A Survey of Proof Complexity from a SAT Solving Perspective

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November 11, 2015

The Satisfiability Problem (SAT)

$$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$$

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- Variables should be set to true or false

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- Constraint $(x \vee \overline{y} \vee z)$: means x or z should be true or y false

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Or is it always the case that some constraint must fail to hold?

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Can we use computers to solve the SAT problem efficiently?

Computational Complexity Theory and SAT Solving

Complexity theory

- Satisfiability of formulas in propositional logic (SAT) foundational problem
- SAT proven NP-complete by Stephen Cook in 1971
- Hence most likely totally intractable
- Just remains to prove this — one of the million-dollar "Millennium Problems"

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Applied SAT solving

- Dramatic performance increase last 15–20 years
- State-of-the-art SAT solvers can deal with real-world formulas containing millions of variables
- But best solvers still based on methods from early 1960s
- Also, tiny formulas known that are totally beyond reach

SAT Solving and Proof Complexity

- How can state-of-the-art SAT solvers decide satisfiability of such huge formulas?
- Why do they work so well? And why do they sometimes miserably fail?
- Best current SAT solvers
  - Based on so-called conflict-driven clause learning (CDCL)
  - Sometimes algebraic reasoning (e.g., Gaussian elimination)
  - Sometimes geometric reasoning (e.g., cardinality constraints)
- How can we analyze the power of these methods?
  Question addressed by research area of **proof complexity**

Outline of This Presentation

This talk: overview of (or crash course in) proof complexity

Focus on connections with current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Algebraic Gröbner basis computations — polynomial calculus
- Geometric pseudo-Boolean solvers — cutting planes

Survey (some of) what is known about these proof systems

Show theoretical "benchmark formulas" used to understand potential and limitations of methods of reasoning

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- Mostly assume formulas k-CNFs (for simplicity of exposition)
  Conversion to 3-CNF (most often) doesn't change much
- N denotes size of formula (# literals, which is $\approx$ # clauses)

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\perp$ derived

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Can represent refutation as

- annotated list or
- directed acyclic graph

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6.	$x \vee \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
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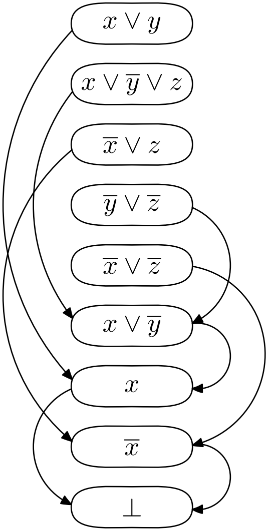

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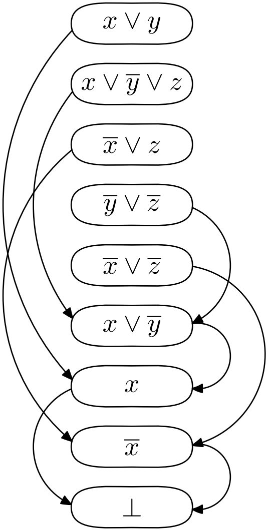
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length = # clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Examples of Hard Formulas w.r.t Resolution Length (1/3)

Pigeonhole principle (PHP) [Hak85]*
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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Even onto functional PHP formula is hard for resolution
"Resolution cannot count"

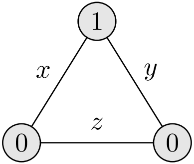
(*) List of full references given at the end of the slides (also available online)

Examples of Hard Formulas w.r.t Resolution Length (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



$$
\begin{aligned}
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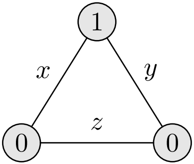

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Requires length $\exp(\Omega(N))$ on well-connected so-called expanders
"Resolution cannot count $\mathrm{mod}\ 2$**"**

Examples of Hard Formulas w.r.t Resolution Length (3/3)

Random k-CNF formulas [CS88]
Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

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Again lower bound $\exp(\Omega(N))$

And more. . .

- k-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera. . .

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Width $=$ size of largest clause in refutation (always $\leq N$)

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Proof: at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [ALN14])

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Width lower bound $\Rightarrow$ length lower bound

Much less obvious...

Width Lower Bounds Imply Length Lower Bounds

Theorem ([BW01])

$$\textit{length} \geq \exp\left(\Omega\left(\frac{(\textit{width})^2}{(\textit{formula size } N)}\right)\right)$$

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For tree-like resolution have length $\geq 2^{\text{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no length lower bounds — possible to tighten analysis? **No!**

Optimality of the Length-Width Lower Bound

Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j}$ = "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

Can also add "total order" axioms

$x_{i,j} \lor x_{j,i}$	totality; either $e_i < e_j$ or $e_j < e_i$

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Refutable in resolution in length $\mathcal{O}(N)$
Requires resolution width $\Omega(\sqrt[3]{N})$ (3-CNF version)

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

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8.	$\overline{x}$	Res$(3, 5)$
9.	$\bot$	Res$(7, 8)$

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

Example: Space at step 7 . . .

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
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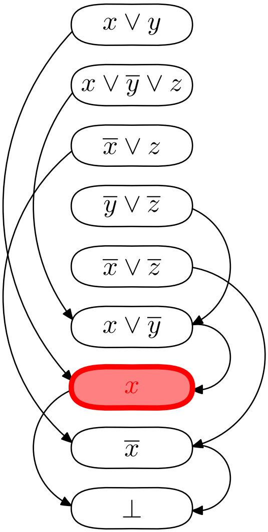

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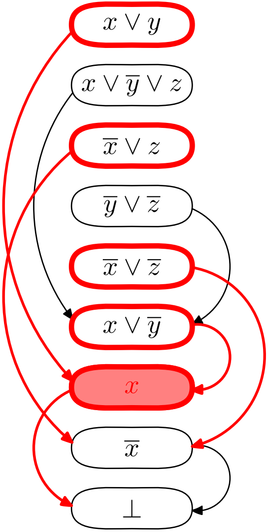
Example: Space at step 7 is 5



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Lower bounds subsequently proven for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random k-CNFs [BG03]

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Results always exactly matching width lower bounds
And proofs of very similar flavour...
Just a coincidence?

Space vs. Width

Theorem ([AD08])

$$\mathit{space} \geq \mathit{width} + \mathcal{O}(1)$$

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Pebbling formulas [BN08]

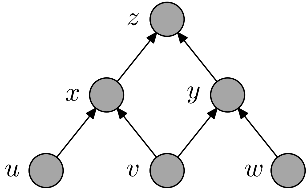
- Can be refuted in width $\mathcal{O}(1)$
- May require space $\Omega(N/\log N)$

A bit more involved to describe than previous benchmarks...

Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



- sources are true
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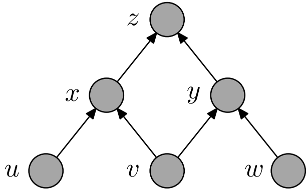

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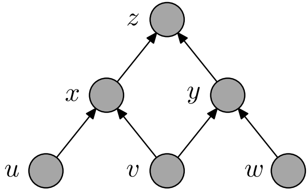

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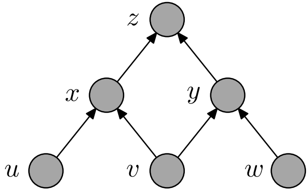

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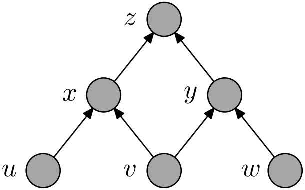

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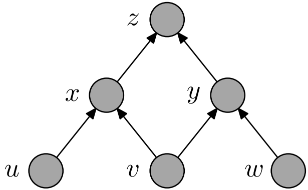

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Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas. **Except...**

Substituted Pebbling Formulas

Won't work — formulas are supereasy (solved by unit propagation)

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} &(x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ &(x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ &(\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ &(\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Now CNF formula inherits pebbling graph properties!

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- Space vs. width: **No!** [Ben09]
- Length vs. width: **No!** [Tha14]
- Length vs. space: Arguably most interesting case
  Length ≈ running time
  Space ≈ memory consumption
  SAT solvers aggressively try to minimize both

Length-Space Trade-offs

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

Holds for

- Substituted pebbling formulas over the right graphs
- Tseitin formulas over long, narrow rectangular grids

So no meaningful simultaneous optimization possible in worst case

Complexity Measures for Resolution: Summary

Recall that N = size of formula

Length

\# clauses in refutation — at most $\exp(N)$

Width

Size of largest clause in refutation — at most N

Space

Max # clauses one needs to remember when "verifying correctness of refutation" — at most N (!)

Proof Complexity Measures and CDCL Proof Search

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

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Recall $\log(\text{length}) \lesssim$ width $\lesssim$ space

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- Lower bound on running time for CDCL
- CDCL polynomially simulates resolution [PD11]
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Space

- In practice, memory consumption important bottleneck
- Space complexity gives lower bound on clause database size
- Plus assumes solver knows **exactly** which clauses to keep $\Rightarrow$ in reality, probably (much) more memory needed

Bridging the Gap Between Theory and Practice?

- CDCL hardness related to width and/or space?
  Preliminary work in [JMNŽ12] — no clear-cut answers
- Or is CDCL as good as general resolution?
  Are [PD11] and [AFT11] results "true in practice"? Doubt it
- CDCL explores only small part of resolution search space —
  Can time-space trade-offs in this talk occur in principle? Yes
- Do such time-space trade-offs occur in practice?
  Great question — on our to-do list

Not all mathematically well-defined questions. . .

Still possible to do experiments and draw interesting conclusions?

Using Theoretical Benchmarks to Shed Light on CDCL?

CDCL performance on theory benchmarks can be surprising:

- Sometimes worse behaviour with heuristics than without
  Pigeonhole principle formulas [Hak85]
- Sometimes "easy" formulas harder than "hard" ones?!
  Zero-one designs [VS10, MN14]
- Sometimes minor changes in internals makes all the difference between supereasy and totally impossible
  Ordering principle formulas [Stå96, BG01]

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Open Problems

- *Could explanations of above phenomena help us understand CDCL better?*
- *Could experiments on easily scalable theoretical benchmarks yield other interesting insights?*

Polynomial Calculus

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
Example: $x \lor y \lor \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

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(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Size, Degree and Space

Clauses turn into monomials

Write out all polynomials as sums of monomials

W.l.o.g. all polynomials multilinear (because of Boolean axioms)

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Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length
total # monomials in refutation counted with repetitions

Degree — analogue of resolution width
largest degree of monomial in refutation

(Monomial) space — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus simulates resolution efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

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Open Problem

Decide whether polynomial calculus is strictly stronger than resolution w.r.t. space

Size vs. Degree

- Degree upper bound ⇒ size upper bound [CEI96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [ALN14]

- Degree lower bound ⇒ size lower bound [IPS99]
  Precursor of [BW01] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [GL10]
  Example: same ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery much less developed)

Examples of Hard Formulas w.r.t. Size (and Degree)

Pigeonhole principle formulas

Follows from [AR03]

Earlier work on other encodings in [Raz98, IPS99]

Hard even with functionality axioms added [MN15]

Tseitin formulas with "wrong modulus"

Can define Tseitin-like formulas counting $\mathrm{mod}\ p$ for $p \neq 2$

Hard if $p \neq$ characteristic of field [BGIP01]

Random k-CNF formulas

Hard in all characteristics except 2 [BI99]

Lower bound for all characteristics in [AR03]

Polynomial Calculus Space

Monomial space lower bounds for

- pigeonhole principle [ABRW02]
- Random k-CNFs [BG15, BBG$^+$15]
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Open Problems

Prove polynomial calculus space lower bounds on

- *Tseitin formulas on any expander*
- *3-CNF version of PHP formulas*

Open Problem (analogue of [AD08])

Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?

Trade-offs for Polynomial Calculus

- Strong, essentially optimal space-degree trade-offs [BNT13]
  Same formulas as for resolution — same parameters
- Strong size-space trade-offs [BNT13]
  Same formulas as for resolution — some loss in parameters

Open Problem

Are there size-degree trade-offs in polynomial calculus?

[Tha14] works only for resolution (so far)

Algebraic SAT Solvers?

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- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work (counts #satisfying assignments — we just want to know whether $\neq 0$)

Cutting Planes

Introduced in [CCT87]

Clauses interpreted as linear inequalities over the reals with integer coefficients

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

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No (useful) analogue of width/degree

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- is strictly stronger w.r.t. space — can refute any CNF in constant space 5 (!) [GPT15] (But coefficients will be exponentially large — what if also coefficient size counted?)

Hard Formulas w.r.t Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with a k-clique is not $(k-1)$-colourable"

Lower bound via interpolation and circuit complexity

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Lower bound via interpolation and circuit complexity

Open Problems

Prove length lower bounds for cutting planes

- *for Tseitin formulas*
- *for random k-CNFs*
- *for any formula using other technique than interpolation*

Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of Tseitin formulas on expanders require large space [GP14]
  (But such short refutations probably don't exist anyway)
- Short cutting planes refutations of (some) pebbling formulas require large space [HN12, GP14] (such refutations exist)

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Open Problems

- *Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial size)*
- *Are there space lower bounds for CP refutations with polynomial-size coefficients?*

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- But given helpful encoding, solvers can do really well (e.g., PHP formulas and zero-one designs) [BBLM14]
- **Roadblock 2(?):** Solvers seem inefficient for systems of inequalities that have rational but not integral solutions (too limited form of division?)
- Not well understood at all — work in progress

Summing up This Presentation

Overview of resolution, polynomial calculus and cutting planes
(More details in survey paper [Nor15])

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Thank you for your attention!

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