Missing details from lecture 2

Recall Tech Lemma 2

If $W(F\restriction_x \vdash 0) \leq w-1$ and
$W(F\restriction_{\bar{x}} \vdash 0) \leq w$ then
$W(F \vdash 0) \leq \max(w, W(F))$

Let F be a k-CNF formula over n variables. Suppose $L(F \vdash 0) = L$.
We want to prove that

$$W(F \vdash 0) \leq k + \sqrt{8n \ln L}$$

where $\ln$ is the natural logarithm.

Fix a min length refutation $\pi : F \vdash 0$ of length $L(\pi) = L$.

Set

$$d = \sqrt{2n \ln L} \qquad (1)$$

$$a = \left(1 - \frac{d}{2n}\right)^{-1} \qquad (2)$$

Since $L \leq 2^{n+1} \ll e^{2n}$ we have $d < 2n$ and hence $a > 1$

Call a clause D FAT if $W(D) \geq d$

Write $fat(\pi)$ to denote # fat clauses in π

To establish the theorem, sufficient to prove:

CLAIM

Let G be any k-CNF over $m \leq n$ variables and suppose $\exists \pi' : G \vdash 0$ such that $fat(\pi') < a^b$ for a as in (2). Then

$$W(G \vdash 0) \leq k + d + b - 1 \qquad (3)$$

From this claim we get

$$W(F \vdash 0) \leq k + \sqrt{8n \ln L}$$

To see this, note that

— $k = W(F)$

— $d = \sqrt{2n \ln L}$

So we just need to work on b.

We have $fat(\pi) \leq L \leq a^{\lfloor \log_a L \rfloor + 1}$

so $b \leq \log_a L + 1 = 1 + \dfrac{\ln L}{\ln a}$

$\left(\text{using } \log_a x = \dfrac{\ln x}{\ln a}\right)$

Furthermore,

$$\ln a = \ln\left(\left(1-\frac{d}{2n}\right)^{-1}\right)$$
$$= -\ln\left(1-\frac{d}{2n}\right)$$
$$\geq d/2n$$

using

$$\ln(1+x) \leq x \quad \text{for } x > -1$$

and

$$d < 2n, \quad \text{and}$$

$$d/2n = \sqrt{\ln \lambda / 2n}$$

so

$$b \leq 1 + \frac{\ln \lambda}{\ln a} \leq 1 + \sqrt{2n \ln \lambda}$$

which gives

$$W(F \vdash 0) \leq k + d + b - 1$$
$$\leq k + 2\sqrt{2n \ln 2} \quad \boxed{\text{Lh}}$$

It remains to prove the claim.

Again we have nested induction over b and m $(\leq n)$

Base cases

1. $m = 1$ or $k = 1$ or $\ell \leq 3$

This means that G contains $x \wedge \bar{x}$ and $W(G \vdash 0) = 1$ and we're done.

2. $m \leq k$ (for $m, k \geq 2$, $\ell \geq 4$)

$\ell > 3 \Rightarrow d > 1$

Refutation width is at most $m \leq k \leq k + \underbrace{d + b - 1}_{\geq 0}$

3. $b = 0 \Rightarrow$ no fat clauses $\Rightarrow$ refutation width at most $d \leq d + \underbrace{k + b - 1}_{\geq 0}$

Inductive step

Suppose (3) holds for

(a) all k-CNFs over $< m$ variables

(b) all k-CNFs over $= m$ variables with $\mathrm{fat}(\pi') < a^{b-1}$.

Consider $\pi : G \vdash 0$ with $\mathrm{fat}(\pi) < a^{b}$

G has $2m$ distinct literals all in all

There are at least $d \cdot \mathrm{fat}(\pi)$ literals in fat clauses.

So by counting there is some literal appearing in at least $\frac{d}{2m}\,\mathrm{fat}(\pi) \geq \frac{d}{2n}\,\mathrm{fat}(\pi)$ fat clauses.

Suppose w.l.o.g. that literal x appears in $\geq \frac{d}{2n} fat(\pi)$ fat clauses.

Consider $\pi\restriction_x : G\restriction_x \vdash 0$ — all these fat clauses are satisfied and disappear.

So $\pi\restriction_x$ has $< \left(1 - \frac{d}{2n}\right) a^b \leq a^{b-1}$ fat clauses, and by induction hypothesis

$$W(G\restriction_x \vdash 0) \leq k + d + b - 2 \quad (4)$$

Now consider

$$\pi\restriction_{\bar{x}} : G\restriction_{\bar{x}} \vdash 0.$$

$\pi\restriction_{\bar{x}}$ has less than a^b fat clauses and $G\restriction_{\bar{x}}$ has $< m$ variables. By induction hypothesis

$$W(G\restriction_{\bar{x}} \vdash 0) \leq k + d + b - 1 \quad (5)$$

Use Tech Lemma 2 on (4) and (5)

$$\Rightarrow \quad W(G \vdash 0) \leq \max\{k, k + d + b - 1\}$$
$$= k + d + b - 1 \quad \text{Q.E.D.} \; \boxed{\text{K}}$$