Supercritical Space-Width Trade-offs for Resolution

Jakob Nordström

KTH Royal Institute of Technology

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Joint work with Christoph Berkholz

Proof Complexity

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

Input: Unsatisfiable formula in conjunctive normal form (CNF)
Output: Polynomial-time verifiable certificate of unsatisfiability

Proof of unsatifiability = **refutation** of formula

Want to measure efficiency of proof system in terms of different complexity measures (size, space, et cetera)

Can be viewed as proving upper and lower bounds for weak nondeterministic models of computation

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- Done when empty clause $\perp$ derived

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Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

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6.	$x \lor \overline{y}$	Res$(2, 4)$
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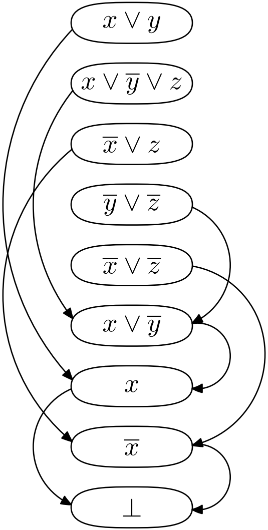

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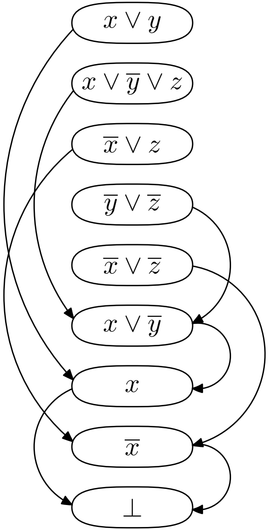
Tree-like resolution if DAG is tree



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Size should strictly speaking measure # symbols
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Set size = length

Width of proof = # literals in largest clause (3 in our example)
Width of refuting F = min width over all proofs for F

Width at most linear, so here obviously care about linear factors

Resolution Space

Space = amount of memory needed when performing refutation

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Clause space at step t: # clauses at steps $\leq t$ used at steps $\geq t$
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Example: Clause space at step 7

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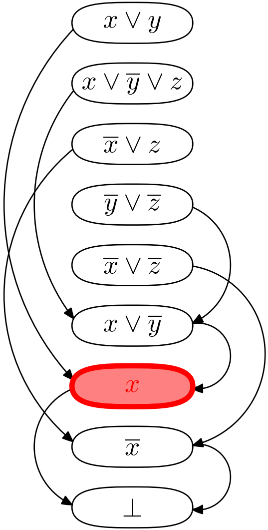

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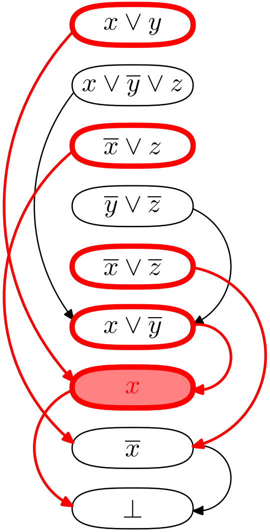

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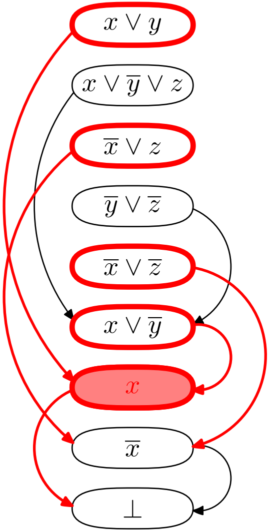

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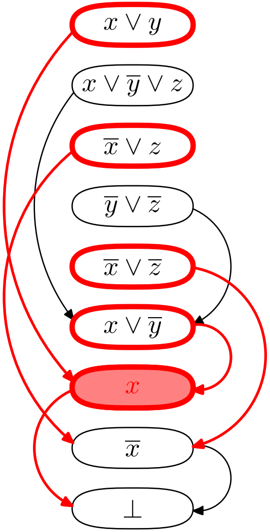
Space of proof = max over all steps
Space of refuting F = min over all proofs



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This talk: focus on width and clause space
But results translate to total space by:

$$\text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width}$$

Lower Bounds via Resolution Width

For n-variable k-CNFs (k constant) it holds that:

width $\leq$ Ω(clause space) [Atserias & Dalmau '03]

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- Can have width $\Theta(\sqrt{n})$ and still size $\mathrm{poly}(n)$
  [Bonet & Galesi '99]
- Can have width $\mathcal{O}(1)$ and still clause space $\Omega(n/\log n)$
  [Ben-Sasson & Nordström '08]

Upper Bounds via Resolution Width

$$\text{size} \quad \leq \quad n^{\mathcal{O}(\text{width})}$$

Upper Bounds via Resolution Width

$$\begin{aligned} \text{size} &\leq n^{\mathcal{O}(\text{width})} \\ \text{time to find refutation} &\leq n^{\mathcal{O}(\text{width})} \end{aligned}$$

for $w \leftarrow 3 \ldots n$ **do**
    Resolve all clauses & keep resolvents with at most w literals
    If $\perp$ has been derived, then output `UNSAT`
end for
Output `SAT`

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Algorithm (and resolution proof) requires time/size $n^{\mathcal{O}(\text{width})}$
Cannot do better in general [Atserias, Lauria, & Nordström '14]
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Recall: can always do clause space $\mathcal{O}(n)$

A Supercritical Space-Width Tradeoff

Theorem

For any $\varepsilon > 0$ *and* $6 \leq w \leq n^{\frac{1}{2}-\varepsilon}$ *exist* n*-variable* CNF*s* F_n *s.t.*

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Use hardness condensation approach in [Razborov '16]:

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Key components:

- Expander graphs
- XORification (substitution with exclusive or)

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- We feel “supercritical” is more descriptive

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Very well-connected so-called expander graphs play leading role in many proof complexity lower bounds

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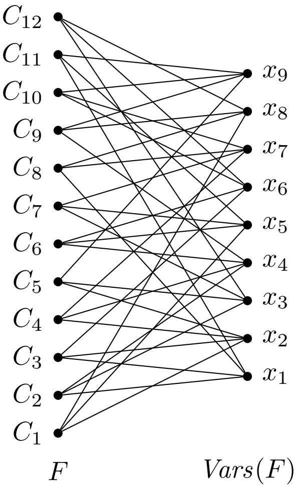

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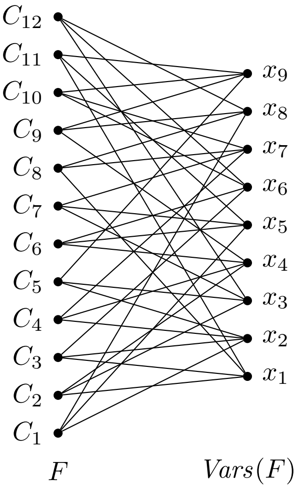

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- width, size, and space in resolution
  [Ben-Sasson & Wigderson '99, Ben-Sasson & Galesi '03]
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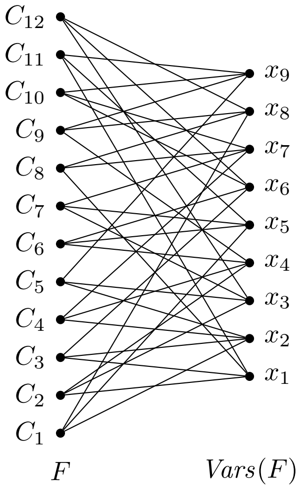

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Can also define more general graphs that capture "underlying combinatorial structure" and extend results [Mikša & Nordström '15]

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- # vars in memory $\geq s$ for $F \Longrightarrow$ clause space $\geq \Omega(s)$ for $F[\oplus_2]$ [Ben-Sasson & Nordström '08]

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Intuition behind proof

- Given resolution refutation π of $F[\oplus_2]$
- Extract the refutation π' of F that π is simulating
- Prove that extraction preserves complexity measures of interest

Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

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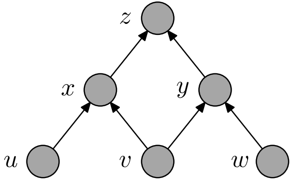

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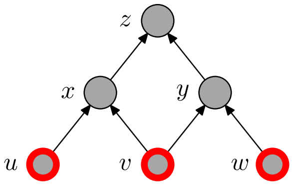

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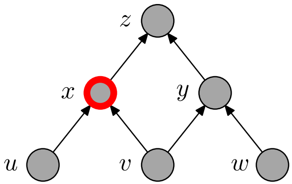

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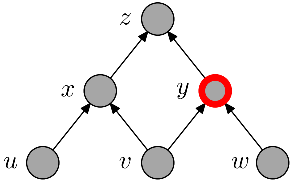

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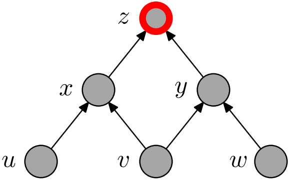

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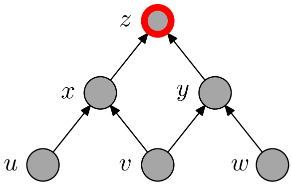

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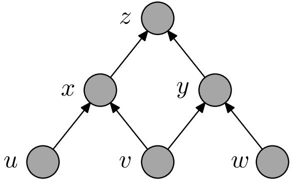

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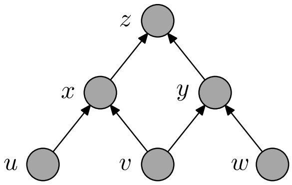

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Easy to refute pebbling formulas in size $\mathcal{O}(n)$ and width $\mathcal{O}(1)$
Pebbling space lower bounds $\Rightarrow$ clause space lower bounds
[Ben-Sasson & Nordström '08, '11]

XOR Substitution with Recycling (1/2)

Suppose

- F CNF formula over variables U
- $\mathcal{G} = (U \,\dot{\cup}\, V, E)$ bipartite graph

Substituted formula $F[\mathcal{G}]$ over variables V:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

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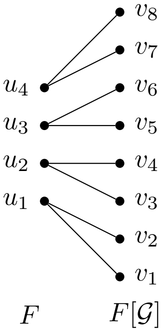

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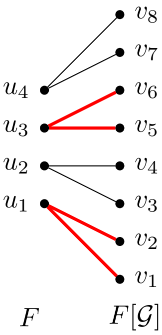

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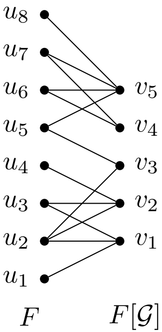

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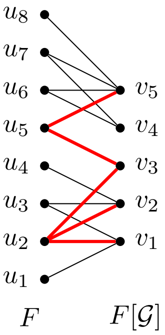

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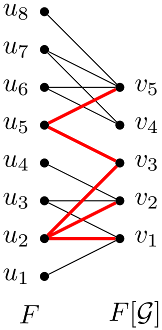

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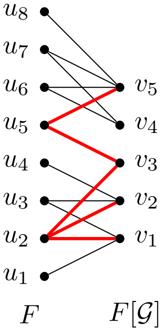

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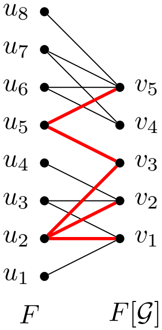

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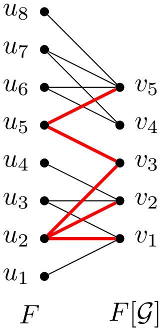

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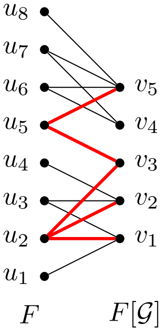

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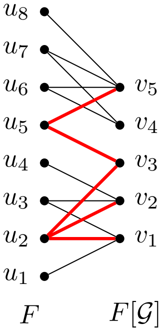

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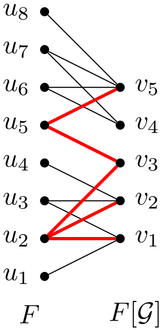

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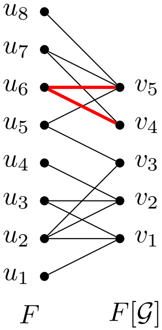

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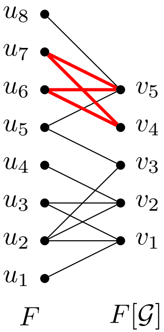

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$\lightning$

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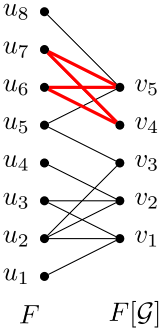

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Solution: Use expander graphs!

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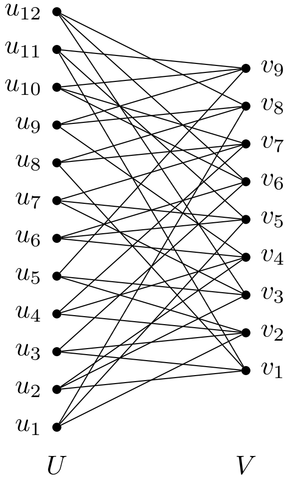
Bipartite Boundary Expander



$\mathcal{G} = (U \dot\cup V, E)$ is (d, r, c)-boundary expander if

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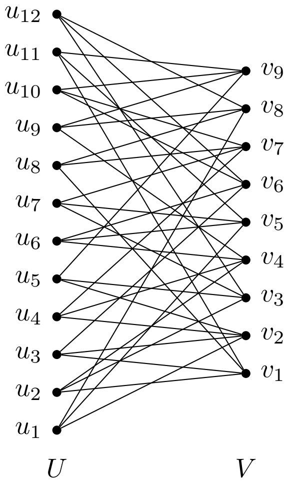

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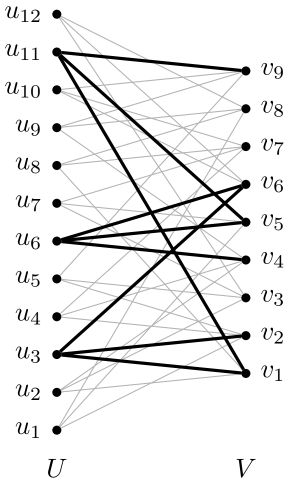

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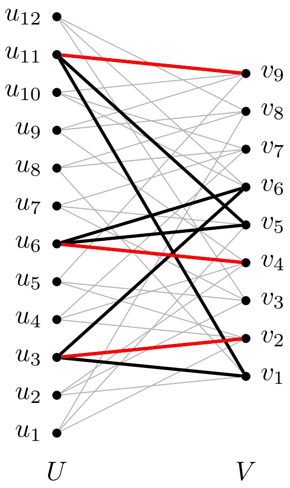

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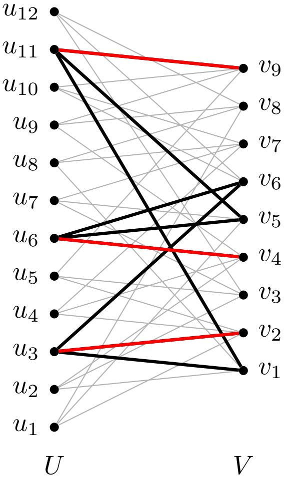

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Lemma ([Razborov '16])

For $\varepsilon > 0$ and n, d with $d \leq |V|^{\frac{1}{2}-\varepsilon}$, $|U| = n$, $|V| = n^{\mathcal{O}(1/d)}$ there are $(d, r, 2)$-boundary expanders $\mathcal{G}$ with $r = d \log n$

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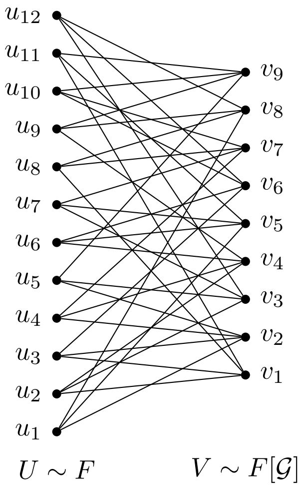

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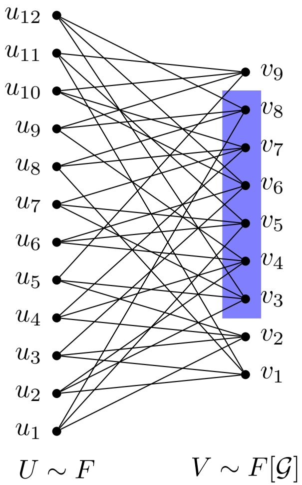

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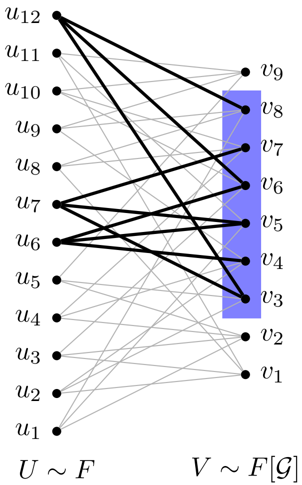

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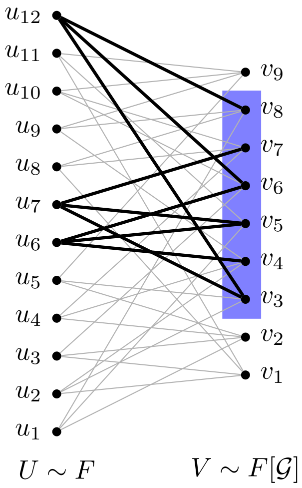

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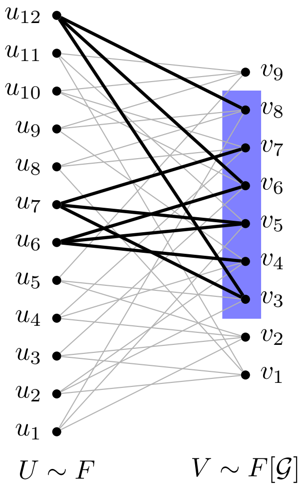

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Some further technical twists needed, but this is main idea of proof

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Where else can this technique be useful?

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Thank you for your attention!