Divide and Conquer: Towards Faster Conflict-Driven Pseudo-Boolean Solving

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Joint work with Jan Elffers

This Is Me...

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. . . And This Is What I Do for a Living

$(x_{1,1} \lor x_{1,2} \lor x_{1,3} \lor x_{1,4} \lor x_{1,5} \lor x_{1,6} \lor x_{1,7}) \land (x_{2,1} \lor x_{2,2} \lor x_{2,3} \lor x_{2,4} \lor x_{2,5} \lor x_{2,6} \lor x_{2,7}) \land (x_{3,1} \lor$

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$x_{5,4} \lor x_{5,5} \lor x_{5,6} \lor x_{5,7}) \land (x_{6,1} \lor x_{6,2} \lor x_{6,3} \lor x_{6,4} \lor x_{6,5} \lor x_{6,6} \lor x_{6,7}) \land (x_{7,1} \lor x_{7,2} \lor x_{7,3} \lor x_{7,4} \lor x_{7,5} \lor$

$x_{7,6} \lor x_{7,7}) \land (x_{8,1} \lor x_{8,2} \lor x_{8,3} \lor x_{8,4} \lor x_{8,5} \lor x_{8,6} \lor x_{8,7}) \land (\overline{x}_{1,1} \lor \overline{x}_{2,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{3,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{4,1}) \land$

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A Fundamental Theoretical Problem...

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

- Variables should be set to true $(= 1)$ or false $(= 0)$
- Constraint $(x \lor \overline{y} \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously

Is there a truth value assignment satisfying all constraints?

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Can computers solve this satisfiability (SAT) problem efficiently?

- Mentioned already in Gödel's famous letter in 1956 to von Neumann (the "father of computer science")
- Intense research in theoretical computer science ever since early 1970s
- Now one of Millennium Prize Problems in mathematics

. . . with Huge Practical Implications

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  - hardware verification
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  - et cetera. . .

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- But. . . There are also small formulas (just $\sim$100 variables) that are completely beyond reach of even the very best solvers
- Limitations of CDCL
  1. Clauses weak formalism for encoding constraints
  2. Method of reasoning used (resolution) also weak

Pseudo-Boolean Reasoning to the Rescue?

- Pseudo-Boolean (PB) linear constraints are stronger than clauses

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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- And pseudo-Boolean reasoning exponentially more powerful in theory
- But PB solvers less efficient than CDCL in practice(!?)

Outline

1. Conflict-Driven Clause Learning
   - CDCL by Example
   - Pseudocode and Analysis
2. Conflict-Driven Pseudo-Boolean Solving
   - Some Preliminaries
   - Pseudo-Boolean Solving Using Saturation
   - Pseudo-Boolean Solving Using Division
3. Open Problems and Future Directions

Slides online at www.csc.kth.se/~jakobn/research/TalkDIKU18.pdf

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DPLL method [DP60, DLL62]

- Assign values to variables (in some smart way)
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Variable Assignments

Two kinds of assignments — illustrate on our example formula:

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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Decision

Free choice to assign value to variable

Notation $w \stackrel{\mathsf{d}}{=} 0$

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Given $w = 0$, clause $\overline{u} \lor w$ forces $u = 0$

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Conflict-Driven Clause Learning

Time to analyse this conflict!

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Could backtrack by flipping last decision

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But want to learn from conflict and cut away as much of search space as possible

Conflict-Driven Clause Learning

Time to analyse this conflict!

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

$w \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{\overline{u} \lor w}{=} 0$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1 \rightarrow x \lor \overline{y}$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

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Case analysis over z for last two clauses:

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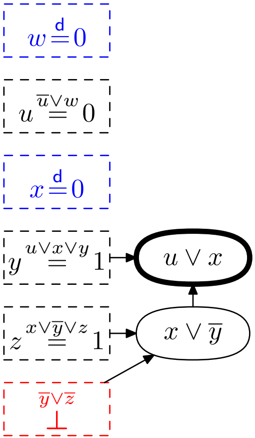

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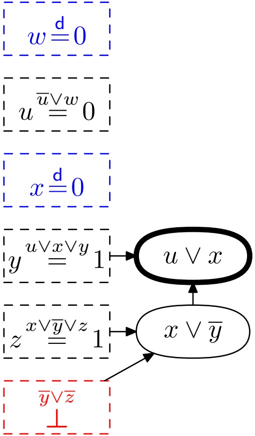
Repeat until only 1 variable after last decision — learn that clause (1UIP) and backjump

Complete Example of CDCL Execution

Backjump: roll back max #decisions so that last variable still flips

$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$



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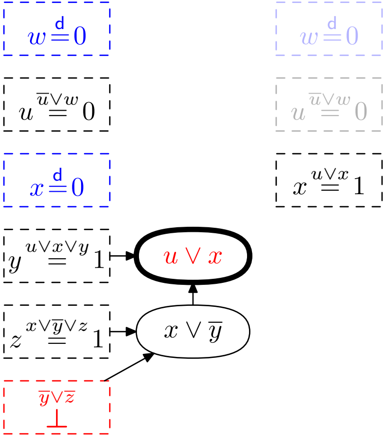
$$(u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$$



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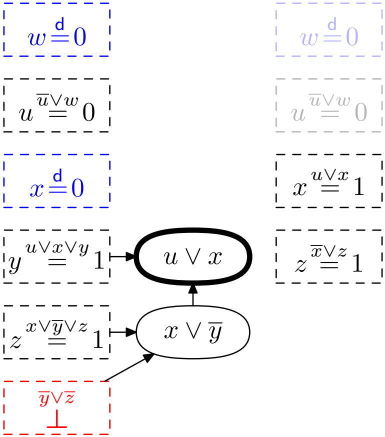

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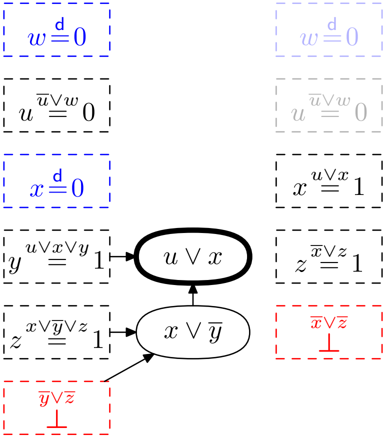

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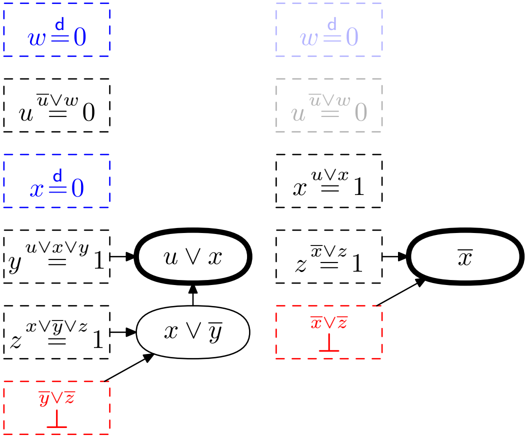

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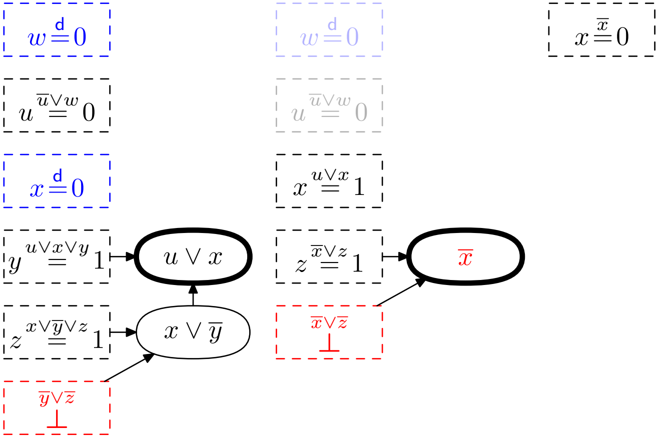

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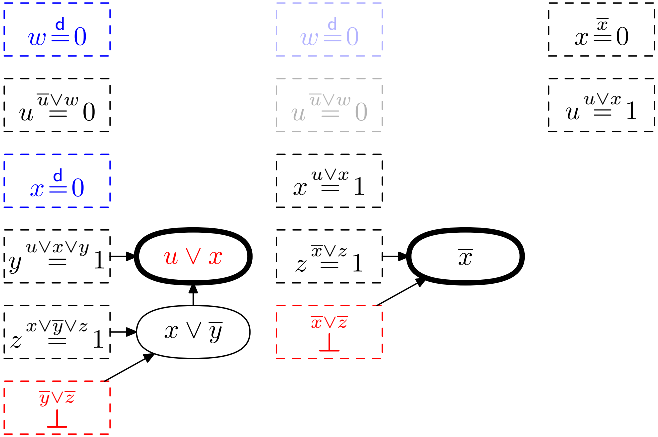

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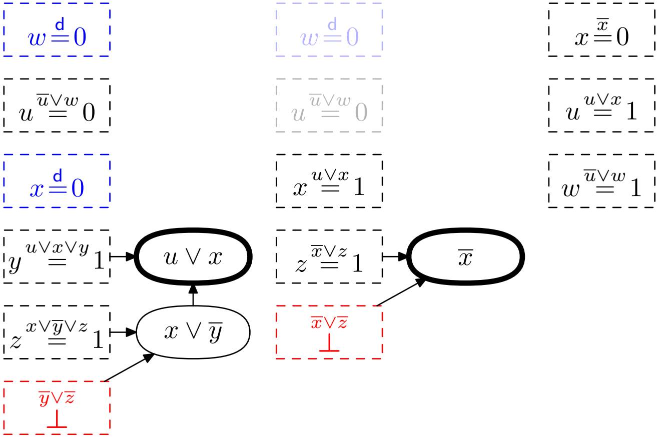

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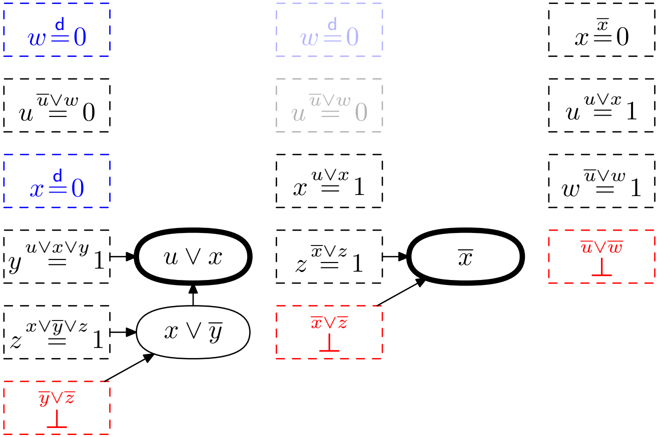

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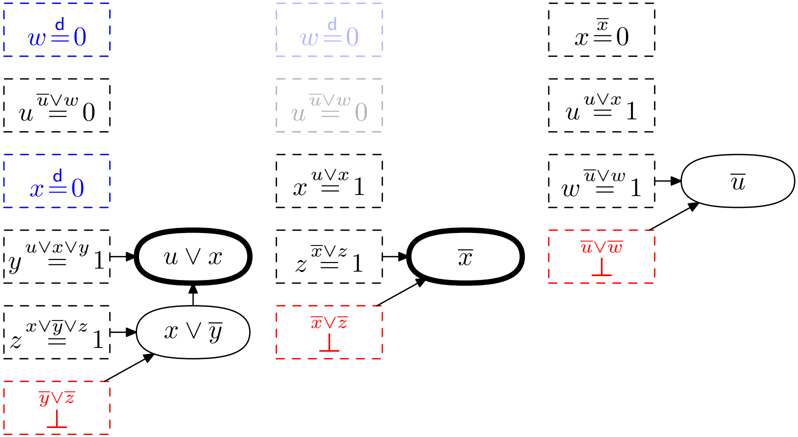

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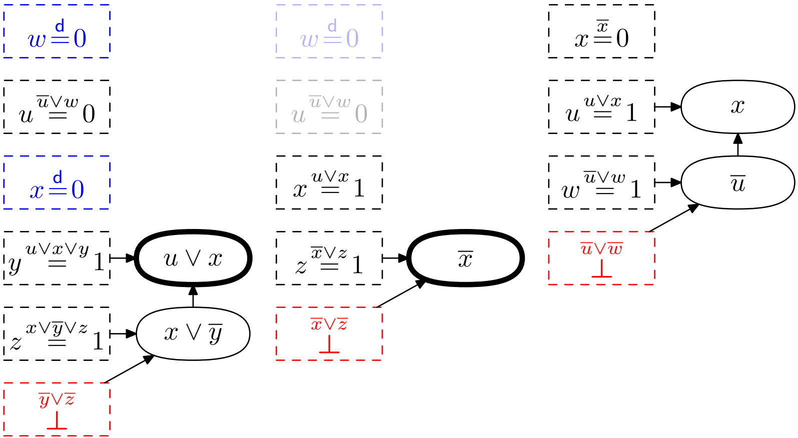

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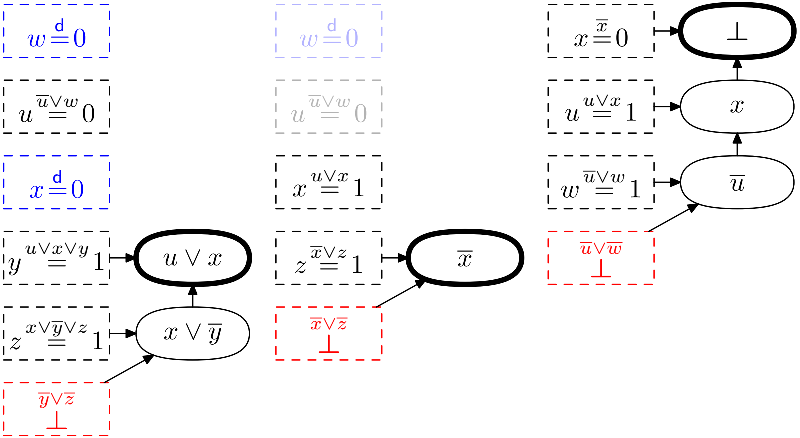

CDCL Main Loop Pseudocode (High Level)

forever do
  if *current assignment falsifies clause* **then**
    apply learning scheme to derive new clause;
    if *learned clause empty* **then** output `UNSATISFIABLE` and exit;
    else
      add learned clause and backjump
    end
  else if *all variables assigned* **then** output `SATISFIABLE` and exit;
  else if *exists unit clause* C *propagating* x *to value* $b \in \{0,1\}$ **then**
    add propagated assignment $x \stackrel{C}{=} b$
  else if *time to restart* **then**
    remove all variable assignments
  else
    if *time for clause database reduction* **then**
      erase (roughly) half of learned clauses in memory
    end
    use decision scheme to choose assignment $x \stackrel{\mathrm{d}}{=} b$;
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- Derive new clauses by **resolution rule**

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- Done when contradiction $\perp$ in form of empty clause derived

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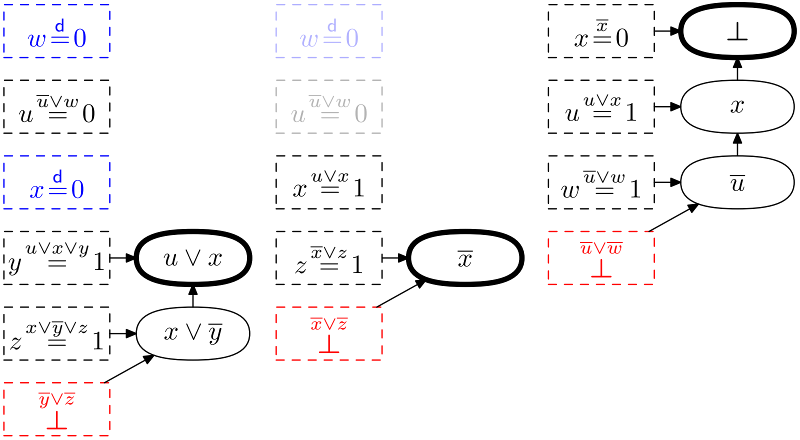
Obtain resolution proof from our example CDCL execution...



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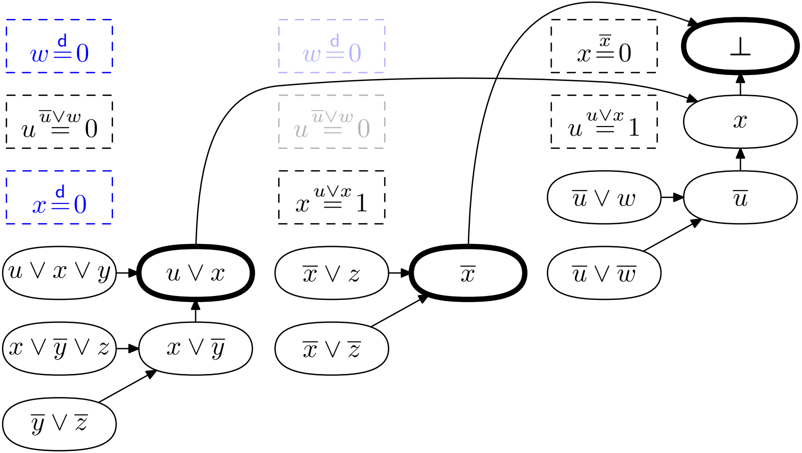
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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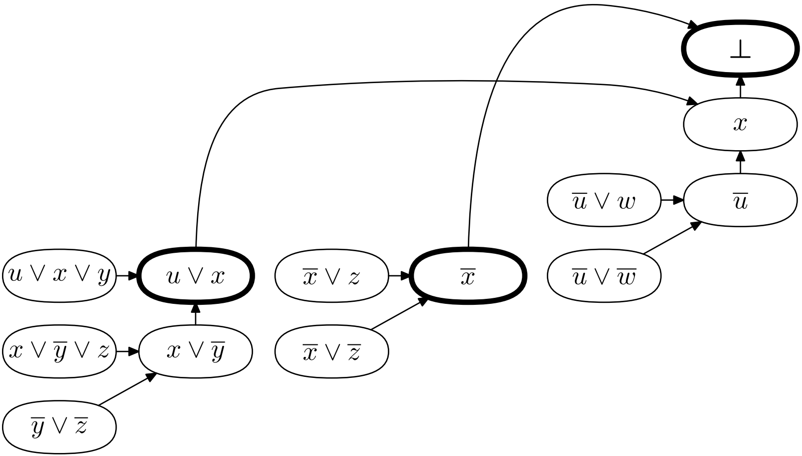

Current state of affairs

- State-of-the-art CDCL solvers often perform amazingly well ("SAT is easy in practice")
- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong lower bounds for "obvious" formulas, e.g., [Hak85, Urq87, BW01, MN14]
- Explore stronger reasoning methods (potential exponential speed-up)
- In particular, pseudo-Boolean solving (a.k.a. 0-1 integer programming) corresponding to cutting planes proof system
- Importantly, extends to pseudo-Boolean optimization (but we won't talk about that)

Pseudo-Boolean Constraints and Normalized Form

In this talk, "pseudo-Boolean" refers to 0-1 integer linear constraints

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- coefficients a_i: non-negative integers
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(All constraints in what follows assumed to be implicitly normalized)

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3. General constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Approaches to Pseudo-Boolean Solving

Conversion to disjunctive clauses

- Lazy approach: learn clauses from PB constraints
  - *Sat4j* [LP10] (one of versions in library)
- Eager approach: re-encode to clauses and run CDCL
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Conflict-Driven Search in a Pseudo-Boolean Setting

Want to do "same thing" as CDCL but with linear constraints

- Variable assignments
  1. Always propagate forced assignment if possible
  2. Otherwise make assignment using decision heuristic
- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to instance
  3. Backjump by rolling back max #decisions so that variable flips

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ρ	$slack(C;\rho)$	comment

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$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack

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Note that constraint can be conflicting though not all variables assigned

Conflict Analysis Invariant

Look at our example CDCL conflict analysis again

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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Assignment "left on trail" always falsifies derived clause

Conflict Analysis Invariant

Look at our example CDCL conflict analysis again

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

$\boxed{w \overset{\mathsf{d}}{=} 0}$

$\boxed{u \overset{\overline{u} \lor w}{=} 0}$

$\boxed{x \overset{\mathsf{d}}{=} 0}$

$\boxed{y \overset{u \lor x \lor y}{=} 1}$

$\boxed{z \overset{x \lor \overline{y} \lor z}{=} 1}$

$\boxed{\overset{\overline{y} \lor \overline{z}}{\bot}}$

Assignment "left on trail" always falsifies derived clause

$\overline{y} \lor \overline{z}$ falsified by
trail $\rho = \{\overline{w}, \overline{u}, \overline{x}, y, z\}$

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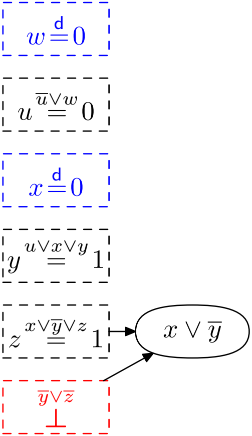

$x \lor \overline{y}$ falsified by trail $\rho' = \{\overline{w}, \overline{u}, \overline{x}, y\}$

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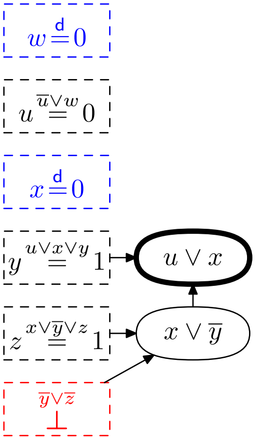

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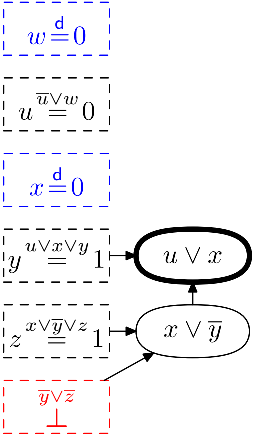

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$\Rightarrow$ every derived constraint "explains" conflict

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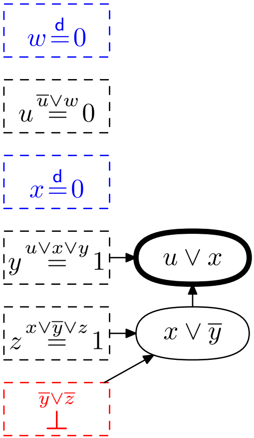

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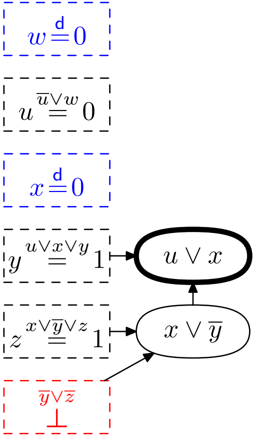

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Learn asserting constraint: after backjump, some variable guaranteed to flip

Generalized Resolution

Can mimic resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

by adding clauses as pseudo-Boolean constraints

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

(Recall $z + \overline{z} = 1$)

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Generalized resolution rule [Hoo88, Hoo92]
Positive linear combination so that some variable cancels

$$\frac{a_1 x_1 + \sum_{i \geq 2} a_i \ell_i \geq A \qquad b_1 \overline{x}_1 + \sum_{i \geq 2} b_i \ell_i \geq B}{\sum_{i \geq 2} \left(\frac{c}{a_1} a_i + \frac{c}{b_1} b_i\right) \ell_i \geq \frac{c}{a_1} A + \frac{c}{b_1} B - c} \; [c = \mathrm{lcm}(a_1, b_1)]$$

Saturation

Actually, don't get quite the right constraint in mimicking of resolution

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Saturation rule

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min\{a_i, A\} \cdot \ell_i \geq A}$$

Sound over integers, not over rationals (need such rules for SAT solving)

Analyze Conflict with Generalized Resolution + Saturation!

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

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(Note: same constraint can propagate several times!)

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Fix (non-obvious): Apply weakening to reason constraints

$$\text{weaken}(\textstyle\sum_i a_i\ell_i \geq A, \ell_j) = \sum_{i\neq j} a_i\ell_i \geq A - a_j$$

Try to Reduce the Reason Constraint

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1. Weaken reason on non-falsified literal (but not last propagated)
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Bummer! Still non-negative slack — not conflicting

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Asserting! Backjump propagates to conflict without decisions $\Rightarrow$ **done**

Reason Reduction Using Saturation [CK05]

$\mathrm{reduceSat}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell, \rho)$

while $\mathit{slack}(\mathsf{resolve}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell); \rho) \geq 0$ **do**

  $\ell' \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ not falsified by ρ;

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- Saturation decreases slack — reach 0 when max #literals weakened

Pseudo-Boolean Conflict Analys

analyzePBconflict(C_{confl}, ρ)

while C_{confl} *not asserting* **do**
  $\ell \leftarrow$ literal assigned last on trail ρ;
  if $\overline{\ell}$ *occurs in* C_{confl} **then**
    $C_{\text{reason}} \leftarrow$ reason(ℓ, ρ);
    $\boldsymbol{C_{\text{reason}} \leftarrow \textbf{reduceSat}(C_{\text{reason}}, C_{\text{confl}}, \ell, \rho)}$;
    $C_{\text{confl}} \leftarrow$ resolve$(C_{\text{confl}}, C_{\text{reason}}, \ell)$;
    $C_{\text{confl}} \leftarrow$ saturate(C_{confl});
  end
  $\rho \leftarrow removeLast(\rho)$;
end
return C_{confl};

The need to reduce the reason is new compared to CDCL
Everything else is the same

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- Compared to clauses harder to detect propagation for constraints like

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- For CNF inputs, degenerates to resolution!
  $\Rightarrow$ CDCL but with super-expensive data structures

The Cutting Planes Proof System

Cutting planes as defined in [CCT87] doesn't use saturation but instead division (a.k.a. Chvátal-Gomory cut)

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A / c \rceil}$$

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- Cutting planes with saturation is **not** [VEG+18]
- Can division yield stronger conflict analysis?

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- Cutting planes with division implicationally complete
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- Can division yield stronger conflict analysis?
  (Used for general integer linear programming in *CutSat* [JdM13])

Using Division to Reduce the Reason

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

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$$\text{resolve } x_3 \dfrac{\text{divide by } 2 \dfrac{\text{weaken } x_4 \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_2 + 2x_3 \geq 3}}{x_1 + x_2 + x_3 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{0 \geq 1}$$

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Terminate immediately!

Reason Reduction Using Division [EN18]

$\mathrm{reduceDiv}(C_{\mathrm{confl}}, C_{\mathrm{reason}}, \ell, \rho)$

$c \leftarrow \mathit{coeff}(C_{\mathrm{reason}}, \ell)$;
while $\mathit{slack}(\mathsf{resolve}(C_{\mathrm{confl}}, \mathsf{divide}(C_{\mathrm{reason}}, c), \ell); \rho) \geq 0$ **do**
    $\ell_j \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ such that $\overline{\ell}_j \notin \rho$ and $c \nmid \mathit{coeff}(C, \ell_j)$;
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So now why does **this** work?

- Sufficient to get reason with slack 0 since
  1. $\mathit{slack}(C_{\mathrm{confl}}; \rho) < 0$
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- Weakening doesn't change slack $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\mathrm{reason}}; \rho) < c$
- After max #weakenings have $0 \leq \mathit{slack}(\text{divide}(C_{\mathrm{reason}}, c); \rho) < 1$

Round-to-1 Reduction used in *RoundingSat*

Reduction method used in *RoundingSat* does max weakening right away

roundToOne(C, ℓ, ρ)

$c \leftarrow \mathit{coeff}(C, \ell)$;
foreach *literal* ℓ_j *in* C **do**
    if $\overline{\ell}_j \notin \rho$ *and* $c \nmid \mathit{coeff}(C, \ell_j)$ **then**
        $C \leftarrow$ weaken(C, ℓ_j);
    end
end
return divide(C, c);

And roundToOne used more aggressively in conflict analysis

RoundingSat Conflict Analysis

$\text{analyzePBconflict}(C_{\text{confl}}, \rho)$

while C_{confl} *contains no or multiple falsified literals on last level* **do**
  if *no current solver decisions* **then**
    output `UNSATISFIABLE` and terminate
  end
  $\ell \leftarrow$ literal assigned last on trail ρ;
  if $\overline{\ell}$ *occurs in* C_{confl} **then**
    $C_{\text{confl}} \leftarrow \text{roundToOne}(C_{\text{confl}}, \overline{\ell}, \rho)$;
    $C_{\text{reason}} \leftarrow \text{roundToOne}(\text{reason}(\ell, \rho), \ell, \rho)$;
    $C_{\text{confl}} \leftarrow \text{resolve}(C_{\text{confl}}, C_{\text{reason}}, \ell)$;
  end
  $\rho \leftarrow \mathit{removeLast}(\rho)$;
end
$\ell \leftarrow$ literal in C_{confl} last falsified by ρ;
return $\text{roundToOne}(C_{\text{confl}}, \ell, \rho)$;

Division vs. Saturation

- Higher conflict speed when PB reasoning doesn't help [EN18]
- Seems to perform better when PB reasoning crucial [EGNV18]
- Keeps coefficients small — can do fixed-precision integer arithmetic
- But still equally hard to detect propagation
- And still degenerates to resolution for CNF inputs

Open Problems I: Some Implementation Challenges

1. Degrees of freedom in PB conflict analysis
   - Skip resolution steps when slack very negative?
   - How much to weaken?
   - Learn general PB constraints or more limited form?
2. Efficient propagation detection for PB constraints
3. Assessment of quality of learned constraints
4. Distance to backjump? (Constraint can be asserting at several levels)

Open Problems II: Some PB Reasoning Challenges

1. Better conflict analysis (also for CDCL)
   Is trivial resolution optimal, or can it pay to be smarter?
2. Natural way to recover from bad encodings (e.g., CNF)
3. Efficient and concise PB proof logging
4. Theoretical potential and limitations poorly understood [VEG+18]
   - Separations of subsystems of cutting planes?
   - In particular, is division strictly stronger than saturation?

Open Problems III: Beyond PB Reasoning

- Sometimes very poor performance even on LPs that are rationally infeasible! (And trivial for mixed integer linear programming solvers)
- But sometimes MIP solvers lost when learning from PB constraints crucial (and when conflict-driven PB solvers shine)
- Borrow techniques from (or merge with) MIP?

Summing up

- Conflict-driven search hugely successful SAT solving paradigm
- This talk: Survey how to port from CDCL to PB constraints
- Potential exponential performance gains haven't materialized so far
- Instead highly nontrivial challenges regarding
  - Efficient implementation
  - Theoretical understanding
- But no obvious reason why efficient PB solvers should not be possible (remember CDCL took 50 years)
- And in any case lots of fun questions to work on! ☺

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Thank you for your attention!

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