Current Research in Proof Complexity: Lecture 2
Length and Width in Resolution

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Goal of Today's Lecture

- Study resolution (some repetition, some new stuff)
- Introduce new measure of proof width
- Show that to obtain lower lower bounds on proof length, it is sufficient to prove (strong) lower bounds on proof width
- Sets the stage for proving exponential lower bounds on resolution proof length (next lecture)

Outline

1. Recap and Some More About Resolution
   - Resolution Basics
   - Proof Length
   - Two Useful Tools
2. Resolution Width
   - Definition of Width
   - Two Technical Lemmas
   - Width is Upper-Bounded by Length

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \ldots \vee a_k$: set of literals
  At most k literals: k-clause
- CNF formula $F = C_1 \wedge \ldots \wedge C_m$: set of clauses
  k-CNF formula: CNF formula consisting of k-clauses
- $Vars(\cdot)$: set of variables in clause or formula
  $Lit(\cdot)$: set of literals in clause or formula
- $F \vDash D$: semantical implication, $\alpha(F)$ true $\Rightarrow \alpha(D)$ true for all truth value assignments α
- $[n] = \{1, 2, \ldots, n\}$

Resolution Revisited

Last time we talked about resolution refutation as sequence of clause configurations $\{\mathbb{D}_0, \ldots, \mathbb{D}_\tau\}$

For all $t \in [\tau]$, the set $\mathbb{D}_t$ is obtained from $\mathbb{D}_{t-1}$ by one of the following derivation steps:

Download $\mathbb{D}_t = \mathbb{D}_{t-1} \cup \{C\}$ for axiom clause $C \in F$

Inference $\mathbb{D}_t = \mathbb{D}_{t-1} \cup \{D\}$ for D inferred by resolution on clauses in $\mathbb{D}_{t-1}$.

Erasure $\mathbb{D}_t = \mathbb{D}_{t-1} \setminus \{D\}$ for some $D \in \mathbb{D}_{t-1}$.

But if we don't care about space, then we can view a resolution refutation as simply a listing of the clauses (i.e., no erasures)

Resolution Proof System (Ignoring Space)

Resolution derivation $\pi : F \vdash A$ of clause A from F:
Sequence of clauses $\pi = \{D_1, \ldots, D_s\}$ such that $D_s = A$ and each line D_i, $1 \leq i \leq s$, is either

- a clause $C \in F$ (an axiom)
- a resolvent derived from clauses D_j, D_k in π (with $j, k < i$) by the resolution rule

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

  resolving on the variable x

Resolution refutation of CNF formula F:
Derivation of empty clause $\perp$ (clause with no literals) from F

Example Resolution Refutation

$$F = (x \vee z) \wedge (\overline{z} \vee y) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res$(1,2)$
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res$(3,4)$
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res$(5,6)$
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res$(7,8)$
5.	$u \vee v$	Axiom	13.	x	Res$(9,10)$
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11,12)$
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res$(13,14)$
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

Resolution Sound and Complete

Resolution is sound and implicationally complete.

Sound If there is a resolution derivation $\pi : F \vdash A$ then $F \vDash A$

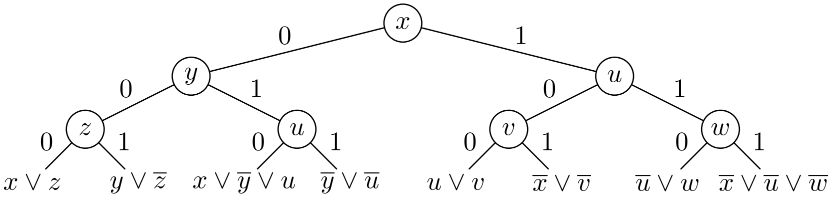
Complete If $F \vDash A$ then there is a resolution derivation $\pi : F \vdash A'$ for some $A' \subseteq A$.

In particular:

$$F \text{ is unsatisfiable} \Leftrightarrow \exists \text{ resolution refutation of } F$$

Completeness of Resolution: Proof by Example

Decision tree:



Resulting resolution refutation:

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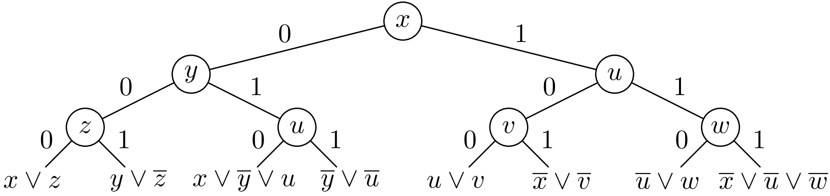

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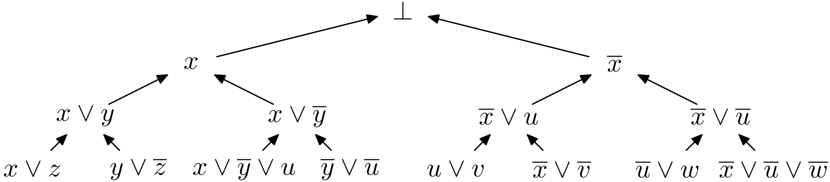

Derivation Graph and Tree-Like Derivations

Derivation graph G_π of a resolution derivation π:
directed acyclic graph (DAG) with

- vertices: clauses of the derivations
- edges: from $B \vee x$ and $C \vee \overline{x}$ to $B \vee C$ for each application of the resolution rule

A resolution derivation π is tree-like if G_π is a tree
(We can make copies of axiom clauses to make G_π into a tree)

Example

Our example resolution proof is tree-like.
(The derivation graph is on the previous slide.)

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Length

- Length of derivation $\pi : F \vdash A$ is $\#$ clauses in π (with repetitions)
- Length of deriving A from F is

$$L(F \vdash A) = \min_{\pi : F \vdash A} \{L(\pi)\}$$

  where minimum taken over all derivations of A

- Length of deriving A from F in *tree-like resolution* is $L_T(F \vdash A)$ (min of all tree-like derivations)

1. $x \lor z$
2. $\overline{z} \lor y$
3. $x \lor \overline{y} \lor u$
4. $\overline{y} \lor \overline{u}$
5. $u \lor v$
6. $\overline{x} \lor \overline{v}$
7. $\overline{u} \lor w$
8. $\overline{x} \lor \overline{u} \lor \overline{w}$
9. $x \lor y$
10. $x \lor \overline{y}$
11. $\overline{x} \lor u$
12. $\overline{x} \lor \overline{u}$
13. x
14. $\overline{x}$
15. $\perp$

} Length 15

Exponential Lower Bound for Proof Length

Goal of this and next lecture to prove:

Theorem (Haken '85)

There is a family of unsatisfiable CNF formulas $\{F_n\}_{n=1}^{\infty}$ *of size* $\Theta(n^3)$ *such that* $L(F_n \vdash \bot) = \exp(\Omega(n))$.

Also known: general resolution is exponentially stronger than tree-like resolution [Bonet et al. '98, Ben-Sasson et al. '04]

Resolution widely used in practice anyway because of nice properties for proof search algorithms (but is probably not automatizable)

Theoretical point of view: we want to understand resolution

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Weakening

In proofs, sometimes convenient to add a derivation rule for weakening

$$\frac{B}{B \lor C}$$

(for arbitrary clauses B, C).

Proposition

Any resolution refutation $\pi : F \vdash \bot$ using weakening can be transformed into a refutation $\pi' : F \vdash \bot$ without weakening in at most the same length.

Proof.

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Restriction

Restriction ρ: partial truth value assignment
Represented as set of literals $\rho = \{a_1, \ldots, a_m\}$ set to true by ρ

For a clause C, the ρ-restriction of C is

$$C\restriction_\rho = \begin{cases} 1 & \text{if } \rho \cap Lit(C) \neq \emptyset \\ C \setminus \{\overline{a} \mid a \in \rho\} & \text{otherwise} \end{cases}$$

where 1 denotes the trivially true clause

For a formula F, define $F\restriction_\rho = \bigwedge_{C \in F} C\restriction_\rho$

For a derivation $\pi = \{D_1, \ldots, D_s\}$, define $\pi\restriction_\rho = \{D_1\restriction_\rho, \ldots, D_s\restriction_\rho\}$
(with all trivial clauses 1 removed)

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(with all trivial clauses 1 removed)

Example Restriction

$\pi =$

1.	$x \vee z$	Axiom in F
2.	$\overline{z} \vee y$	Axiom in F
3.	$x \vee \overline{y} \vee u$	Axiom in F
4.	$\overline{y} \vee \overline{u}$	Axiom in F
5.	$u \vee v$	Axiom in F
6.	$\overline{x} \vee \overline{v}$	Axiom in F
7.	$\overline{u} \vee w$	Axiom in F
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom in F
9.	$x \vee y$	Res(1, 2)
10.	$x \vee \overline{y}$	Res(3, 4)
11.	$\overline{x} \vee u$	Res(5, 6)
12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
13.	x	Res(9, 10)
14.	$\overline{x}$	Res(11, 12)
15.	$\bot$	Res(13, 14)

$\pi\restriction_x =$

1.	1	
2.	$\overline{z} \vee y$	Axiom in $F\restriction_x$
3.	1	
4.	$\overline{y} \vee \overline{u}$	Axiom in $F\restriction_x$
5.	$u \vee v$	Axiom in $F\restriction_x$
6.	$\overline{v}$	Axiom in $F\restriction_x$
7.	$\overline{u} \vee w$	Axiom in $F\restriction_x$
8.	$\overline{u} \vee \overline{w}$	Axiom in $F\restriction_x$
9.	1	
10.	1	
11.	u	Res(5, 6)
12.	$\overline{u}$	Res(7, 8)
13.	1	
14.	$\bot$	Res(11, 12)

Restrictions Preserve Resolution Derivations

Proposition

If $\pi : F \vdash A$ *is a resolution derivation and* ρ *is a restriction on* $Vars(F)$*, then* $\pi\restriction_\rho$ *is a derivation of* $A\restriction_\rho$ *from* $F\restriction_\rho$*, possibly using weakening.*

Proof.

Easy proof by induction over the resolution derivation. □

In particular, if $\pi : F \vdash \bot$ then $\pi\restriction_\rho$ can be transformed into a resolution refutation of $F\restriction_\rho$ *without weakening* in at most the same length as π.

Width

- Width $W(C)$ of clause C is $|C|$, i.e., # literals
- Width of formula F or derivation π is width of the widest clause in the formula / derivation
- Width of deriving A from F is

$$W(F \vdash A) = \min_{\pi : F \vdash A} \{ W(\pi) \}$$

(No difference between tree-like and general resolution)

Always $W(F \vdash \bot) \leq \left| Vars(F) \right|$

1. $x \vee z$
2. $\overline{z} \vee y$
3. $x \vee \overline{y} \vee u$
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12. $\overline{x} \vee \overline{u}$
13. x
14. $\overline{x}$
15. $\bot$

Width 3

Width and Length

A narrow resolution proof is necessarily short.

For a proof in width w, $\left(2 \cdot |Vars(F)|\right)^w$ is an upper bound on the number of possible clauses.

Ben-Sasson & Wigderson proved (sort of) that the converse also holds.

Theorem (Very informal)

If there is a short resolution refutation of F, then there is a resolution refutation in small width as well.

Making this theorem precise, and proving it, is today's goal.

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Technical Lemma 1

Lemma

If $W(F\lceil_x \vdash A) \le w$ *then* $W(F \vdash A \vee \overline{x}) \le \max\{w+1, W(F)\}$ *(possibly by use of the weakening rule).*

Proof.

- Suppose $\pi = \{D_1, \ldots, D_s\}$ derives A from $F\lceil_x$ in width $W(\pi) \le w$
- Start new derivation π' by listing all clauses in F
- Then list clauses in π but with literal $\overline{x}$ added to all clauses
- Claim: this makes π' into a legal derivation of $A \vee \overline{x}$ from F (possibly with weakening)
- Given this claim, obviously $W(\pi') \le \max\{w+1, W(F)\}$ and the last line in π' is $A \vee \overline{x}$ □

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If $W(F\restriction_x \vdash A) \leq w$ *then* $W(F \vdash A \vee \overline{x}) \leq \max\{w+1, W(F)\}$ *(possibly by use of the weakening rule).*

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Proof of Technical Lemma 1 (continued)

Proof of claim.

Need to show that each clause in π' can be derived from previous clauses by resolution and/or weakening.

First half of π' just listing axioms in F — clearly OK
Consider second half with clauses $D_i \lor \overline{x}$ for $D_i \in \pi$ with $\overline{x}$ added

Let $F_{\overline{x}} = \{C \in F \mid \overline{x} \in Lit(C)\}$ be set of all clauses of F containing $\overline{x}$

Three cases:

1. $D_i \in F_{\overline{x}}\restriction_x$: This means that $D_i \lor \overline{x} \in F$, which is OK
2. $D_i \in F\restriction_x \setminus F_{\overline{x}}\restriction_x$: This means that $D_i \in F$, so $D_i \lor \overline{x}$ can be derived by weakening
3. D_i not axiom in $F\restriction_x$: Then derived from $D_j, D_k \in \pi$ by resolution. By induction $D_j \lor \overline{x}$, $D_k \lor \overline{x} \in \pi'$ derivable; resolve to get $D_i \lor \overline{x}$ □

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Proof of Technical Lemma 1 (continued)

Proof of claim.

Need to show that each clause in π' can be derived from previous clauses by resolution and/or weakening.

First half of π' just listing axioms in F — clearly OK
Consider second half with clauses $D_i \lor \overline{x}$ for $D_i \in \pi$ with $\overline{x}$ added

Let $F_{\overline{x}} = \{C \in F \mid \overline{x} \in Lit(C)\}$ be set of all clauses of F containing $\overline{x}$

Three cases:

1. $D_i \in F_{\overline{x}}\restriction_x$: This means that $D_i \lor \overline{x} \in F$, which is OK
2. $D_i \in F\restriction_x \setminus F_{\overline{x}}\restriction_x$: This means that $D_i \in F$, so $D_i \lor \overline{x}$ can be derived by weakening
3. D_i not axiom in $F\restriction_x$: Then derived from $D_j, D_k \in \pi$ by resolution. By induction $D_j \lor \overline{x}$, $D_k \lor \overline{x} \in \pi'$ derivable; resolve to get $D_i \lor \overline{x}$ □

Technical Lemma 2

Lemma

If

- $W(F\restriction_x \vdash \bot) \le w - 1$ *and*
- $W(F\restriction_{\overline{x}} \vdash \bot) \le w$

then

- $W(F \vdash \bot) \le \max\{w, W(F)\}$.

Proof.

1. Derive $\overline{x}$ in width $\le w$ by Technical Lemma 1.
2. Resolve $\overline{x}$ with all clauses $C \in F$ containing literal x to get $F\restriction_{\overline{x}}$ in width $\le W(F)$.
3. Derive $\bot$ from $F\restriction_{\overline{x}}$ in width $\le w$ (by assumption). □

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Warm-Up: Tree-Like Resolution

Theorem (Ben-Sasson & Wigderson '99)

For tree-like resolution, the width of refuting a CNF formula F is bounded from above by

$$W(F \vdash \bot) \leq W(F) + \log_2 L_T(F \vdash \bot).$$

Corollary

For tree-like resolution, the length of refuting a CNF formula F is bounded from below by

$$L_T(F \vdash \bot) \geq 2^{(W(F \vdash \bot) - W(F))}.$$

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Proof for Tree-Like Resolution (1 / 2)

Proof by nested induction over b and # variables n that

$$L_T(F \vdash \bot) \leq 2^b \;\Rightarrow\; W(F \vdash \bot) \leq W(F) + b$$

Base cases:
$b = 0 \Rightarrow$ proof of length 1 $\Rightarrow$ empty clause $\bot \in F$
$n = 1 \Rightarrow$ formula over 1 variable, i.e., $x \wedge \overline{x} \Rightarrow \exists$ proof of width 1

Induction step:
Suppose for formula F with n variables that π is tree-like refutation in length $\leq 2^b$

Last step in refutation $\pi : F \vdash \bot$ is $\frac{x \quad \overline{x}}{\bot}$ for some x

Let π_x and $\pi_{\overline{x}}$ be the tree-like subderivations of x and $\overline{x}$, respectively

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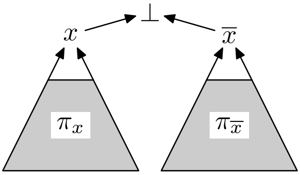
Proof for Tree-Like Resolution (2 / 2)

Since $L(\pi) = L(\pi_x) + L(\pi_{\overline{x}}) + 1 \leq 2^b$ (true since π is tree-like), one of π_x and $\pi_{\overline{x}}$ has length $\leq 2^{b-1}$

Suppose w.l.o.g. $L(\pi_{\overline{x}}) \leq 2^{b-1}$



1. $\pi_{\overline{x}}\restriction_x$ is a refutation of $F\restriction_x$ in length $\leq 2^{b-1}$
   $\Rightarrow$ by induction $W(F\restriction_x \vdash \bot) \leq W\big(F\restriction_x\big) + b - 1 \leq W(F) + b - 1$

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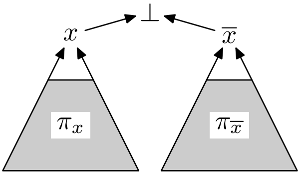

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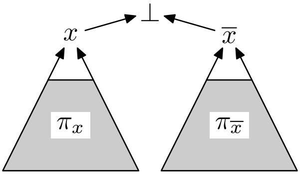

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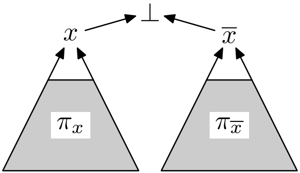

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Are Short Proofs Narrow?

- Proof on previous slide transform short refutation into narrow one
- But construction leads to exponential blow-up in length
- Look at step 2 in Tech Lemma 2 — every time we need $\overline{x}$ have to do whole derivation in step 1 again (because of tree-likeness)
- Potentially blows up length exponentially, and by [Ben-Sasson '02] this can't be avoided
- So short proofs are not narrow after all. . . (At least not tree-like proofs)

The General Case

Theorem (Ben-Sasson & Wigderson '99)

The width of refuting a CNF formula F over n variables in general resolution is bounded from above by

$$W(F \vdash \bot) \leq W(F) + \mathcal{O}\left(\sqrt{n \log L(F \vdash \bot)}\right).$$

Note: $2^{n+\mathcal{O}(1)}$ maximal possible proof length, so bound is

$$W(F \vdash \bot) \lessapprox W(F) + \sqrt{\log(\text{max possible}) \cdot \log L(F \vdash \bot)}$$

Kind of ugly bound — possible to do better?
Will return to this question in coming lecture

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The General Case: Corollary

Corollary

For general resolution, the length of refuting a CNF formula F over n variables is bounded from below by

$$L(F \vdash \bot) \geq \exp\left(\Omega\left(\frac{(W(F \vdash \bot) - W(F))^2}{n}\right)\right).$$

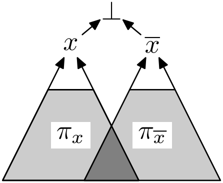
Has been used to simplify many length lower bound proofs in resolution (and to prove a couple of new ones)

Need $W(F \vdash \bot) - W(F) = \omega\left(\sqrt{n \log n}\right)$ to get superpolynomial bounds

(Not a) Proof of the General Case

Proof for tree-like resolution breaks down in general case

Not true that $L(\pi) = L(\pi_x) + L(\pi_{\overline{x}}) + 1$
Subderivations π_x and $\pi_{\overline{x}}$ may share clauses!



Instead

- Look at very wide clauses in π
- Eliminate many of them by applying restriction setting commonly occurring literal to true
- More complicated inductive argument
- Still exponential blow-up in length

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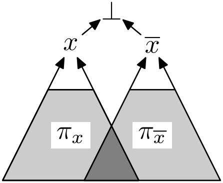

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An Intriguing Open Problem

Open Problem

Can the exponential length blow-up in [Ben-Sasson & Wigderson '99] for general resolution be avoided?

I.e., given short resolution refutation, can we find a refutation that is both narrow and short? (With at most polynomial blow-up, say)

Or is there a length-width trade-off, so that decreasing width must always increase length in worst case?

Would be very interesting to know the answer

And there are simpler variants of this open problem that would also be very interesting to solve

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Proof of the General Case

On the board
(if we have the time)

But we didn't, so see handwritten notes instead