Current Research in Proof Complexity: Lecture 4
Space and Width in Resolution

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Goal of Today's Lecture

- Study space in resolution and prove some basic facts
- Then once again take detour of studying width instead
- Show that width is a lower bound for space
- So optimal space lower bounds follow from width lower bounds
- Finally start discussing trade-offs between width and space
- Make detour into pebble games (for the first but not the last time)

Outline

1. Definition of Space
2. Some Basic Properties
3. Combinatorial Characterization of Width
4. Space is Greater than Width
5. Pebble Games and Pebbling Formulas

Introducing Space

- Memory usage is a major concern in practical SAT solving
- Question raised by Haken at workshop in Toronto 1998 whether proof complexity could say anything about this
- Formal measure of proof space introduced in [Esteban & Torán '99] (maximal # clauses in memory while verifying proof)
- Generalized and developed further in [Alekhnovich et al. '00] Name clause space adopted to distinguish from other space measures
- But we'll sometimes be sloppy — just "space" usually means "clause space" unless stated otherwise

Resolution Derivation (When We Care About Space)

Sequence of sets of clauses, or clause configurations, $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $\mathbb{C}_0 = \emptyset$ and $\mathbb{C}_t$ follows from $\mathbb{C}_{t-1}$ by:

Download $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C\}$ for clause $C \in F$ (axiom)

Erasure $\mathbb{C}_t = \mathbb{C}_{t-1} \setminus \{C\}$ for clause $C \in \mathbb{C}_{t-1}$

Inference $\mathbb{C}_t = \mathbb{C}_{t-1} \cup \{C \vee D\}$ for clause $C \vee D$ inferred by resolution rule from $C \vee x, D \vee \overline{x} \in \mathbb{C}_{t-1}$

Resolution derivation $\pi : F \vdash D$ of clause D from F:
Derivation $\{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ such that $D \in \mathbb{C}_\tau$

Resolution refutation of F:
Derivation $\pi : F \vdash \bot$ of empty clause $\bot$ from F

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	$\bot$	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
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6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor z$

Download axiom 1: $x \lor z$

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$x \lor z$

$\overline{z} \lor y$

Download axiom 1: $x \lor z$

Download axiom 2: $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

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8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor z$

$\overline{z} \lor y$

Download axiom 1: $x \lor z$

Download axiom 2: $\overline{z} \lor y$

Infer $x \lor y$ from

$x \lor z$ and $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
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$x \lor z$
$\overline{z} \lor y$
$x \lor y$

Download axiom 1: $x \lor z$
Download axiom 2: $\overline{z} \lor y$
Infer $x \lor y$ from
$x \lor z$ and $\overline{z} \lor y$

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$x \lor z$
$\overline{z} \lor y$
$x \lor y$

Download axiom 1: $x \lor z$
Download axiom 2: $\overline{z} \lor y$
Infer $x \lor y$ from
$x \lor z$ and $\overline{z} \lor y$
Erase the clause $x \lor z$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
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7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$\overline{z} \vee y$

$x \vee y$

Download axiom 1: $x \vee z$

Download axiom 2: $\overline{z} \vee y$

Infer $x \vee y$ from

$x \vee z$ and $\overline{z} \vee y$

Erase the clause $x \vee z$

Example (Our Favourite Resolution Refutation Again)

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$\overline{z} \lor y$

$x \lor y$

Download axiom 2: $\overline{z} \lor y$

Infer $x \lor y$ from

$x \lor z$ and $\overline{z} \lor y$

Erase the clause $x \lor z$

Erase the clause $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom
2.	$\overline{z} \lor y$	Axiom
3.	$x \lor \overline{y} \lor u$	Axiom
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5.	$u \lor v$	Axiom
6.	$\overline{x} \lor \overline{v}$	Axiom
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8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom
9.	$x \lor y$	Res(1, 2)
10.	$x \lor \overline{y}$	Res(3, 4)
11.	$\overline{x} \lor u$	Res(5, 6)
12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
13.	x	Res(9, 10)
14.	$\overline{x}$	Res(11, 12)
15.	$\perp$	Res(13, 14)

$x \lor y$

Download axiom 2: $\overline{z} \lor y$
Infer $x \lor y$ from
$x \lor z$ and $\overline{z} \lor y$
Erase the clause $x \lor z$
Erase the clause $\overline{z} \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
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8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$x \vee y$

$x \vee \overline{y} \vee u$

Infer $x \vee y$ from
$x \vee z$ and $\overline{z} \vee y$
Erase the clause $x \vee z$
Erase the clause $\overline{z} \vee y$
Download axiom 3: $x \vee \overline{y} \vee u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
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$x \vee y$
$x \vee \overline{y} \vee u$
$\overline{y} \vee \overline{u}$

$x \vee z$ and $\overline{z} \vee y$
Erase the clause $x \vee z$
Erase the clause $\overline{z} \vee y$
Download axiom 3: $x \vee \overline{y} \vee u$
Download axiom 4: $\overline{y} \vee \overline{u}$

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$x \lor y$
$x \lor \overline{y} \lor u$
$\overline{y} \lor \overline{u}$

Erase the clause $\overline{z} \lor y$
Download axiom 3: $x \lor \overline{y} \lor u$
Download axiom 4: $\overline{y} \lor \overline{u}$
Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

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$x \lor y$
$x \lor \overline{y} \lor u$
$\overline{y} \lor \overline{u}$
$x \lor \overline{y}$

Erase the clause $\overline{z} \lor y$
Download axiom 3: $x \lor \overline{y} \lor u$
Download axiom 4: $\overline{y} \lor \overline{u}$
Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

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$x \lor y$
$x \lor \overline{y} \lor u$
$\overline{y} \lor \overline{u}$
$x \lor \overline{y}$

Download axiom 3: $x \lor \overline{y} \lor u$
Download axiom 4: $\overline{y} \lor \overline{u}$
Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$
Erase the clause $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
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$x \lor y$
$\overline{y} \lor \overline{u}$
$x \lor \overline{y}$

Download axiom 3: $x \lor \overline{y} \lor u$
Download axiom 4: $\overline{y} \lor \overline{u}$
Infer $x \lor \overline{y}$ from
$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$
Erase the clause $x \lor \overline{y} \lor u$

Example (Our Favourite Resolution Refutation Again)

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4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
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7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$x \vee y$
$\overline{y} \vee \overline{u}$
$x \vee \overline{y}$

Download axiom 4: $\overline{y} \vee \overline{u}$
Infer $x \vee \overline{y}$ from
$x \vee \overline{y} \vee u$ and $\overline{y} \vee \overline{u}$
Erase the clause $x \vee \overline{y} \vee u$
Erase the clause $\overline{y} \vee \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
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7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor y$

$x \lor \overline{y}$

Download axiom 4: $\overline{y} \lor \overline{u}$

Infer $x \lor \overline{y}$ from

$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$

Erase the clause $x \lor \overline{y} \lor u$

Erase the clause $\overline{y} \lor \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
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8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor y$
$x \lor \overline{y}$

$x \lor \overline{y} \lor u$ and $\overline{y} \lor \overline{u}$
Erase the clause $x \lor \overline{y} \lor u$
Erase the clause $\overline{y} \lor \overline{u}$
Infer x from
$x \lor y$ and $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
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$x \vee y$
$x \vee \overline{y}$
x

$x \vee \overline{y} \vee u$ and $\overline{y} \vee \overline{u}$
Erase the clause $x \vee \overline{y} \vee u$
Erase the clause $\overline{y} \vee \overline{u}$
Infer x from
$x \vee y$ and $x \vee \overline{y}$

Example (Our Favourite Resolution Refutation Again)

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2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

$x \vee y$
$x \vee \overline{y}$
x

Erase the clause $x \vee \overline{y} \vee u$
Erase the clause $\overline{y} \vee \overline{u}$
Infer x from
$x \vee y$ and $x \vee \overline{y}$
Erase the clause $x \vee y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor \overline{y}$

x

Erase the clause $x \lor \overline{y} \lor u$

Erase the clause $\overline{y} \lor \overline{u}$

Infer x from

$x \lor y$ and $x \lor \overline{y}$

Erase the clause $x \lor y$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

$x \lor \overline{y}$

x

Erase the clause $\overline{y} \lor \overline{u}$

Infer x from

$x \lor y$ and $x \lor \overline{y}$

Erase the clause $x \lor y$

Erase the clause $x \lor \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\bot$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x

Erase the clause $\overline{y} \vee \overline{u}$
Infer x from
$x \vee y$ and $x \vee \overline{y}$
Erase the clause $x \vee y$
Erase the clause $x \vee \overline{y}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x

$u \vee v$

Infer x from

$x \vee y$ and $x \vee \overline{y}$

Erase the clause $x \vee y$

Erase the clause $x \vee \overline{y}$

Download axiom 5: $u \vee v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x

$u \vee v$

$\overline{x} \vee \overline{v}$

$x \vee y$ and $x \vee \overline{y}$

Erase the clause $x \vee y$

Erase the clause $x \vee \overline{y}$

Download axiom 5: $u \vee v$

Download axiom 6: $\overline{x} \vee \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x

$u \vee v$

$\overline{x} \vee \overline{v}$

Erase the clause $x \vee \overline{y}$

Download axiom 5: $u \vee v$

Download axiom 6: $\overline{x} \vee \overline{v}$

Infer $\overline{x} \vee u$ from

$u \vee v$ and $\overline{x} \vee \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$u \lor v$

$\overline{x} \lor \overline{v}$

$\overline{x} \lor u$

Erase the clause $x \lor \overline{y}$

Download axiom 5: $u \lor v$

Download axiom 6: $\overline{x} \lor \overline{v}$

Infer $\overline{x} \lor u$ from

$u \lor v$ and $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$u \lor v$

$\overline{x} \lor \overline{v}$

$\overline{x} \lor u$

Download axiom 5: $u \lor v$

Download axiom 6: $\overline{x} \lor \overline{v}$

Infer $\overline{x} \lor u$ from

$u \lor v$ and $\overline{x} \lor \overline{v}$

Erase the clause $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\bot$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor \overline{v}$

$\overline{x} \lor u$

Download axiom 5: $u \lor v$

Download axiom 6: $\overline{x} \lor \overline{v}$

Infer $\overline{x} \lor u$ from

$u \lor v$ and $\overline{x} \lor \overline{v}$

Erase the clause $u \lor v$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1,2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3,4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5,6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7,8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9,10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11,12)$
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res$(13,14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor \overline{v}$

$\overline{x} \lor u$

Download axiom 6: $\overline{x} \lor \overline{v}$

Infer $\overline{x} \lor u$ from

$u \lor v$ and $\overline{x} \lor \overline{v}$

Erase the clause $u \lor v$

Erase the clause $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor u$

Download axiom 6: $\overline{x} \lor \overline{v}$
Infer $\overline{x} \lor u$ from
$u \lor v$ and $\overline{x} \lor \overline{v}$
Erase the clause $u \lor v$
Erase the clause $\overline{x} \lor \overline{v}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\bot$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor u$

$\overline{u} \lor w$

Infer $\overline{x} \lor u$ from

$u \lor v$ and $\overline{x} \lor \overline{v}$

Erase the clause $u \lor v$

Erase the clause $\overline{x} \lor \overline{v}$

Download axiom 7: $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res$(1, 2)$
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res$(3, 4)$
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res$(5, 6)$
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res$(7, 8)$
5.	$u \lor v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res$(13, 14)$
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor u$

$\overline{u} \lor w$

$\overline{x} \lor \overline{u} \lor \overline{w}$

$u \lor v$ and $\overline{x} \lor \overline{v}$

Erase the clause $u \lor v$

Erase the clause $\overline{x} \lor \overline{v}$

Download axiom 7: $\overline{u} \lor w$

Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor u$

$\overline{u} \lor w$

$\overline{x} \lor \overline{u} \lor \overline{w}$

Erase the clause $\overline{x} \lor \overline{v}$

Download axiom 7: $\overline{u} \lor w$

Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$

Infer $\overline{x} \lor \overline{u}$ from

$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x
$\overline{x} \lor u$
$\overline{u} \lor w$
$\overline{x} \lor \overline{u} \lor \overline{w}$
$\overline{x} \lor \overline{u}$

Erase the clause $\overline{x} \lor \overline{v}$
Download axiom 7: $\overline{u} \lor w$
Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$
Infer $\overline{x} \lor \overline{u}$ from
$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x
$\overline{x} \lor u$
$\overline{u} \lor w$
$\overline{x} \lor \overline{u} \lor \overline{w}$
$\overline{x} \lor \overline{u}$

Download axiom 7: $\overline{u} \lor w$
Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$
Infer $\overline{x} \lor \overline{u}$ from
$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$
Erase the clause $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x
$\overline{x} \lor u$
$\overline{x} \lor \overline{u} \lor \overline{w}$
$\overline{x} \lor \overline{u}$

Download axiom 7: $\overline{u} \lor w$
Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$
Infer $\overline{x} \lor \overline{u}$ from
$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$
Erase the clause $\overline{u} \lor w$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res$(1, 2)$
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res$(3, 4)$
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res$(5, 6)$
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res$(7, 8)$
5.	$u \vee v$	Axiom	13.	x	Res$(9, 10)$
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res$(11, 12)$
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res$(13, 14)$
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x
$\overline{x} \vee u$
$\overline{x} \vee \overline{u} \vee \overline{w}$
$\overline{x} \vee \overline{u}$

Download axiom 8: $\overline{x} \vee \overline{u} \vee \overline{w}$
Infer $\overline{x} \vee \overline{u}$ from
$\overline{u} \vee w$ and $\overline{x} \vee \overline{u} \vee \overline{w}$
Erase the clause $\overline{u} \vee w$
Erase the clause $\overline{x} \vee \overline{u} \vee \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
3.	$x \lor \overline{y} \lor u$	Axiom	11.	$\overline{x} \lor u$	Res(5, 6)
4.	$\overline{y} \lor \overline{u}$	Axiom	12.	$\overline{x} \lor \overline{u}$	Res(7, 8)
5.	$u \lor v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \lor \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \lor w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \lor \overline{u} \lor \overline{w}$	Axiom			

x

$\overline{x} \lor u$

$\overline{x} \lor \overline{u}$

Download axiom 8: $\overline{x} \lor \overline{u} \lor \overline{w}$

Infer $\overline{x} \lor \overline{u}$ from

$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$

Erase the clause $\overline{u} \lor w$

Erase the clause $\overline{x} \lor \overline{u} \lor \overline{w}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \vee z$	Axiom	9.	$x \vee y$	Res(1, 2)
2.	$\overline{z} \vee y$	Axiom	10.	$x \vee \overline{y}$	Res(3, 4)
3.	$x \vee \overline{y} \vee u$	Axiom	11.	$\overline{x} \vee u$	Res(5, 6)
4.	$\overline{y} \vee \overline{u}$	Axiom	12.	$\overline{x} \vee \overline{u}$	Res(7, 8)
5.	$u \vee v$	Axiom	13.	x	Res(9, 10)
6.	$\overline{x} \vee \overline{v}$	Axiom	14.	$\overline{x}$	Res(11, 12)
7.	$\overline{u} \vee w$	Axiom	15.	$\perp$	Res(13, 14)
8.	$\overline{x} \vee \overline{u} \vee \overline{w}$	Axiom			

x

$\overline{x} \vee u$

$\overline{x} \vee \overline{u}$

$\overline{u} \vee w$ and $\overline{x} \vee \overline{u} \vee \overline{w}$

Erase the clause $\overline{u} \vee w$

Erase the clause $\overline{x} \vee \overline{u} \vee \overline{w}$

Infer $\overline{x}$ from

$\overline{x} \vee u$ and $\overline{x} \vee \overline{u}$

Example (Our Favourite Resolution Refutation Again)

1.	$x \lor z$	Axiom	9.	$x \lor y$	Res(1, 2)
2.	$\overline{z} \lor y$	Axiom	10.	$x \lor \overline{y}$	Res(3, 4)
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x
$\overline{x} \lor u$
$\overline{x} \lor \overline{u}$
$\overline{x}$

$\overline{u} \lor w$ and $\overline{x} \lor \overline{u} \lor \overline{w}$
Erase the clause $\overline{u} \lor w$
Erase the clause $\overline{x} \lor \overline{u} \lor \overline{w}$
Infer $\overline{x}$ from
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$\bot$

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Infer $\bot$ from
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Clause Space

- Clause space of resolution derivation $\pi = \{\mathbb{C}_0, \ldots, \mathbb{C}_\tau\}$ is max # clauses in any configuration

$$Sp(\pi) = \max_{t \in [\tau]} \{ |\mathbb{C}_t| \}$$

- Clause space of deriving D from F is

$$Sp(F \vdash D) = \min_{\pi : F \vdash D} \{ Sp(\pi) \}$$

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As for length, space measures in general and tree-like resolution differ

We concentrate on the interesting case: general resolution

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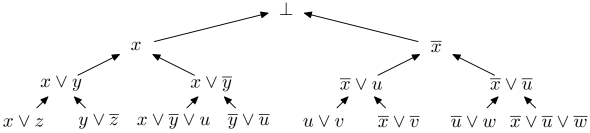
Space $\lessapprox$ # variables

Consider decision tree for F



n variables $\Rightarrow$ height of decision tree at most n

By induction: Clause at root of subtree of height h derivable in space $h+2$

- Derive left child clause in space $h+1$ and keep in memory
- Derive right child clause in space $1+(h+1)$
- Resolve the two children clauses to get root clause

(See example derivation a few slides back)

Theorem

$Sp(F \vdash \perp) \leq \left|Vars(F)\right| + 2$

Space $\lesssim$ # variables

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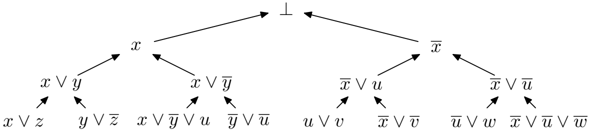

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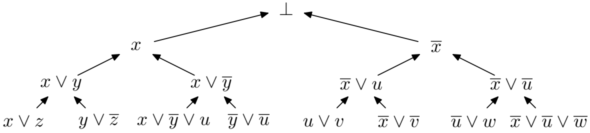

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Minimally Unsatisfiable CNF formula

Definition

An unsatisfiable CNF formula F is minimally unsatisfiable if removing any clause from F makes it satisfiable.

Example

$$F = (x \vee z) \wedge (\overline{z} \vee y) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

is minimally unsatisfiable (but tedious to verify)

$$F\restriction_x = (\overline{z} \vee y) \wedge (\overline{y} \vee \overline{u}) \wedge (u \vee v) \wedge \overline{v} \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$$

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A Theorem About Minimally Unsatisfiable CNFs

Theorem (Tarsi's lemma)

Any minimally unsatisfiable CNF formula must have more clauses than variables.

The proof uses matching arguments, so Hall will come in handy again

Theorem (Hall's Marriage Theorem)

Let $G = (U \dot{\cup} V, E)$ be a bipartite graph. Then there is a matching of U into V if and only if for all subsets $U' \subseteq U$ it holds that $|N(U')| \geq |U'|$.

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Proof of Tarsi's lemma

- Bipartite graph with clauses F left, variables V right, edge if variable occurs in clause (ignore signs)
- Unsatisfiable $\Rightarrow$ no matching
- So by Hall's thm $\exists\ S \subseteq F$ with $|S| > N(S)$ — fix maximal such S
- If $S = F$ we're done, so suppose not
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- If $S' \subseteq F \setminus S$ then by maximality $|S'| \leq |N(S') \setminus N(S)|$
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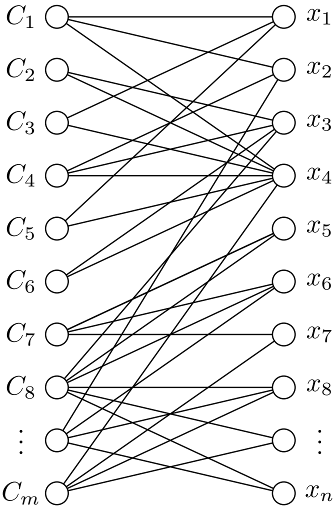
- Whole F satisfiable $\Rightarrow$ contradiction



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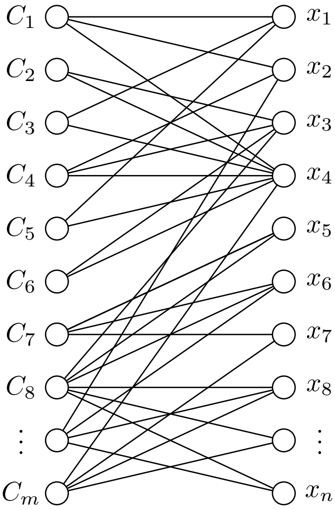

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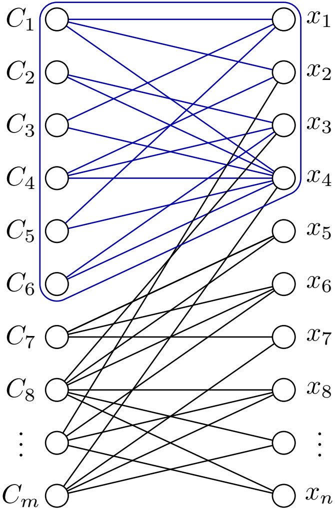

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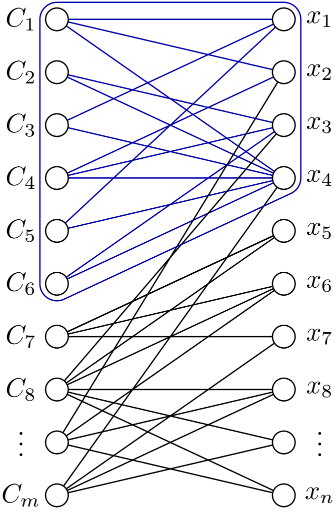

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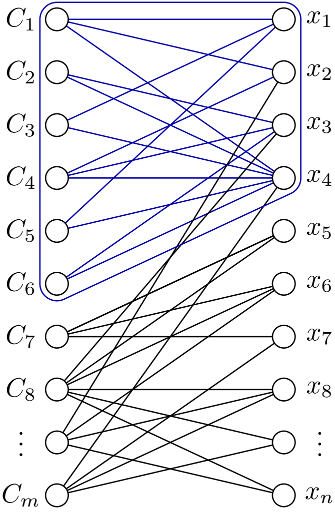

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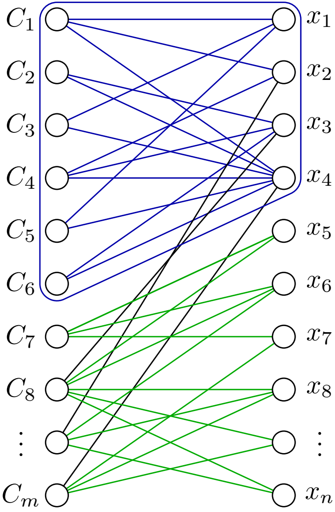

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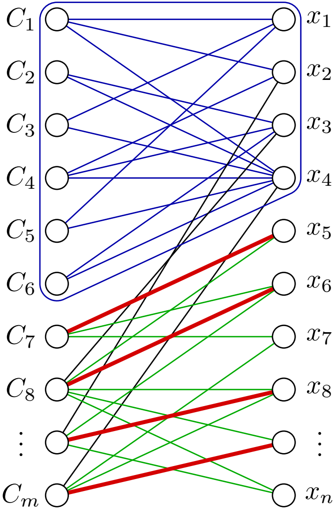

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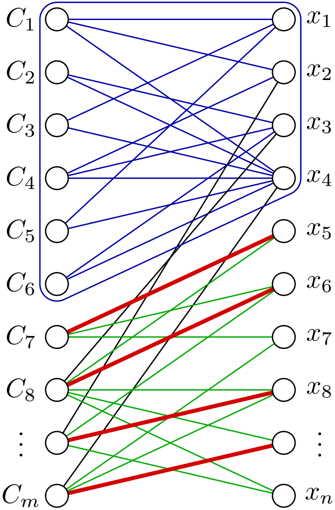

Space $\lessapprox$ # clauses

Theorem

$Sp(F \vdash \bot) \leq |F| + 1$

Proof.

- Pick minimally unsatisfiable $F' \subseteq F$
- We know $\left|F'\right| > \left|Vars(F')\right|$
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Proof.

- Pick minimally unsatisfiable $F' \subseteq F$
- We know $\left|F'\right| > \left|Vars(F')\right|$
- Use bound in terms of # variables to get refutation in space $\leq \left|Vars(F')\right| + 2 \leq \left|F'\right| + 1 \leq \left|F\right| + 1$ □

The Parameter Range of Interest for Space

We just showed

$$Sp(F \vdash \bot) \leq \mathsf{min}\{|F| + 1, |Vars(F)| + 2\}$$

So clause space at most linear

The interesting questions is:

- Which formulas require this much space? (Are there such formulas?)
- Which formulas can be refuted in, say, just logarithmic space?
- Or even constant space?

Tight Lower Bounds on Clause Space

Theorem (Alekhnovich et al. '00, Torán '99)

There is a polynomial-size family $\{F_n\}_{n=1}^{\infty}$ of unsatisfiable 3-CNF formulas such that $Sp(F \vdash \bot) = \Omega(|F|) = \Omega(|Vars(F)|)$.

But the history of clause space lower bounds is interesting

- PHP formulas? — same lower bound as for width
- Tseitin formulas? — same lower bound as for width
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A Natural Conjecture

Conjecture

Could it be that width is always a lower bound on clause space?

Remained open...
But seemed more and more plausible

Until one day...

A beautiful paper out of the blue that

- Resolved this question completely (the answer is "yes")
- Did so by providing elegant combinatorial characterization of width
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Informal Description of Existential Pebble Game

Game between Spoiler and Duplicator over CNF formula F

Duplicator claims formula is satisfiable

Spoiler wants to disprove this, but suffers from light, selective senility (can only keep p variable assignments in memory)

In each round, Spoiler

- picks a variable to which Duplicator must assign a value, or
- forgets a variable (can choose which)

In each round, Duplicator

- assigns value to chosen variable to get partial assignment to variables in Spoiler's memory not falsifying F, or
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Formal Definition

Duplicator wins the Boolean existential p-pebble game over the CNF formula F if there is a nonempty family $\mathcal{H}$ of partial truth value assignments that do not falsify any clause in F and for which the following holds:

1. If $\alpha \in \mathcal{H}$ then $|\alpha| \leq p$.
2. If $\alpha \in \mathcal{H}$ and $\beta \subseteq \alpha$ then $\beta \in \mathcal{H}$.
3. If $\alpha \in \mathcal{H}$, $|\alpha| < p$ and $x \in Vars(F)$ then there exists a $\beta \in \mathcal{H}$ such that $\alpha \subseteq \beta$ and x is in the domain of β.

$\mathcal{H}$ is called a winning strategy for Duplicator.

If there is no winning strategy for Duplicator, Spoiler wins the game.

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Constructive Strategies

If there is a winning strategy for Duplicator, then there is a deterministic winning strategy that for each $\alpha \in \mathcal{H}$ and each move of Spoiler defines a move β for Duplicator.

Proposition

If Duplicator has no winning strategy, then there is a winning strategy (in the form of a partial function from partial truth value assignments to variable queries/deletions) for Spoiler.

Proof sketch.

The number of possible deterministic strategies for Duplicator is finite, so Spoiler can build a strategy by evaluating all possible responses to sequences of queries and deletions. □

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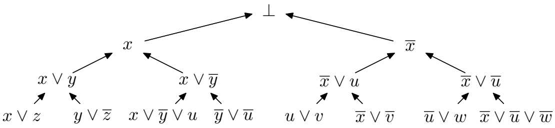
Existential Pebble Game Characterizes Width

It turns out that the Boolean existential p-pebble game exactly characterizes resolution width.

Theorem (Atserias & Dalmau '03)

The CNF formula F has a resolution refutation of width $\leq p$
if and only if
Spoiler wins the existential $(p+1)$-pebble game on F.

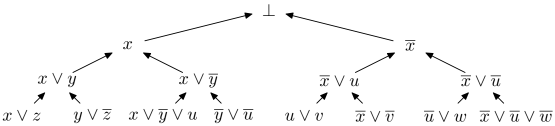
Narrow Proof Yields Winning Strategy for Spoiler



- Given $\pi : F \vdash \bot$ with DAG G_π.
- Spoiler starts at the vertex for $\bot$ and inductively queries the variable resolved upon to to get there
- Spoiler moves to the assumption clause D falsified by Duplicator's answer and forgets all variables not in D
- Repeat for the new clause et cetera
- Sooner or later Spoiler reaches a falsified axiom, having used no more than $W(\pi) + 1$ variables simultaneously (+1 is for the variable resolved on)

Narrow Proof Yields Winning Strategy for Spoiler

- Given $\pi : F \vdash \perp$ with DAG G_π.



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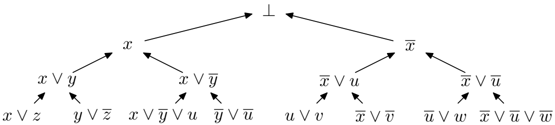

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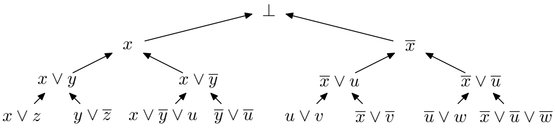

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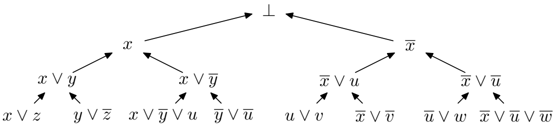

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Winning Strategy for Spoiler Yields Narrow Proof

Given strategy for Spoiler, build DAG G_π as follows:

- Start with $\perp$ vertex. For x the first variable queried, make vertices x, $\overline{x}$ with edges to $\perp$.
- Inductively, let ρ_v be the unique minimal partial truth value assignment falsifying the clause D_v at v.
- If move on ρ_v is deletion of y, make new vertex $D_v \setminus \{y, \overline{y}\}$ with edge to D_v. Otherwise, if y is queried, make new vertices $D \vee y$, $D \vee \overline{y}$ with edges to D.
- In the (finite) DAG G constructed, all sources are (weakenings of) axioms of F, and by induction G describes a resolution derivation with weakening.
- If we eliminate the weakening we get a derivation in width at most p, since if $|\rho_v| = p + 1$ the next move for Spoiler must be a deletion.

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Spoiler Strategy for Tight Proofs

The lower bound on space in terms of width follows from the fact that Spoiler can use proofs in small space to construct winning strategies with few pebbles.

Lemma

Let F be an unsatisfiable CNF formula with

- $W(F) = w$ *and*
- $Sp(F \vdash \bot) = s$.

Then

- *Spoiler wins the existential* $(s+w-2)$*-pebble game on* F.

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Proof of Lemma (1 / 2)

Given: proof $\pi = \{\mathbb{C}_0 = \emptyset, \mathbb{C}_1, \ldots, \mathbb{C}_\tau\}$ with $\bot \in \mathbb{C}_\tau$ in space s

Spoiler constructs a strategy by inductively defining
partial truth value assignments ρ_t such that
ρ_t satisfies $\mathbb{C}_t$ by setting (at most) one literal per clause to true.

W.l.o.g. axiom downloads occur only for $\mathbb{C}_t$ of size $|\mathbb{C}_t| \leq s - 2$.

One memory slot must be saved for resolvent, otherwise next step will be an erasure and we can reverse the order of these two derivation steps

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Proof of Lemma (2 / 2)

- At download of $C \in F$, Spoiler queries Duplicator about all variables in C and keep the literal satisfying it, using at most $(s-2)+w$ pebbles.
- When a clause is deleted, Spoiler deletes the corresponding literal satisfying the clause from ρ_t if necessary (i.e., if $|\rho_t| = |\mathbb{C}_t|$).
- For inference steps, Spoiler sets $\rho_t = \rho_{t-1}$ since by induction ρ_{t-1} must satisfy the resolvent.

Now ρ_τ cannot satisfy $\mathbb{C}_\tau$ since $\bot \in \mathbb{C}_\tau$, so Duplicator must fail at some time prior to τ.

Thus Spoiler has a winning strategy with $\leq (s-2)+w$ pebbles.

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Lower Bound on Space in Terms of Width

Theorem (Atserias & Dalmau '03)

For any unsatisfiable k-CNF formula F (k fixed) it holds that

$$Sp(F \vdash \bot) - 3 \geq W(F \vdash \bot) - W(F).$$

Proof.

Combine the facts that:

- If Spoiler wins the existential $(p+1)$-pebble game on F, then $W(F \vdash \bot) \leq p$.
- If $W(F) = w$ and $Sp(F \vdash \bot) = s$, then Spoiler wins the existential $(s+w-2)$-pebble game on F.

It follows that $W(F \vdash \bot) \leq Sp(F \vdash \bot) + W(F) - 3$. □

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Some Interesting Corollaries

- This theorem allows us to rederive all optimal (i.e., linear) lower bounds on space known
- Namely, just use lower bounds on width in [Ben-Sasson & Wigderson] and then the space $\geq$ width inequality from [Atserias & Dalmau]
- Natural question: Do space and width always coincide?
- Will get back to this question later in the course

Another Interesting Corollary

If a k-CNF formula is easy w.r.t. space, it is also easy w.r.t length

Corollary

If a k-CNF formula F over n variables is refutable in constant space, then F is also refutable in polynomial length.

Proof.

- Constant space $\Rightarrow$ constant width
- There are only polynomially many distinct clauses of constant width
- This is an upper bound on the length by simple counting $\square$

Open Problems (1/2)

Open Problem

Is it true that a logarithmic upper bound on space implies a polynomial upper bound on length?

Open Problem

Is it true that a resolution refutation in constant space also w.l.o.g. has polynomial length?

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Open Problems (2/2)

Open Problem

Is there a direct proof showing that constant clause space implies polynomial length, and explaining why this is true?

Open Problem

Is there a constructive, explicit version of [Atserias & Dalmau '03] that can tell us how to convert space-efficient refutations to narrow refutations?

Open Problems (2/2)

Open Problem

Is there a direct proof showing that constant clause space implies polynomial length, and explaining why this is true?

Open Problem

Is there a constructive, explicit version of [Atserias & Dalmau '03] that can tell us how to convert space-efficient refutations to narrow refutations?

Not an Open Problem

Is it true that the space-efficient refutation can itself also be narrow?

In general, NO.

Actually proven before [Atserias & Dalmau '03] in [Ben-Sasson '02]

[Ben-Sasson '02] interesting paper for other reasons as well

Will start discussing it today but won't have time to finish

A Detour into More Combinatorial Games

Want to find formulas that exhibit space-width trade-off

Turns out such formulas can be constructed by using pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many other references)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

Some quick graph terminology

- DAGs consist of vertices with directed edges between them
- vertices with no incoming edges: sources
- vertices with no outgoing edges: sinks

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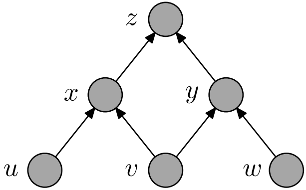
The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	0
Current # pebbles	0
Max # pebbles so far	0

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all predecessors have pebbles

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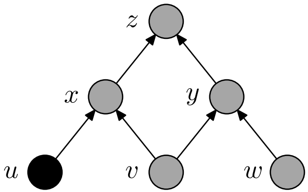

# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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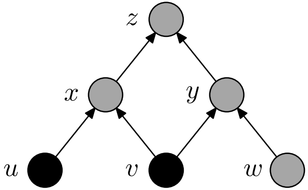

# moves	2
Current # pebbles	2
Max # pebbles so far	2

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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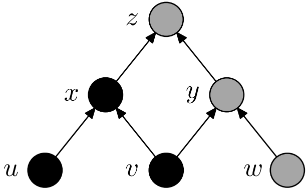

# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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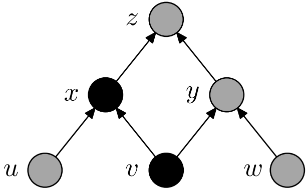
Goal: get single black pebble on sink vertex z of G



# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
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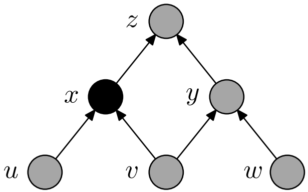
The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
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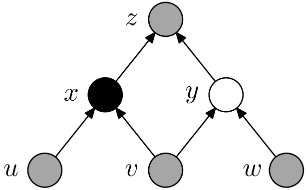
The Black-White Pebble Game

Goal: get single black pebble on sink vertex z of G



# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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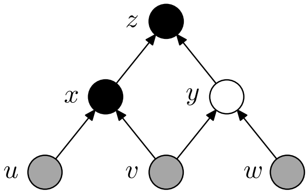
Goal: get single black pebble on sink vertex z of G



# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
2. Can always remove black pebble from vertex
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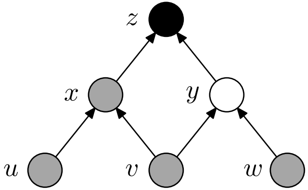
Goal: get single black pebble on sink vertex z of G



# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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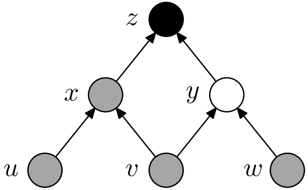
Goal: get single black pebble on sink vertex z of G



# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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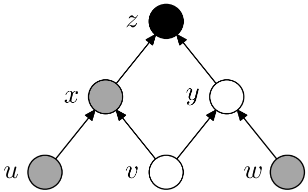

# moves	9
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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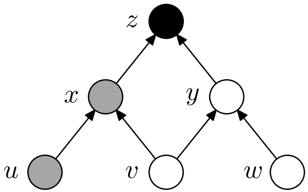
Goal: get single black pebble on sink vertex z of G



# moves	10
Current # pebbles	4
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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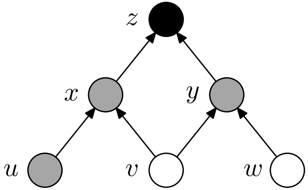

# moves	11
Current # pebbles	3
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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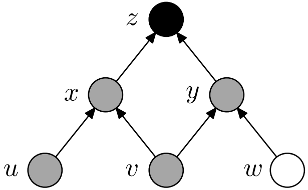
Goal: get single black pebble on sink vertex z of G



# moves	12
Current # pebbles	2
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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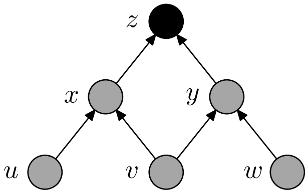
Goal: get single black pebble on sink vertex z of G



# moves	13
Current # pebbles	1
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex v if all predecessors (vertices with edges to v) have pebbles on them
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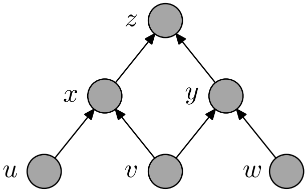
Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



- sources are true
- truth propa- gates upwards
- but sink is false

Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

We want to show that pebbling properties of DAGs somehow carry over to resolution refutations of pebbling contradictions

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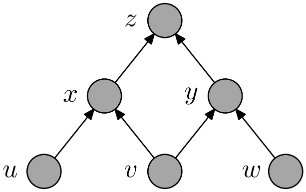

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Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation (where one can guess partial results and verify later)

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified

"Know z assuming v, w"

Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clause $\boxed{\overline{v} \vee \overline{w} \vee z}$

So translate clauses to pebbles by:
unnegated variable ⇒ black pebble
negated variable ⇒ white pebble

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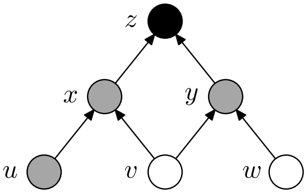

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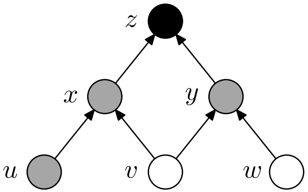

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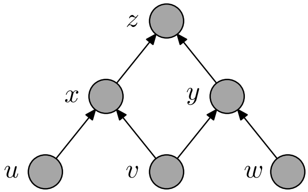
Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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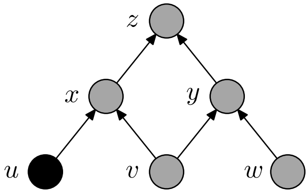


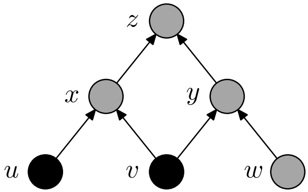
---

u

Write down axiom 1: u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

u
v

Write down axiom 1: u
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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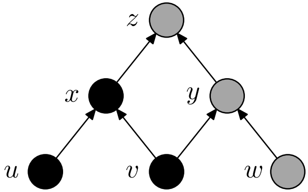
7. $\overline{z}$



---

u

v

$\overline{u} \lor \overline{v} \lor x$

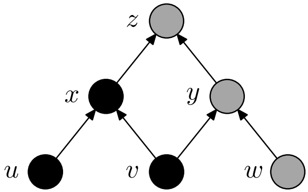
Write down axiom 1: u

Write down axiom 2: v

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

u

v

$\overline{u} \vee \overline{v} \vee x$

Write down axiom 1: u

Write down axiom 2: v

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

u and $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
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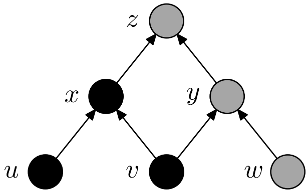

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u
v
$\overline{u} \vee \overline{v} \vee x$
$\overline{v} \vee x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
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4. $\overline{u} \vee \overline{v} \vee x$
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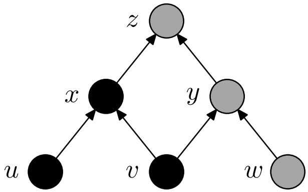

---

u

v

$\overline{u} \vee \overline{v} \vee x$

$\overline{v} \vee x$

Write down axiom 2: v

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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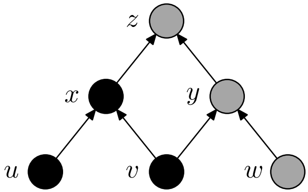

---

u

v

$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the clause $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
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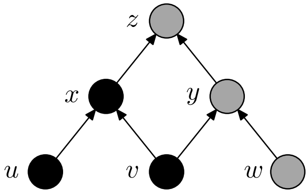

---

u

v

$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Erase the clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
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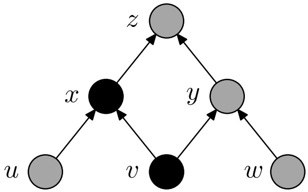
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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v

$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$

Infer $\overline{v} \vee x$ from

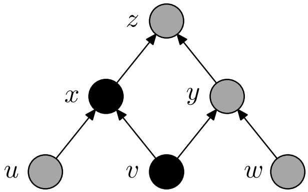
u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

Erase the clause u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

v

$\overline{v} \vee x$

u and $\overline{u} \vee \overline{v} \vee x$

Erase the clause $\overline{u} \vee \overline{v} \vee x$

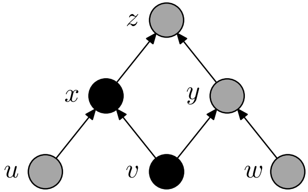
Erase the clause u

Infer x from

v and $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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v

$\overline{v} \lor x$

x

u and $\overline{u} \lor \overline{v} \lor x$

Erase the clause $\overline{u} \lor \overline{v} \lor x$

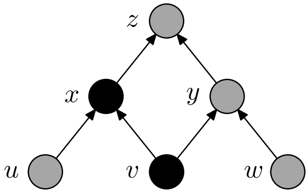
Erase the clause u

Infer x from

v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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v

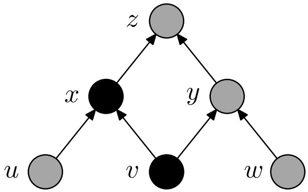
$\overline{v} \vee x$

x

Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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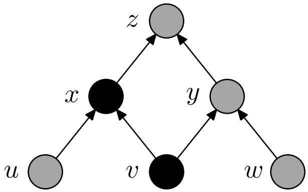
v
x

Erase the clause $\overline{u} \vee \overline{v} \vee x$
Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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v

x

Erase the clause u

Infer x from

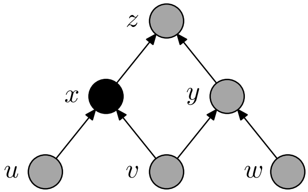
v and $\overline{v} \vee x$

Erase the clause $\overline{v} \vee x$

Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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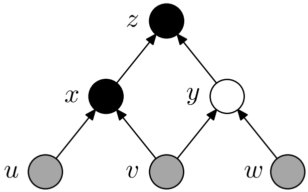
x

Erase the clause u
Infer x from
v and $\overline{v} \vee x$
Erase the clause $\overline{v} \vee x$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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x

$\overline{x} \lor \overline{y} \lor z$

Infer x from

v and $\overline{v} \lor x$

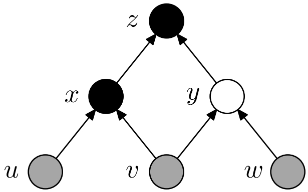
Erase the clause $\overline{v} \lor x$

Erase the clause v

Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



---

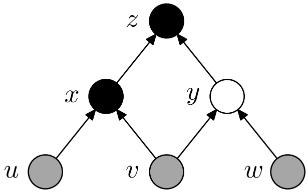
x

$\overline{x} \vee \overline{y} \vee z$

Erase the clause $\overline{v} \vee x$
Erase the clause v
Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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x

$\overline{x} \lor \overline{y} \lor z$

$\overline{y} \lor z$

Erase the clause $\overline{v} \lor x$

Erase the clause v

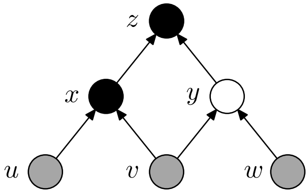
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from

x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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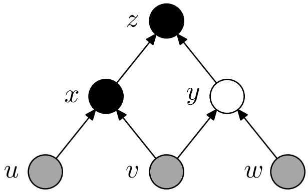
x
$\overline{x} \vee \overline{y} \vee z$
$\overline{y} \vee z$

Erase the clause v
Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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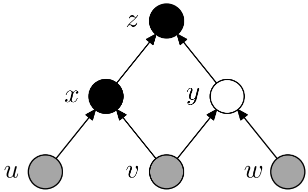
x

$\overline{y} \lor z$

Erase the clause v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the clause $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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x

$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

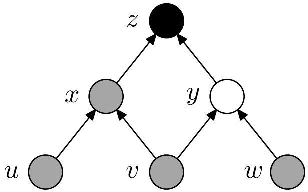
x and $\overline{x} \vee \overline{y} \vee z$

Erase the clause $\overline{x} \vee \overline{y} \vee z$

Erase the clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$

Infer $\overline{y} \vee z$ from

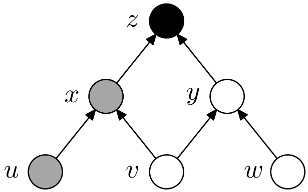
x and $\overline{x} \vee \overline{y} \vee z$

Erase the clause $\overline{x} \vee \overline{y} \vee z$

Erase the clause x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

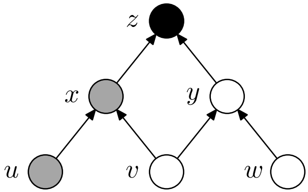


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the clause $\overline{x} \vee \overline{y} \vee z$
Erase the clause x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{y} \lor z$

$\overline{v} \lor \overline{w} \lor y$

Erase the clause $\overline{x} \lor \overline{y} \lor z$

Erase the clause x

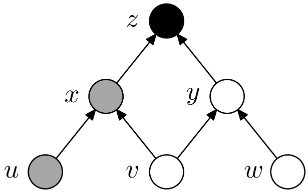
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Infer $\overline{v} \lor \overline{w} \lor z$ from

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

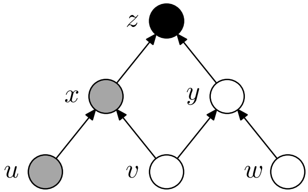


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause $\overline{x} \lor \overline{y} \lor z$
Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

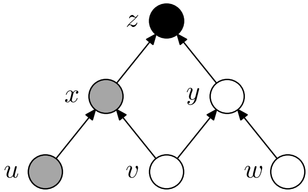


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$
$\overline{v} \vee \overline{w} \vee z$

Erase the clause x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

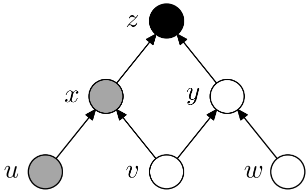


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Erase the clause x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{y} \vee z$

$\overline{v} \vee \overline{w} \vee z$

Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$

Infer $\overline{v} \vee \overline{w} \vee z$ from

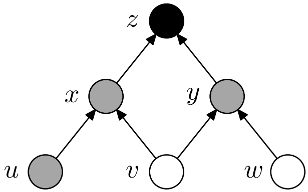
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Erase the clause $\overline{v} \vee \overline{w} \vee y$

Erase the clause $\overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

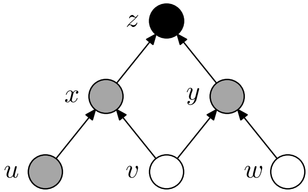


$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{v} \lor \overline{w} \lor y$
Erase the clause $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



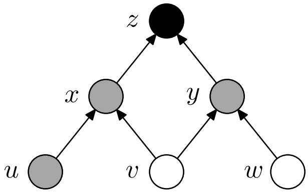
$\overline{v} \vee \overline{w} \vee z$

v

Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{v} \vee \overline{w} \vee z$

v

w

$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Erase the clause $\overline{v} \vee \overline{w} \vee y$

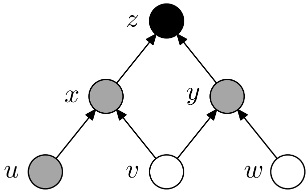
Erase the clause $\overline{y} \vee z$

Write down axiom 2: v

Write down axiom 3: w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

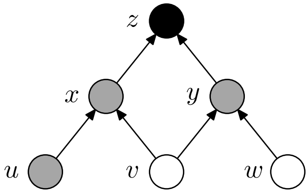


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Erase the clause $\overline{v} \vee \overline{w} \vee y$
Erase the clause $\overline{y} \vee z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

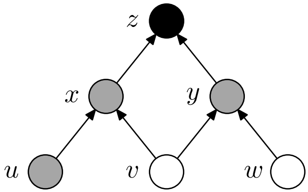


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{v} \lor \overline{w} \lor z$

v

w

$\overline{z}$

$\overline{w} \lor z$

Write down axiom 2: v

Write down axiom 3: w

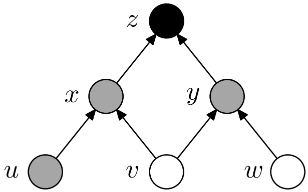
Write down axiom 7: $\overline{z}$

Infer $\overline{w} \lor z$ from

v and $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

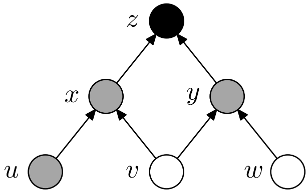


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

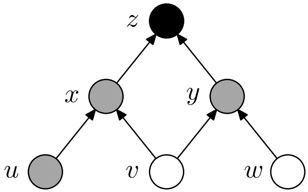


$\overline{v} \vee \overline{w} \vee z$
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

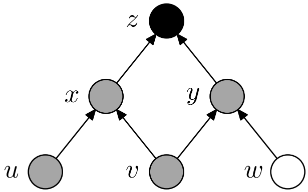


$\overline{v} \vee \overline{w} \vee z$
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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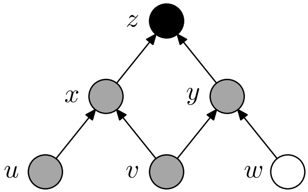
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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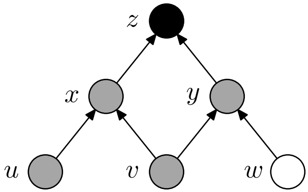
w
$\overline{z}$
$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$
Erase the clause v
Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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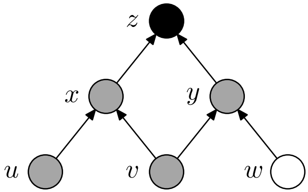
w
$\overline{z}$
$\overline{w} \vee z$
z

v and $\overline{v} \vee \overline{w} \vee z$
Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



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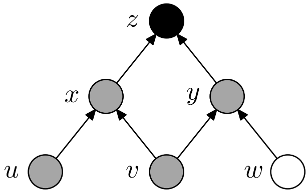
w
$\overline{z}$
$\overline{w} \vee z$
z

Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{z}$

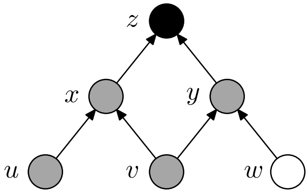
$\overline{w} \vee z$

z

Erase the clause v
Erase the clause $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the clause w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{z}$

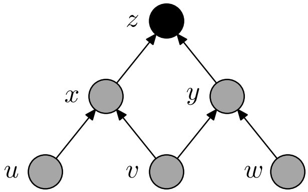
$\overline{w} \vee z$

z

Erase the clause $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the clause w
Erase the clause $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

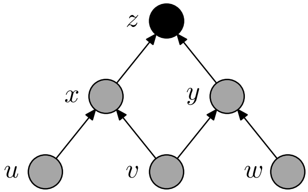


$\overline{z}$
z

Erase the clause $\overline{v} \lor \overline{w} \lor z$
Infer z from
    w and $\overline{w} \lor z$
Erase the clause w
Erase the clause $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{z}$

z

w and $\overline{w} \vee z$

Erase the clause w

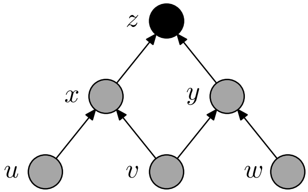
Erase the clause $\overline{w} \vee z$

Infer $\perp$ from

$\overline{z}$ and z

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{z}$

z

$\bot$

w and $\overline{w} \lor z$

Erase the clause w

Erase the clause $\overline{w} \lor z$

Infer $\bot$ from

$\overline{z}$ and z

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- $\#$ *moves* $= \mathcal{O}($*refutation length*$)$
- $\#$ *pebbles* $= \mathcal{O}($*total space*$)$

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length* $= \mathcal{O}(\#$ *moves*$)$
- *total space* $= \mathcal{O}(\#$ *pebbles*$)$

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- $\#$ *moves* $= \mathcal{O}$(*refutation length*)
- $\#$ *pebbles* $= \mathcal{O}$(*total space*)

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length* $= \mathcal{O}(\#$ *moves*)
- *total space* $= \mathcal{O}(\#$ *pebbles*)

To Be Continued...

- Time to wrap up for today
- But we will study these formulas further next time