papers [Nor13, Nor15])

- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all

Open problems motivated by applied SAT solving

- Can proof complexity measures shed more light on the hardness (or easiness) of SAT?
- Is it possible to build efficient SAT solvers based on stronger proof systems than resolution?

## Summing up This Presentation

Overview of resolution, polynomial calculus and cutting planes
(More details in survey papers [Nor13, Nor15])

- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all

Open problems motivated by applied SAT solving

- Can proof complexity measures shed more light on the hardness (or easiness) of SAT?
- Is it possible to build efficient SAT solvers based on stronger proof systems than resolution?

Thank you for your attention!

# References I


[ABRW02] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, 2002. Preliminary version in *STOC '00*.

[AD08] Albert Atserias and Víctor Dalmau. A combinatorial characterization of resolution width. *Journal of Computer and System Sciences*, 74(3):323–334, May 2008. Preliminary version in *CCC '03*.

[AFT11] Albert Atserias, Johannes Klaus Fichte, and Marc Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. *Journal of Artificial Intelligence Research*, 40:353–373, January 2011. Preliminary version in *SAT '09*.

[ALN14] Albert Atserias, Massimo Lauria, and Jakob Nordström. Narrow proofs may be maximally long. In *Proceedings of the 29th Annual IEEE Conference on Computational Complexity (CCC '14)*, pages 286–297, June 2014.

# References II


[AR03] Michael Alekhnovich and Alexander A. Razborov. Lower bounds for polynomial calculus: Non-binomial case. *Proceedings of the Steklov Institute of Mathematics*, 242:18–35, 2003. Available at http://people.cs.uchicago.edu/~razborov/files/misha.pdf. Preliminary version in *FOCS '01*.

[AR08] Michael Alekhnovich and Alexander A. Razborov. Resolution is not automatizable unless W[P] is tractable. *SIAM Journal on Computing*, 38(4):1347–1363, October 2008. Preliminary version in *FOCS '01*.

[AS09] Gilles Audemard and Laurent Simon. Predicting learnt clauses quality in modern SAT solvers. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI '09)*, pages 399–404, July 2009.

[BBG+15] Patrick Bennett, Ilario Bonacina, Nicola Galesi, Tony Huynh, Mike Molloy, and Paul Wollan. Space proof complexity for random 3-CNFs. Technical Report 1503.01613, arXiv.org, April 2015.

# References III


[BBI12] Paul Beame, Chris Beck, and Russell Impagliazzo. Time-space tradeoffs in resolution: Superpolynomial lower bounds for superlinear space. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 213–232, May 2012.

[BCMM05] Paul Beame, Joseph C. Culberson, David G. Mitchell, and Cristopher Moore. The resolution complexity of random graph $k$-colorability. *Discrete Applied Mathematics*, 153(1-3):25–47, December 2005.

[Ben09] Eli Ben-Sasson. Size-space tradeoffs for resolution. *SIAM Journal on Computing*, 38(6):2511–2525, May 2009. Preliminary version in *STOC '02*.

[BG01] María Luisa Bonet and Nicola Galesi. Optimality of size-width tradeoffs for resolution. *Computational Complexity*, 10(4):261–276, December 2001. Preliminary version in *FOCS '99*.

[BG03] Eli Ben-Sasson and Nicola Galesi. Space complexity of random formulae in resolution. *Random Structures and Algorithms*, 23(1):92–109, August 2003. Preliminary version in *CCC '01*.

# References IV


[BG13] Ilario Bonacina and Nicola Galesi. Pseudo-partitions, transversality and locality: A combinatorial characterization for the space measure in algebraic proof systems. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science (ITCS '13)*, pages 455–472, January 2013.

[BGIP01] Samuel R. Buss, Dima Grigoriev, Russell Impagliazzo, and Toniann Pitassi. Linear gaps between degrees for the polynomial calculus modulo distinct primes. *Journal of Computer and System Sciences*, 62(2):267–289, March 2001. Preliminary version in *CCC '99*.

[BI99] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. In *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science (FOCS '99)*, pages 415–421, October 1999. Journal version in [BI10].

[BI10] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. *Computational Complexity*, 19:501–519, 2010. Preliminary version in *FOCS '99*.

# References V


[Bie10] Armin Biere. Lingeling, Plingeling, PicoSAT and PrecoSAT at SAT Race 2010. Technical Report 10/1, FMV Reports Series, Institute for Formal Models and Verification, Johannes Kepler University, August 2010.

[BIS07] Paul Beame, Russell Impagliazzo, and Ashish Sabharwal. The resolution complexity of independent sets and vertex covers in random graphs. *Computational Complexity*, 16(3):245–297, October 2007.

[BLLM14] Armin Biere, Daniel Le Berre, Emmanuel Lonca, and Norbert Manthey. Detecting cardinality constraints in CNF. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 285–301. Springer, July 2014.

[BN08] Eli Ben-Sasson and Jakob Nordström. Short proofs may be spacious: An optimal separation of space and length in resolution. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08)*, pages 709–718, October 2008.

# References VI

[BN11] Eli Ben-Sasson and Jakob Nordström. Understanding space in proof complexity: Separations and trade-offs via substitutions. In *Proceedings of the 2nd Symposium on Innovations in Computer Science (ICS '11)*, pages 401–416, January 2011.

[BNT13] Chris Beck, Jakob Nordström, and Bangsheng Tang. Some trade-off results for polynomial calculus. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC '13)*, pages 813–822, May 2013.

[BS97] Roberto J. Bayardo Jr. and Robert Schrag. Using CSP look-back techniques to solve real-world SAT instances. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI '97)*, pages 203–208, July 1997.

[BW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, March 2001. Preliminary version in *STOC '99*.

# References VII


[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[CEI96] Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing (STOC '96)*, pages 174–183, May 1996.

[Chv73] Vašek Chvátal. Edmond polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4(1):305–337, 1973.

[CS88] Vašek Chvátal and Endre Szemerédi. Many hard examples for resolution. *Journal of the ACM*, 35(4):759–768, October 1988.

[DLL62] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem proving. *Communications of the ACM*, 5(7):394–397, July 1962.

[DP60] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7(3):201–215, 1960.

# References VIII


[dRNV16] Susanna F. de Rezende, Jakob Nordström, and Marc Vinyals. How limited interaction hinders real communication (and what it means for proof and circuit complexity). In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS '16)*, pages 466–485, October 2016. To appear.

[ES04] Niklas Eén and Niklas Sörensson. An extensible SAT-solver. In *6th International Conference on Theory and Applications of Satisfiability Testing (SAT '03), Selected Revised Papers*, volume 2919 of *Lecture Notes in Computer Science*, pages 502–518. Springer, 2004.

[ET01] Juan Luis Esteban and Jacobo Torán. Space bounds for resolution. *Information and Computation*, 171(1):84–97, 2001. Preliminary versions of these results appeared in *STACS '99* and *CSL '99*.

[FLM+13] Yuval Filmus, Massimo Lauria, Mladen Mikša, Jakob Nordström, and Marc Vinyals. Towards an understanding of polynomial calculus: New separations and lower bounds (Extended abstract). In *Proceedings of the 40th International Colloquium on Automata, Languages and Programming (ICALP '13)*, volume 7965 of *Lecture Notes in Computer Science*, pages 437–448. Springer, July 2013.

# References IX


[FLN$^+$12] Yuval Filmus, Massimo Lauria, Jakob Nordström, Neil Thapen, and Noga Ron-Zewi. Space complexity in polynomial calculus (Extended abstract). In *Proceedings of the 27th Annual IEEE Conference on Computational Complexity (CCC '12)*, pages 334–344, June 2012.

[GL10] Nicola Galesi and Massimo Lauria. Optimality of size-degree trade-offs for polynomial calculus. *ACM Transactions on Computational Logic*, 12:4:1–4:22, November 2010.

[Gom63] Ralph E. Gomory. An algorithm for integer solutions of linear programs. In R.L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, New York, 1963.

[GP14] Mika Göös and Toniann Pitassi. Communication lower bounds via critical block sensitivity. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing (STOC '14)*, pages 847–856, May 2014.

# References X


[GPT15] Nicola Galesi, Pavel Pudlák, and Neil Thapen. The space complexity of cutting planes refutations. In *Proceedings of the 30th Annual Computational Complexity Conference (CCC '15)*, volume 33 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 433–447, June 2015.

[Hak85] Armin Haken. The intractability of resolution. *Theoretical Computer Science*, 39(2-3):297–308, August 1985.

[HN12] Trinh Huynh and Jakob Nordström. On the virtue of succinct proofs: Amplifying communication complexity hardness to time-space trade-offs in proof complexity (Extended abstract). In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC '12)*, pages 233–248, May 2012.

[IPS99] Russell Impagliazzo, Pavel Pudlák, and Jiří Sgall. Lower bounds for the polynomial calculus and the Gröbner basis algorithm. *Computational Complexity*, 8(2):127–144, 1999.

# References XI

[JMNŽ12] Matti Järvisalo, Arie Matsliah, Jakob Nordström, and Stanislav Živný. Relating proof complexity measures and practical hardness of SAT. In *Proceedings of the 18th International Conference on Principles and Practice of Constraint Programming (CP '12)*, volume 7514 of *Lecture Notes in Computer Science*, pages 316–331. Springer, October 2012.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th Design Automation Conference (DAC '01)*, pages 530–535, June 2001.

[MN14] Mladen Mikša and Jakob Nordström. Long proofs of (seemingly) simple formulas. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 121–137. Springer, July 2014.

[MN15] Mladen Mikša and Jakob Nordström. A generalized method for proving polynomial calculus degree lower bounds. In *Proceedings of the 30th Annual Computational Complexity Conference (CCC '15)*, volume 33 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 467–487, June 2015.

# References XII


[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, May 1999. Preliminary version in *ICCAD '96*.

[Nor13] Jakob Nordström. Pebble games, proof complexity and time-space trade-offs. *Logical Methods in Computer Science*, 9:15:1–15:63, September 2013.

[Nor15] Jakob Nordström. On the interplay between proof complexity and SAT solving. *ACM SIGLOG News*, 2(3):19–44, July 2015.

[PD11] Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artificial Intelligence*, 175:512–525, February 2011. Preliminary version in *CP '09*.

[Pud97] Pavel Pudlák. Lower bounds for resolution and cutting plane proofs and monotone computations. *Journal of Symbolic Logic*, 62(3):981–998, September 1997.

[Raz98] Alexander A. Razborov. Lower bounds for the polynomial calculus. *Computational Complexity*, 7(4):291–324, December 1998.

# References XIII


[Rii93] Søren Riis. *Independence in Bounded Arithmetic*. PhD thesis, University of Oxford, 1993.

[Spe10] Ivor Spence. sgen1: A generator of small but difficult satisfiability benchmarks. *Journal of Experimental Algorithmics*, 15:1.2:1.1–1.2:1.15, March 2010.

[Stå96] Gunnar Stålmarck. Short resolution proofs for a sequence of tricky formulas. *Acta Informatica*, 33(3):277–280, May 1996.

[Tha14] Neil Thapen. A trade-off between length and width in resolution. Technical Report TR14-137, Electronic Colloquium on Computational Complexity (ECCC), October 2014.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, January 1987.

[VS10] Allen Van Gelder and Ivor Spence. Zero-one designs produce small hard SAT instances. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 388–397. Springer, July 2010.