On the Virtue of Succinct Proofs: Amplifying Communication Complexity Hardness to Time-Space Trade-offs in Proof Complexity

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Mumbai, India
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Joint work with Trinh Huynh

The SAT Problem in Theory and Practice

The Satisfiability Problem (SAT)

Given a formula F in propositional logic formula, is it true no matter how one assigns values to its variables?

- NP-complete and so probably intractable in worst case
- But enormous progress on applied algorithms last 10-15 years
- Surprising fact 1: State-of-the-art SAT solvers can deal with real-world instances containing millions of variables
- Surprising fact 2: Best SAT solvers today still based on methods from early 1960s (i.e., DPLL and resolution)
- Algebraic and geometric methods more efficient in theory but **not** so far in practice

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- Key resources for solvers: time and memory
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- Lower bounds for optimal algorithms

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Hope to understand potential and limitation of SAT solvers by studying corresponding proof systems

Complexity measures also natural and interesting in their own right

This talk: Size-space trade-offs for algebraic and geometric systems

Outline

Some Terminology and Notation

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: all clauses of size $\leq k = \mathcal{O}(1)$
  All formulas in this talk k-CNFs (cleanest and most interesting case)
- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- Refer to clauses of CNF formula as axioms
  (as opposed to conclusions derived from these clauses)

The Theoretical Model

- Proof system operates with lines of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when contradiction is derived

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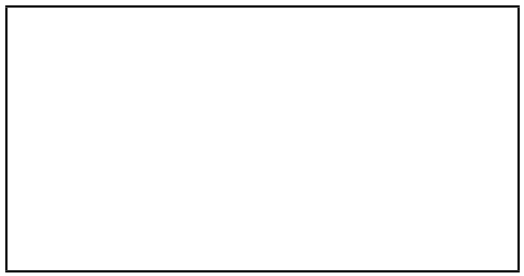

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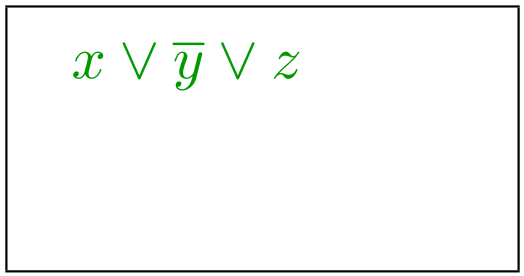

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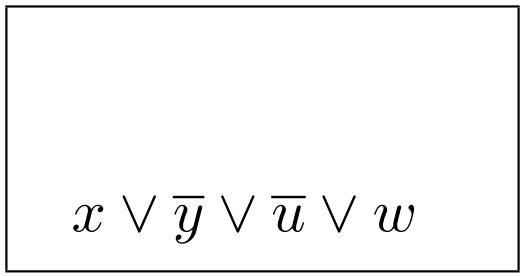

Complexity Measures: Length, Size and Space

Length

derivation steps

Size

$\approx$ total # symbols in proof counted with repetitions

Space

$\approx$ max size of blackboard to carry out proof

(e.g., space 3 for this blackboard)

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Note that:

1. These are somewhat informal definitions — see paper for (standard) details
2. Length and size can be very different — won't really distinguish between them too much in this talk

Resolution

Basis for the most successful SAT solvers to date (DPLL method plus clause learning; a.k.a. CDCL)

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- Optimal (exponential) lower bounds on size
  [Urquhart '87; Chvátal & Szemerédi '88]
- Optimal (linear) lower bounds on clause space
  [Torán '99; Alekhnovich, Ben-Sasson, Razborov & Wigderson '00]
- Strong size-space trade-offs
  [Ben-Sasson & N. '11; Beame, Beck & Impagliazzo '12]

Polynomial Calculus (or Actually PCR [ABRW '00])

Clauses interpreted as polynomial equations over finite field

E.g., $x \vee y \vee \overline{z}$ translated to $x'y'z = 0$

Show no common root by deriving $1 = 0$

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Boolean axioms $\dfrac{}{x^2 - x = 0}$ *Negation* $\dfrac{}{x + x' = 1}$

Linear combination $\dfrac{p = 0 \quad q = 0}{\alpha p + \beta q = 0}$ *Multiplication* $\dfrac{p = 0}{xp = 0}$

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- Optimal (exponential) lower bounds on size
  [Alekhnovich-Razborov '01] and others
- Only recently lower bounds on monomial space for k-CNFs
  [Filmus, Lauria, N., Ron-Zewi & Thapen '12] building on [ABRW '00]
  Very recent optimal bounds in [Bonacina & Galesi '13]
- No size-space trade-offs

Cutting Planes

Clauses interpreted as linear inequalities

E.g., $x \vee y \vee \overline{z}$ translated to $x + y + (1 - z) \geq 1$

Show inconsistent by deriving $0 \geq 1$

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

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- Only one (exponential) lower bounds on size [Pudlák '97]
- No lower bounds on line space
- No size-space trade-offs

Trade-offs for Polynomial Calculus and Cutting Planes

We make some progress on understanding space and size-space trade-offs in polynomial calculus and cutting planes

Theorem (Informal)

There are k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\Theta(n)$ such that

- *resolution can refute F_n in length $\mathcal{O}(n)$ (and hence so can polynomial calculus and cutting planes)*
- *any polynomial calculus or cutting planes refutation of F_n in length L and space s must have*

$$s \log L \gtrapprox \sqrt[4]{n}$$

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Nice bonus: lower bounds hold for semantic versions of proof systems where anything implied by blackboard can be inferred in just one step

Proof Ingredients

- Communication complexity
- Pebbling
- Lifting
- Critical block sensitivity

Two-Player Randomized Communication Complexity

- Alice has private input x and private source of randomness
- Bob has private input y and private source of randomness
- Both have unbounded computational powers
- Want to compute $f(x, y)$ by sending messages back and forth
- Output correct for any x and y except with error probability ε
- Communication cost: max # bits communicated on any x and y

Falsified Clause Search Problem

Fix:

- unsatisfiable CNF formula F
- (devious) partition of $Vars(F)$ between Alice and Bob

Falsified clause search problem $Search(F)$

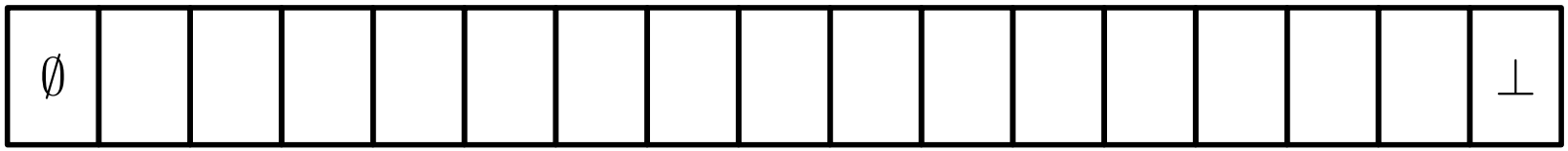
Input: Assignment α to $Vars(F)$ split between Alice and Bob

Output: Clause $C \in F$ such that $\alpha(C) = 0$

Actually, computing not function but relation — more about that later

Succinct Refutations Yield Efficient Protocols

Evaluate blackboard configurations of a refutation of F under α



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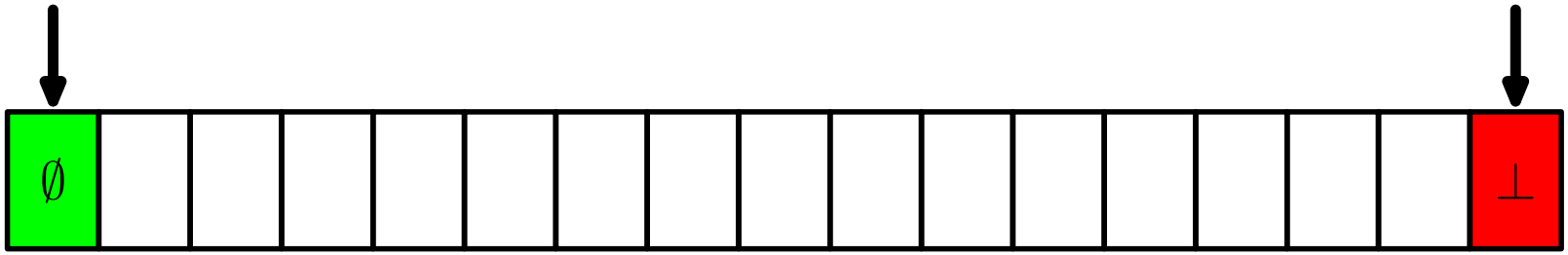

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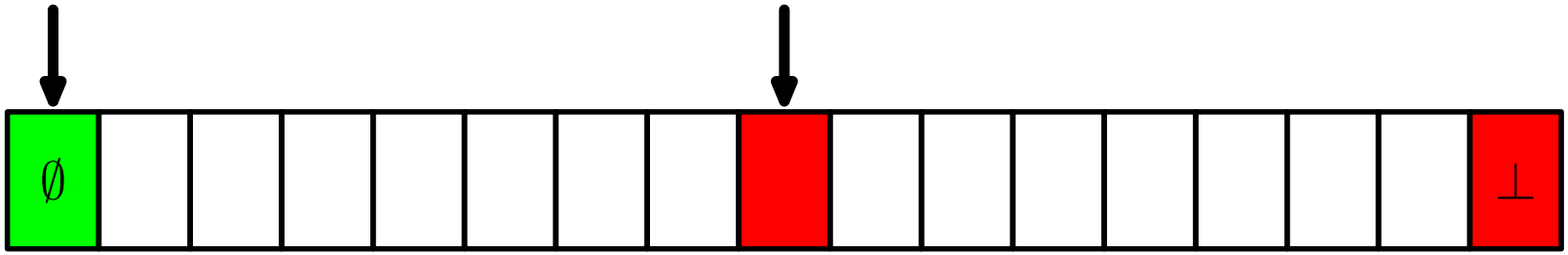

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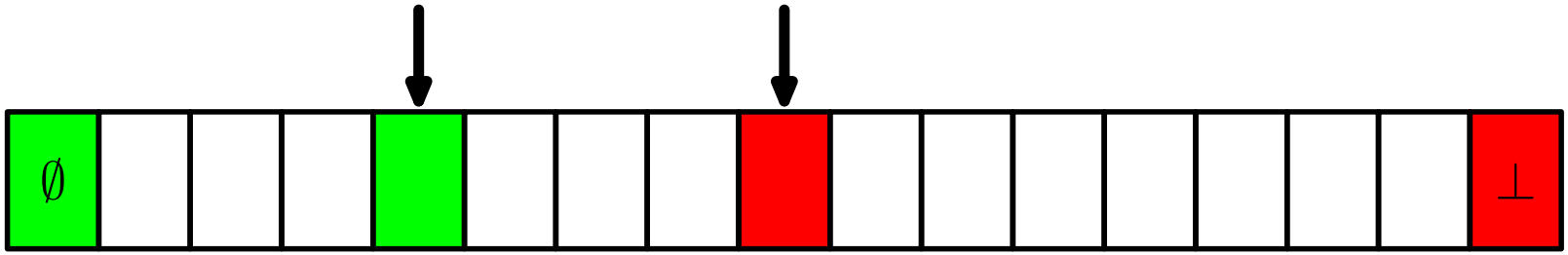

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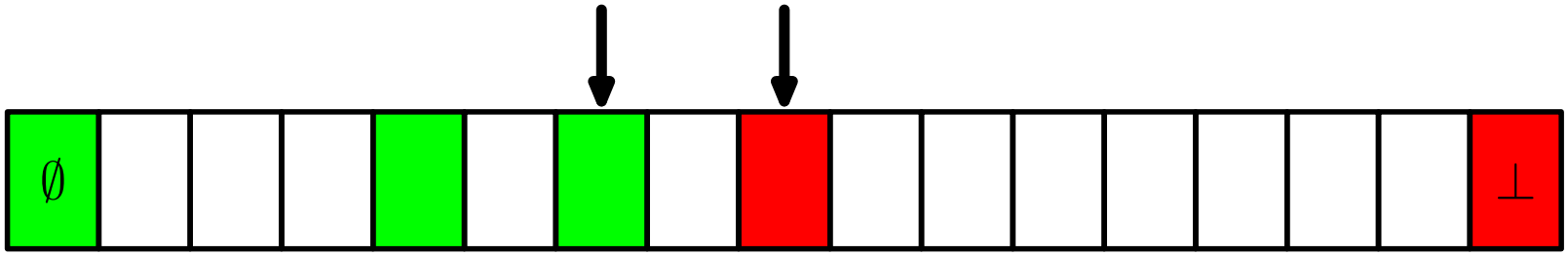

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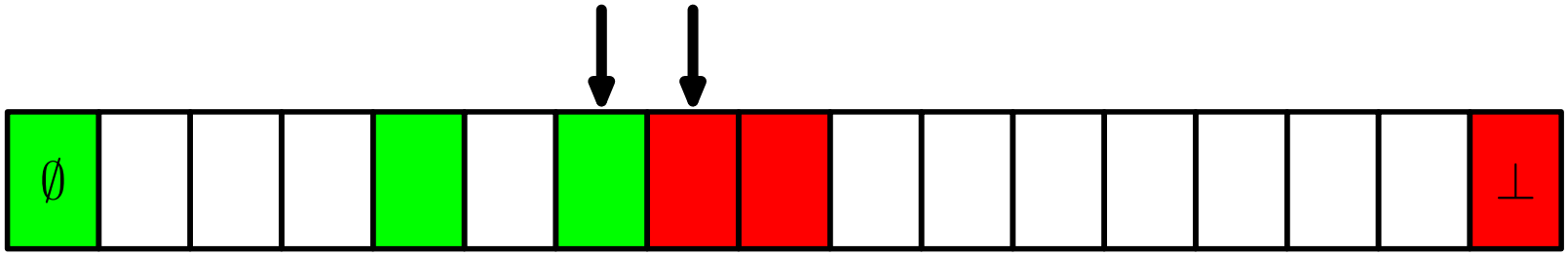

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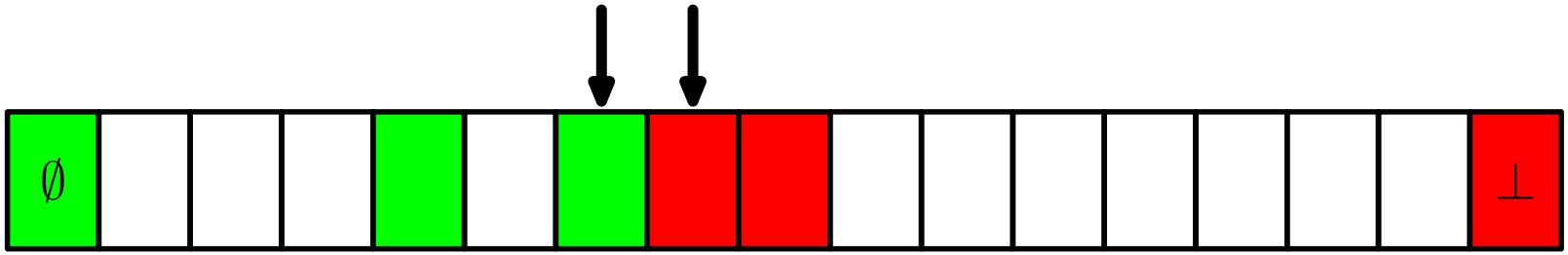

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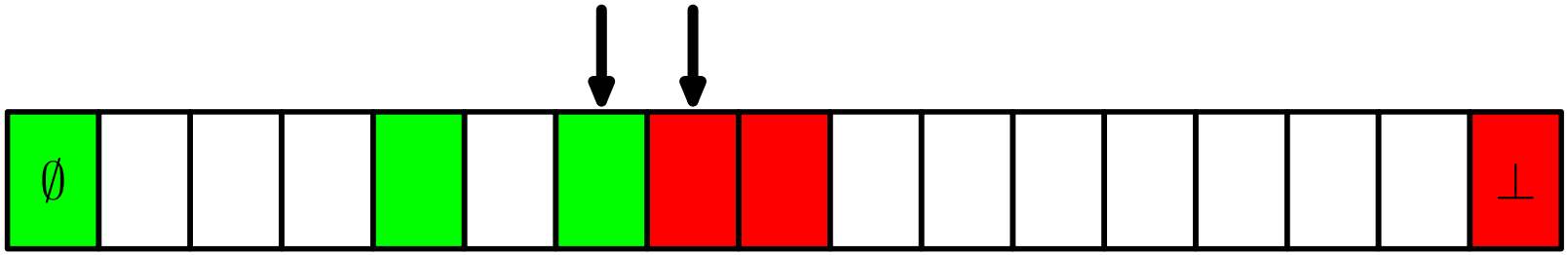

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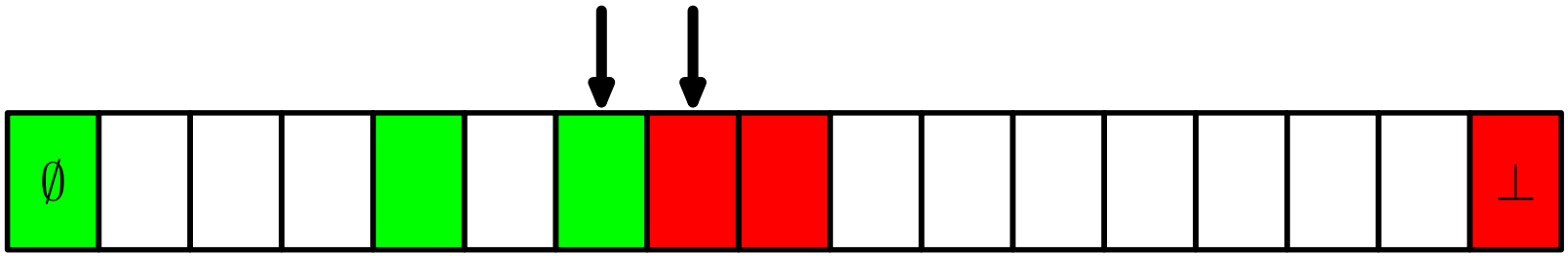

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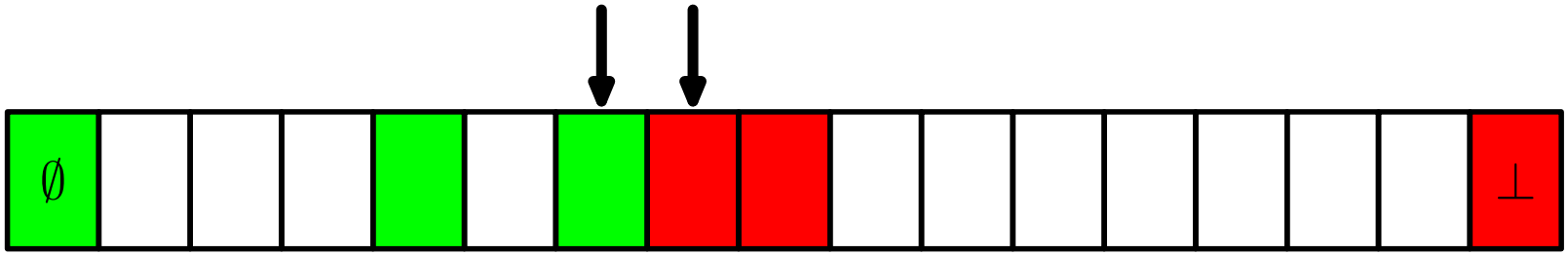

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(E.g. for polynomial calculus Alice and Bob simply evaluate their part of each monomial and exchange values — cutting planes bit more involved but can be done)

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- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

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- Time needed for calculation: # pebbling moves
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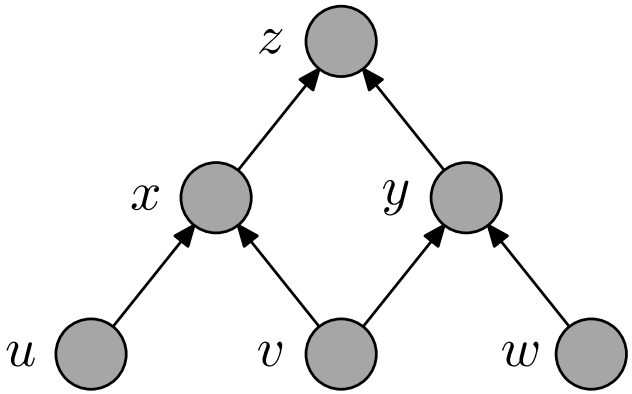
Some quick graph terminology

- DAGs consist of vertices with directed edges between them
- vertices with no incoming edges: sources
- vertices with no outgoing edges: sinks

The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



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Current # pebbles	0
Max # pebbles so far	0

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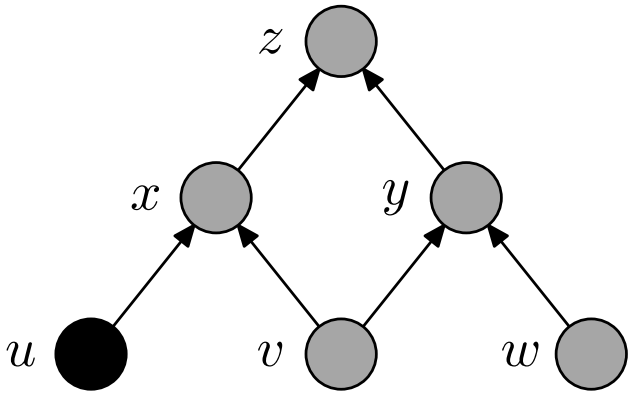

# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them

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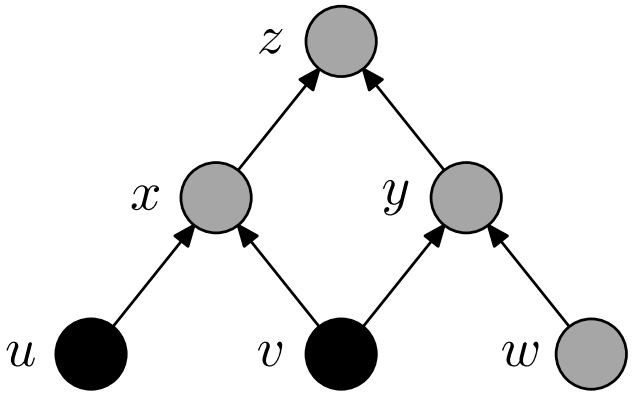

# moves	2
Current # pebbles	2
Max # pebbles so far	2

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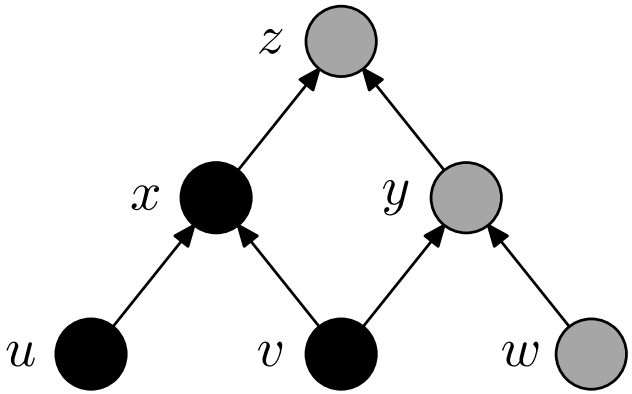

# moves	3
Current # pebbles	3
Max # pebbles so far	3

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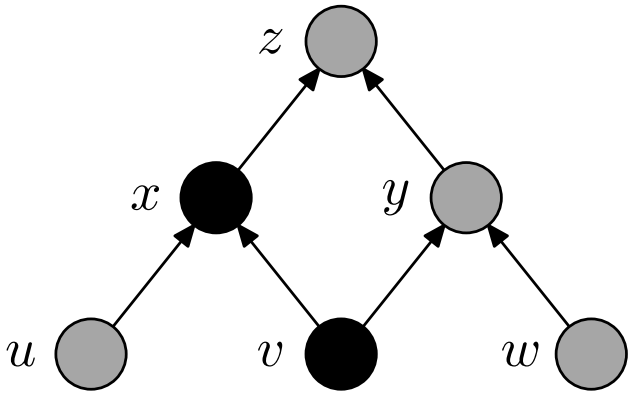

# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex

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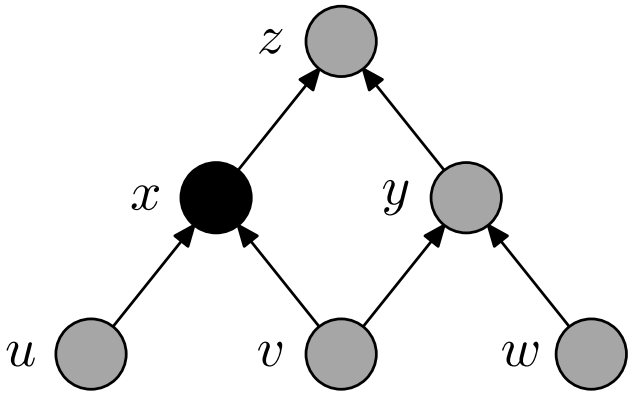

# moves	5
Current # pebbles	1
Max # pebbles so far	3

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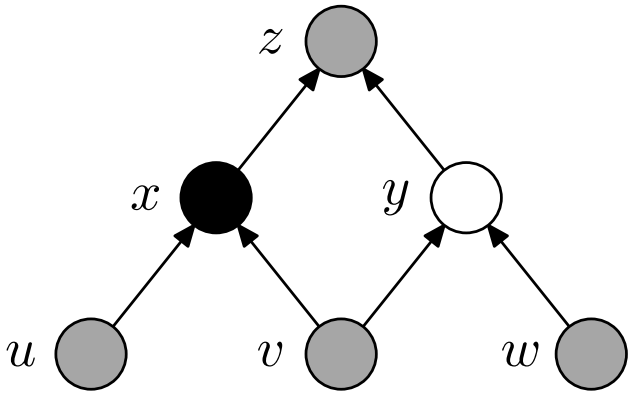

# moves	6
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Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex

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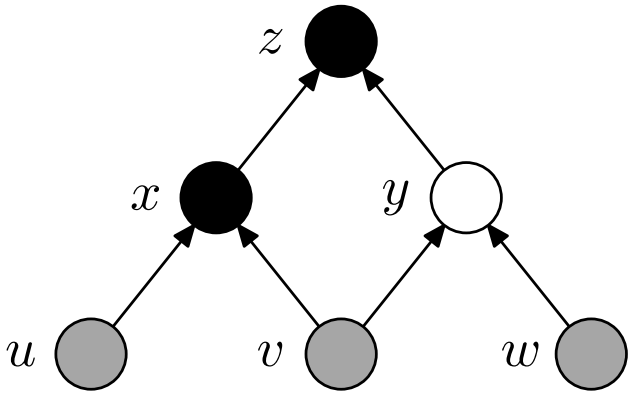
Goal: get single black pebble on sink vertex z of G

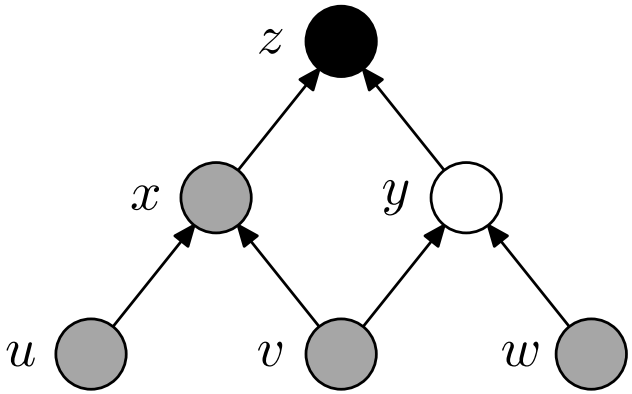


# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex

The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	8
Current # pebbles	2
Max # pebbles so far	3

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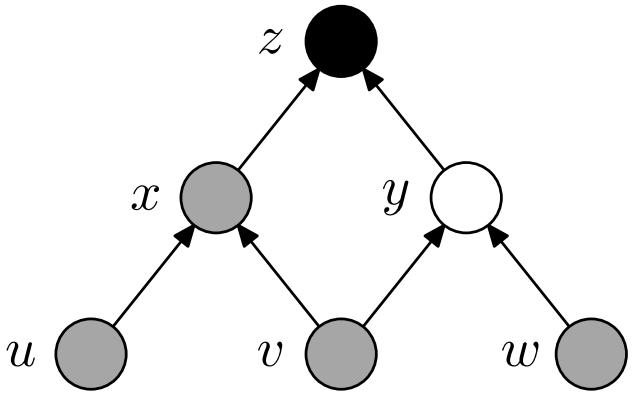

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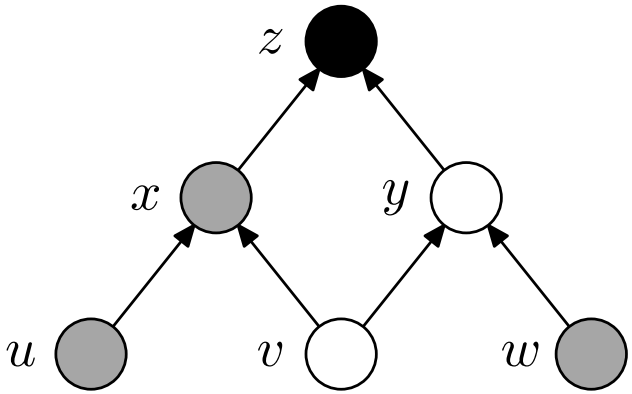

# moves	9
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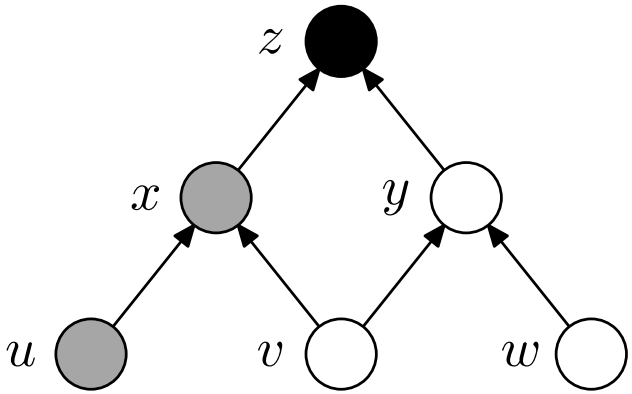
Goal: get single black pebble on sink vertex z of G



# moves	10
Current # pebbles	4
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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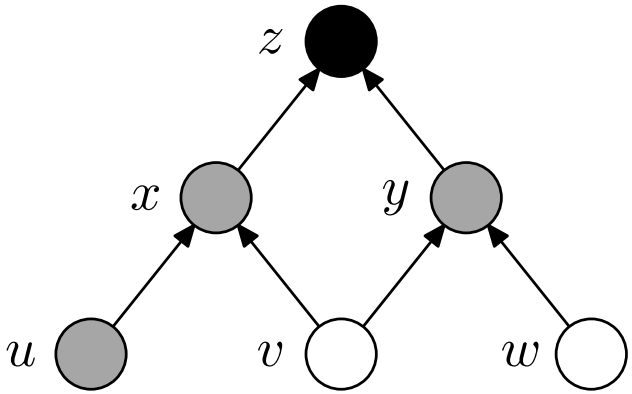
Goal: get single black pebble on sink vertex z of G



# moves	11
Current # pebbles	3
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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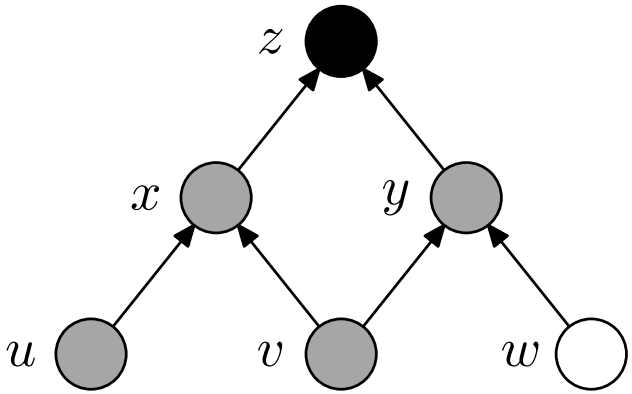

# moves	12
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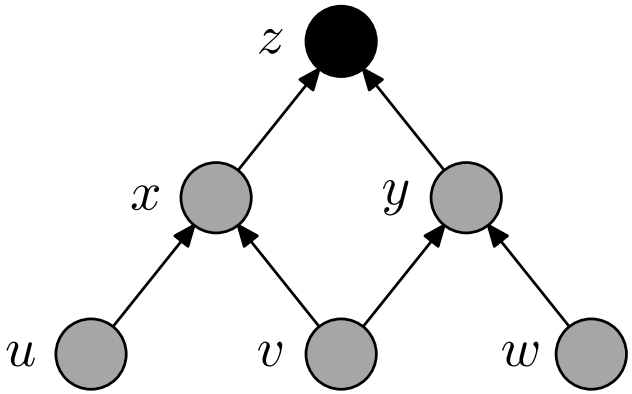

# moves	13
Current # pebbles	1
Max # pebbles so far	4

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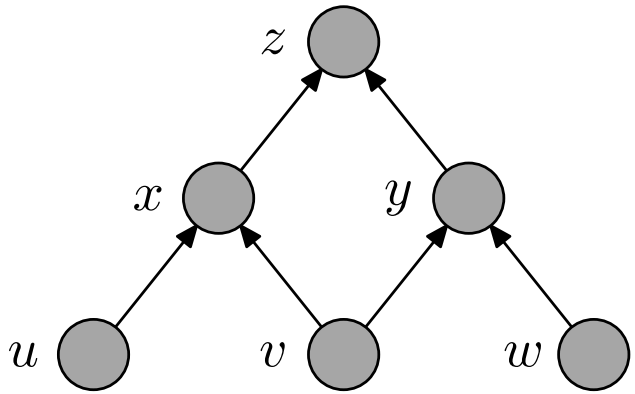
Pebbling Contradiction

CNF formulas encoding pebble game played on DAG G

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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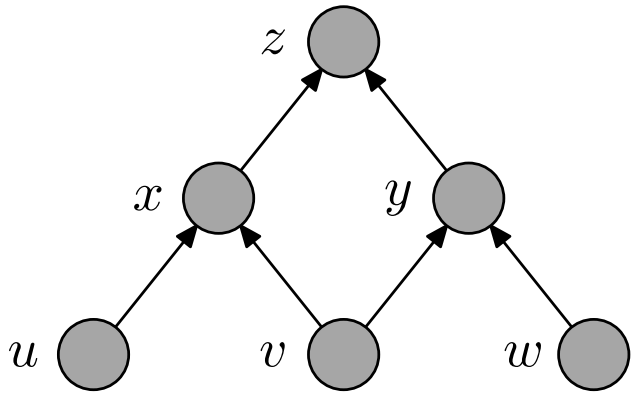

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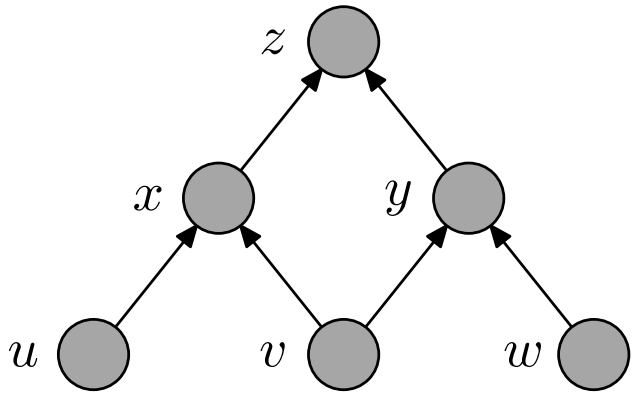

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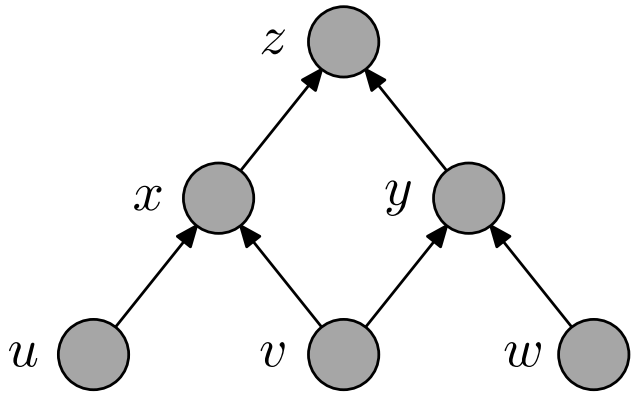

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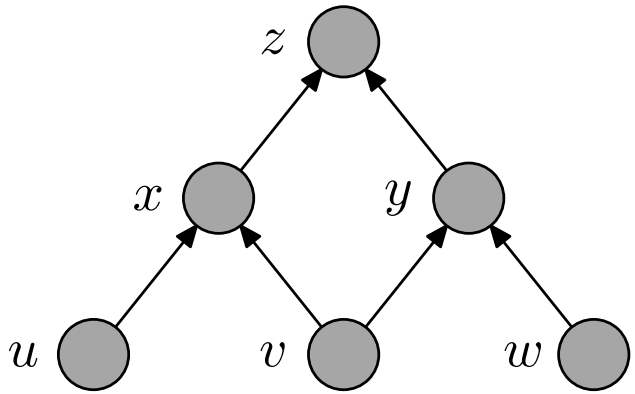

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Appeared in various contexts in [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and other papers

Used in [N. '06, N. & Håstad '08, Ben-Sasson & N. '08, '11] to study space and size-space trade-offs in resolution

Inherit some DAG properties, but not enough — make formulas harder!

Lifting of Functions

Construct hard communication problems by "hardness amplification" using lifting

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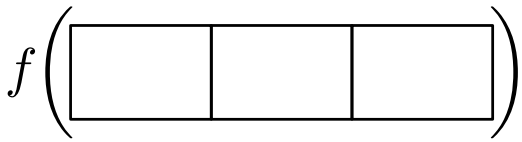

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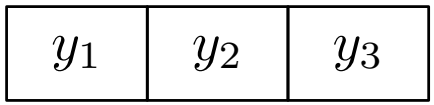

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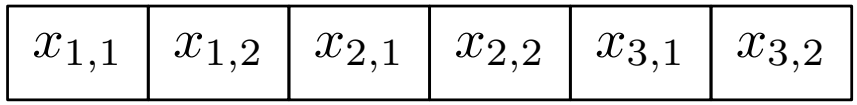

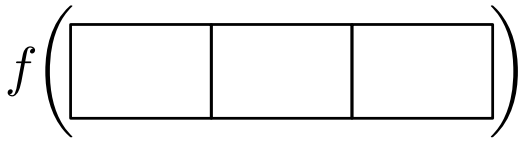

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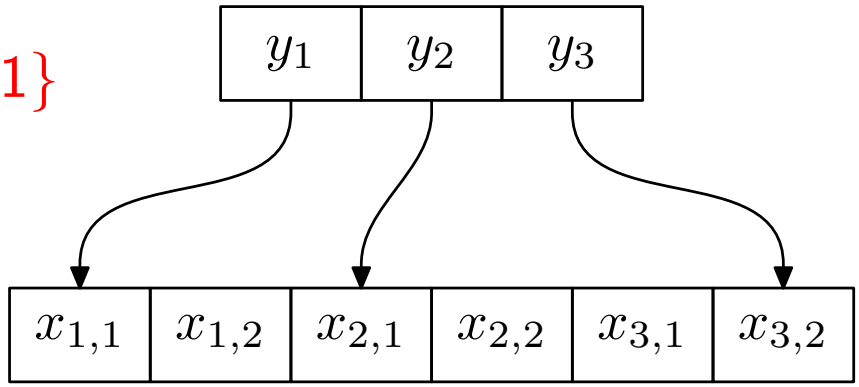

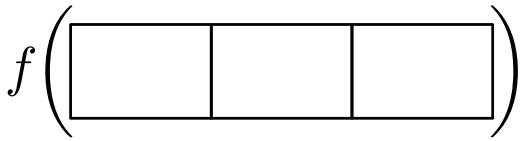

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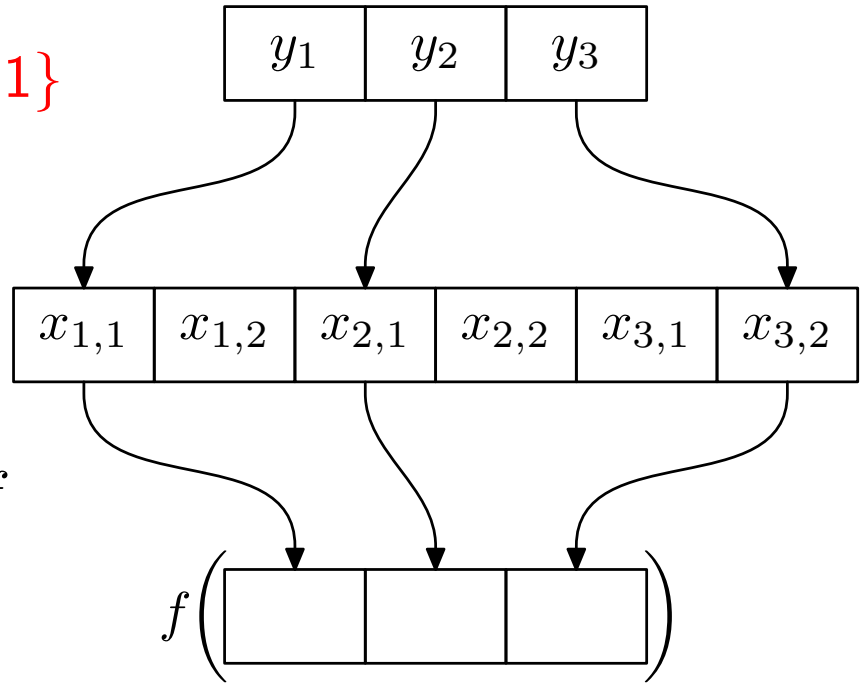

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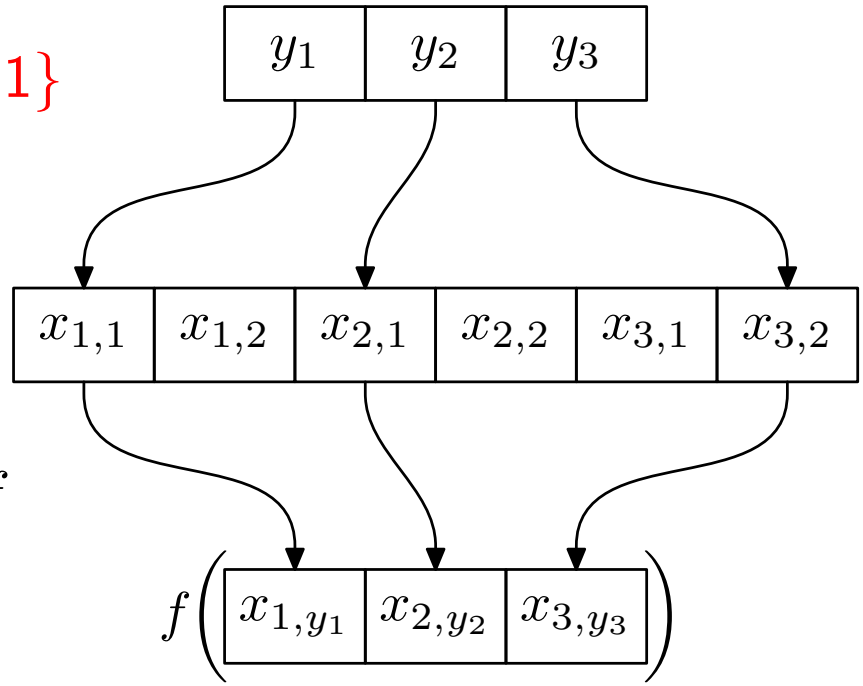
Length-ℓ lifting of f defined as $Lift_\ell(f)(x,y) := f(x_{1,y_1}, \ldots, x_{m,y_m})$



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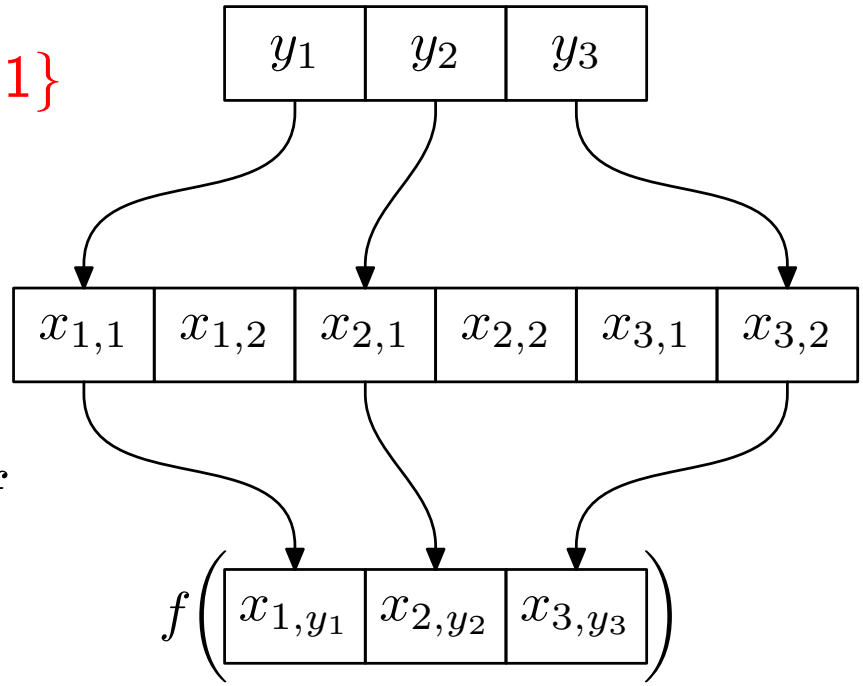

Building on ideas from e.g. [Sherstov '08, Beame, Huynh & Pitassi '10]

Critical Block Sensitivity of Search Problems

- Block sensitivity of f on α: # disjoint blocks of α that flip f if flipped
- Problem: falsified clause search problem defines relation, not function
- Study block sensitivity of search problems
- In addition restrict to critical inputs (where relation is "function-like" in that there is only one right answer)
- Prove randomized communication complexity lower bounds in terms of critical block sensitivity of search problems
- Proof uses information-theoretic approach inspired by [Bar-Yossef, Jayram, Kumar & Sivakumar '04]

Communication Complexity Results

We prove two technical lemmas:

Lemma 1

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Lemma 2

Search problems for pebbling formulas constructed from specific family of pyramid graphs have large critical block sensitivity

Putting the Pieces Together

- Encode lifting of search problem for CNF as new formula $Lift(F)$ (as in [Beame, Huynh & Pitassi '10])

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- But communication complexity of lifted search problem lower-bounded by critical block sensitivity (Lemma 1)
- Plug in lower bound for pyramid pebbling formulas (Lemma 2)
  $\Rightarrow$ trade-off for lifted pebbling formulas

Block Sensitivity in More Detail

Block sensitivity of f at α:

$$f\left(\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\right) = 0$$

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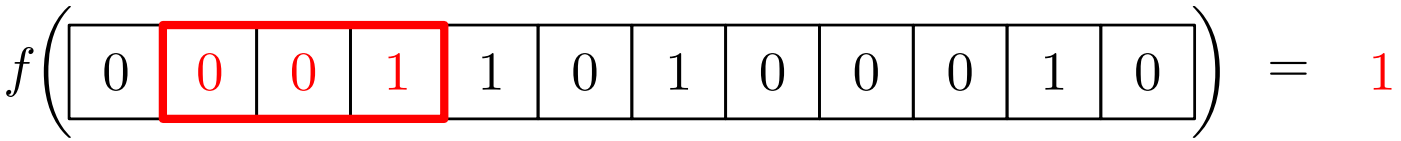

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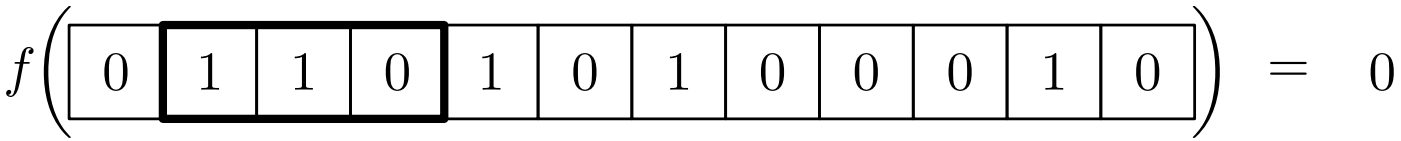

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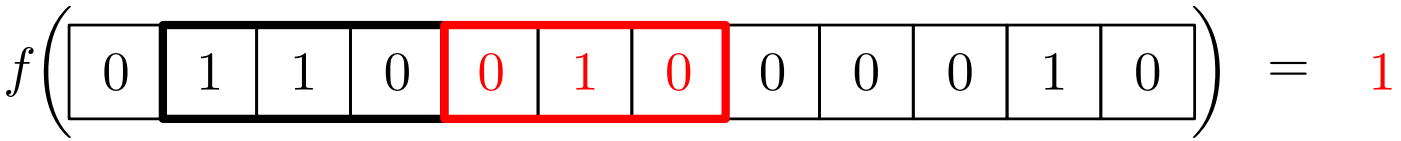

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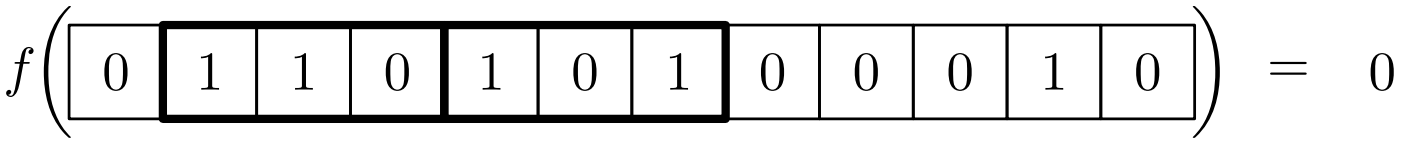

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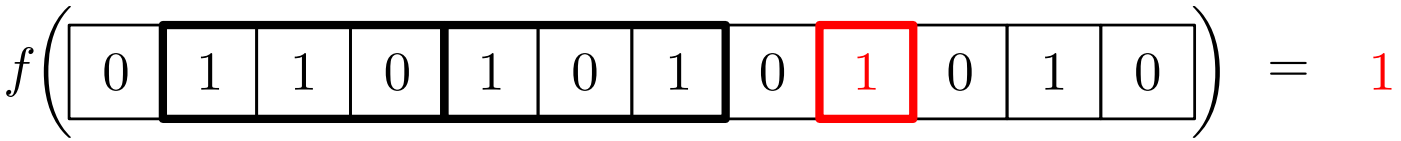

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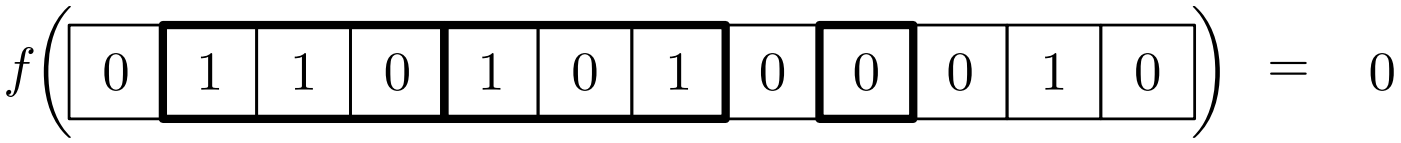

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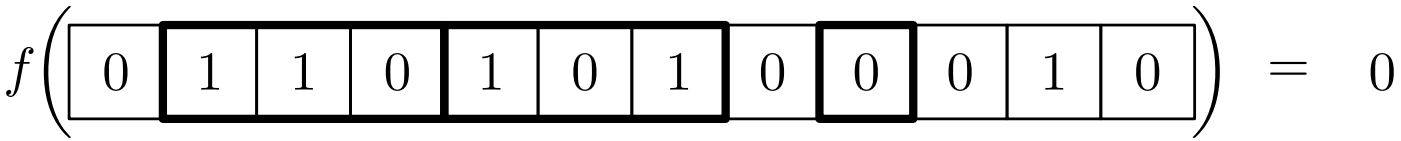

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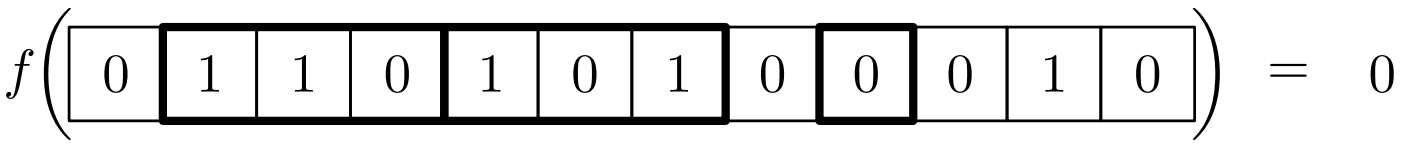

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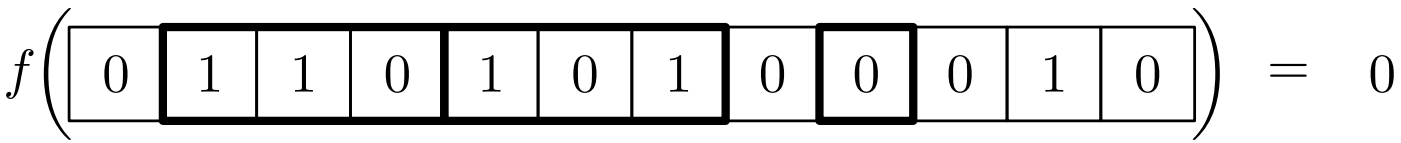

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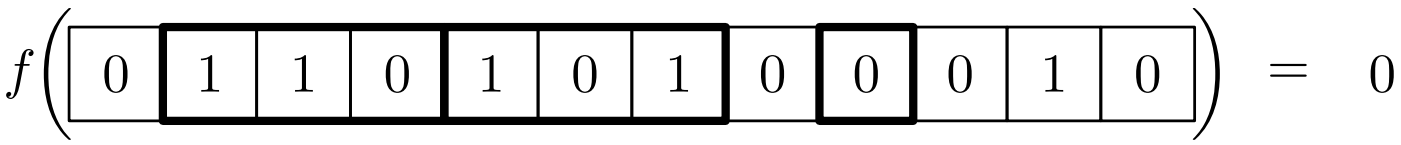

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$bs_{crit}(S)$: block sensitivity over critical assignments $\mathcal{A}$ of best f solving S

Lifting and Critical Block Sensitivity

Lemma 1 (more formal version)

Suppose $S \subseteq \{0,1\}^m \times Q$ is a search problem and $\ell \geq 3$. Then any consistent* randomized protocol solving $Lift_\ell(S)$, where Alice receives the main x-variables and Bob receives the selector y-variables, requires $\Omega(bs_{crit}(S))$ bits of communication.

Proof is by

- information theory tools
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(*) Consistent means Alice and Bob always gives the same correct answer for any fixed (x, y) (although there might be many answers to choose from)

Critical Block Sensitivity Lower Bound for Search Problem

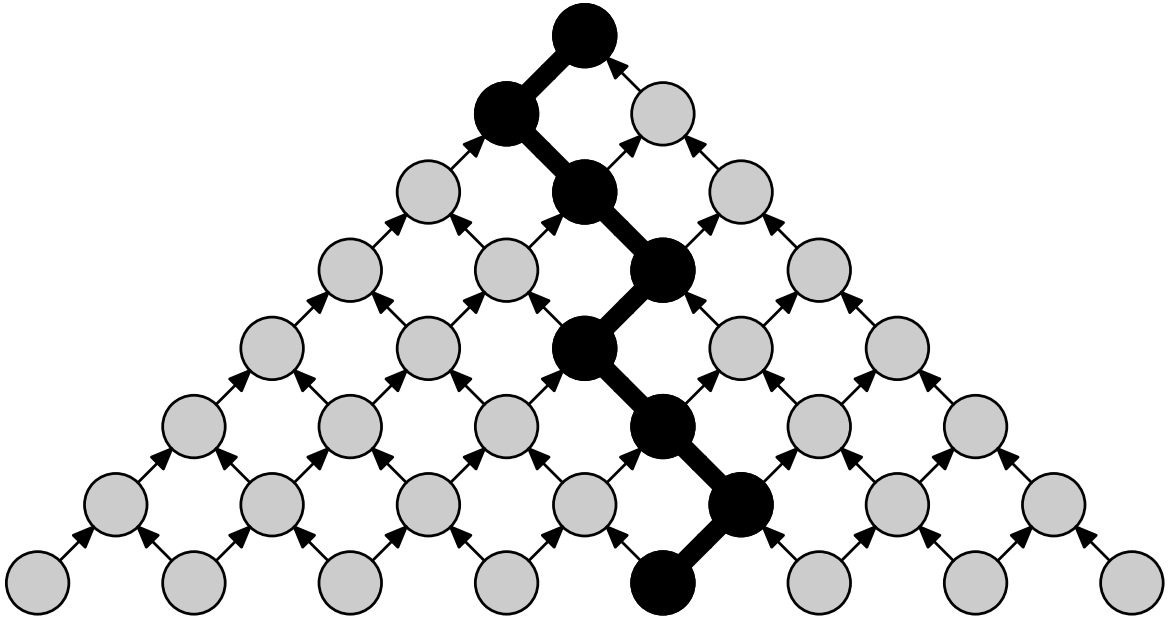
Skip proof of Lemma 1 — very technical

Instead try to give some flavour of the proof of Lemma 2

Lemma 2 (more formal version)

For pebbling formulas on pyramid graphs of height h, the falsified clause search problem has critical block sensitivity $\Omega\left(\sqrt{h}\right)$

Critical Assignments for Pyramid Pebbling Formulas



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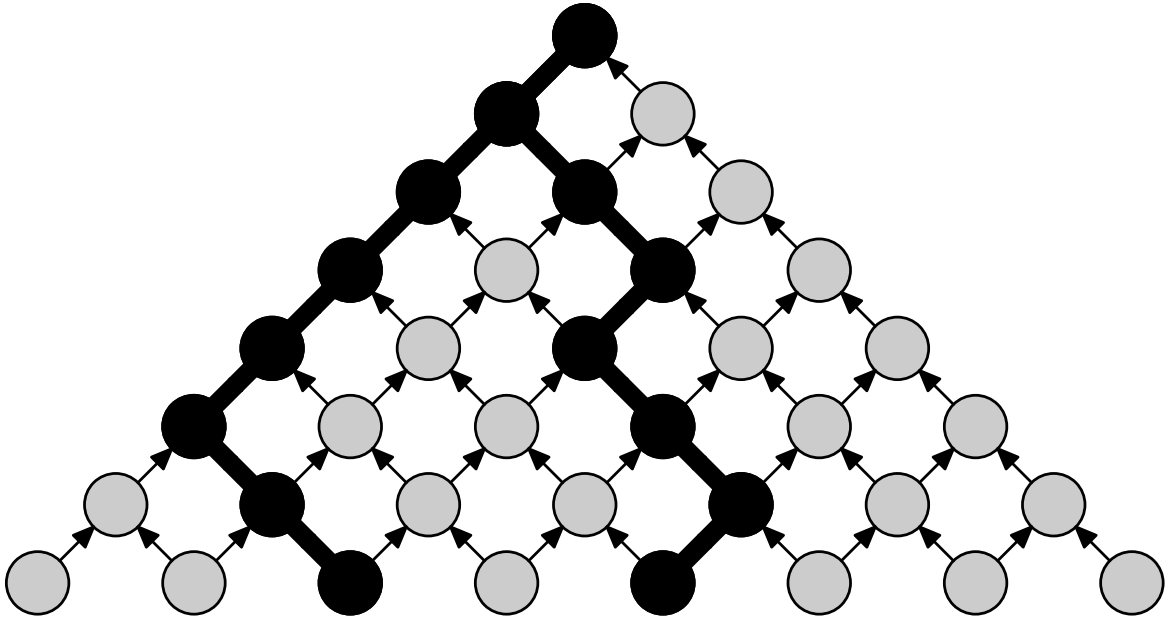

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Bicritical assignments falsify two different paths

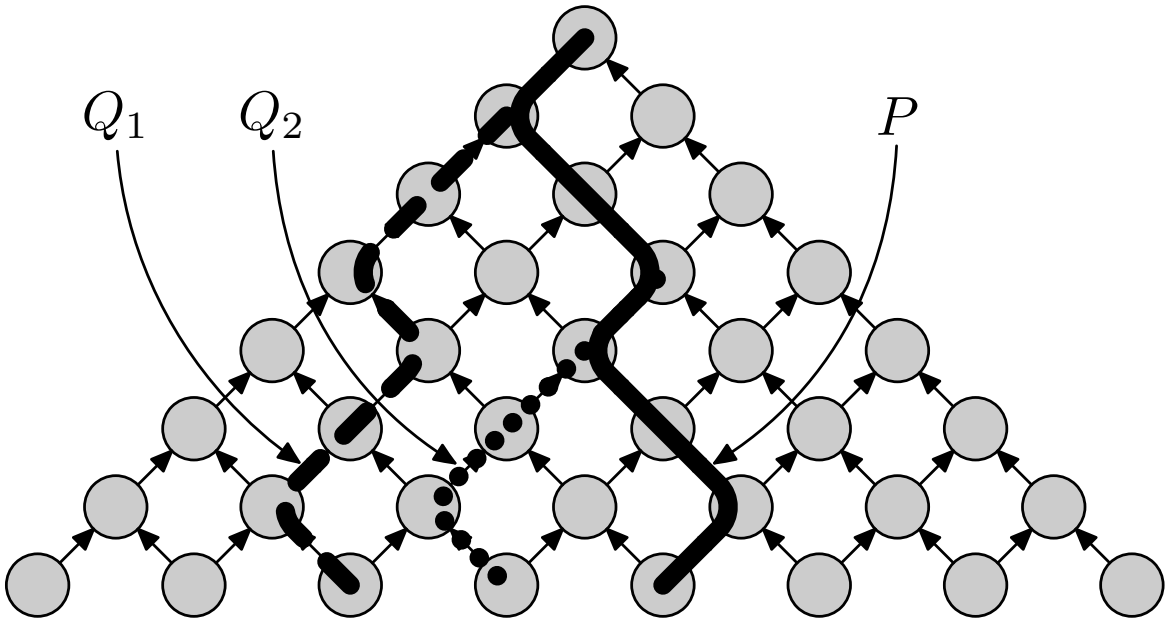
$\Rightarrow$ two possible correct answers

Path Graph

Build directed graph G with vertex set $=$ source-to-sink paths P

Come up with combinatorial construction of edges satisfying:

- edge (P, Q) only if P and Q merge and stay together
- in addition, if (P, Q_1) and (P, Q_2) edges, then $Q_1 \cap Q_2 \subseteq P$
- G is in fact undirected — (P, Q) edge only if (Q, P) edge



Dense Path Graph $\Rightarrow$ High Critical Block Sensitivity

Lemma 3

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Lemma 4

For pyramid of height h on n vertices, can get degree $\Omega\left(\sqrt{h}\right) = \Omega\left(\sqrt[4]{n}\right)$

More General Trade-offs?

Our proofs only work for formulas generated from pyramid graphs

For resolution, correspondence between pebbling and size-space trade-offs holds for arbitrary graphs

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Recently achieved for polynomial calculus in [Beck, N. & Tang '13]
Uses different techniques; in particular random restrictions
$\Rightarrow$ not tight results as for resolution, so room for further improvements

Still open for cutting planes (random restrictions don't work)

Unconditional Space Lower Bounds?

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Can log *length factor be removed from results to yield unconditional space lower bounds?*

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Again answer known to be "yes" for resolution

But [Beck, N. & Tang '13] still has log factor for polynomial calculus

Underlying question: For how wide a family of proof systems do pebbling properties of graphs carry over to CNF size-space trade-offs?

Take-Home Message

- Modern SAT solvers enormously successful in practice — key issue is to minimize time and memory consumption
- Modelled by proof size and space in proof complexity
- We show trade-offs indicating that simultaneous optimization impossible for well-known algebraic and geometric proof systems
- Future theoretical work: Understand size and space in these proof systems better
- Future practical work: Build efficient algebraic or geometric SAT solvers!

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Thank you for your attention!