ÖVERSIKT FÖRELÄSNING 7/2

1. Trade-offs for resolution in sublinear space

Pebbling formulas – trade-offs between var space and # downloads
– always refutable in length $O(n)$ and width $O(1)$ simultaneously

Formula substitution: Naive simulation blows up var space $\leq B$ to clause space $\leq 2^B$

Substitution space thm: If subst funct f is non-authoritarian, then this blow-up is necessary
Proved via projection

Get ⇒ Separation between length & space (of tree-like resolution)

Length – space trade-offs (time-space trade-offs)
Works also for k-DNF res

2. Projection works from any proof syst P

Need projection that is space-faithful between Sp_P and $VarSp_R$.

But for any "reasonable" proof system, R_{proj} is space-faithful for $VarSp_P$ even wrt $f = id$!

And if P can simulate resolution efficiently. Means we also get right upper bounds

Recall that width and (clause) space have interesting relations in resolution

1) Space small $\Rightarrow$ width small
2) Width small – space can still be very large
3) Strong trade-off between the two

Since we have analogies:

resolution	PCR
clause space	monomial space
width	degree

Natural to ask what holds in PCR.

<u>Question 1</u> Wide open

<u>Question 2</u> Open, but we "know" right answer (pebbling formulas should have similar properties)

<u>Question 3</u> Just use observations above!

Let G_n = graphs in [GT78] $\text{BW-Peb}(G_n) = \Omega(n/\log n)$

Then

$$\text{VarS}_{P_{PCR}}(\text{Peb}(G_n) \vdash \bot) = \Omega(n/\log n)$$

PCR can simulate res, so $\text{Deg}_{PCR}(\text{Peb}(G_n) \vdash \bot) = O(1)$

$$S_{P_{PCR}}(\text{Peb}(G_n) \vdash \bot) = O(1)$$

But for any refutation π

$$\text{VarS}_{P_{PCR}}(\pi) \leq S_{P_{PCR}}(\pi) \cdot \text{Deg}_{PCR}(\pi)$$

So if $\pi : \text{Peb}(G_n) \vdash \bot$, then

$$Sp(\pi) \cdot Deg(\pi) = \Omega(n/\log n).$$

3 Sublinear space trade-offs for PCR

Given any G and any f,
$Peb(G)[f]$ can be refuted in both resolution and PC in linear length/size and constant width/degree simultaneously
Also, both resolution and PC can simulate black pebbling

Let us prove lower bounds for PCR

Let m be monomial derived from $Peb(G)[\oplus_2]$
Pick random restriction ρ that for each x sets one of x_1 or x_2 to $0/1$ uniformly and independently at random

Suppose $x_1 \in Vars(m)$ ~~m~~ ~~but~~ $x_2 \notin Vars(m)$
then ρ sets x_1 so that m vanishes with prob $1/4$, m "survives" with prob $3/4$
If $x_1, x_2 \in Vars(m)$,
m survives with prob $1/2 < (3/4)^2$

So
$$Pr_\rho [m\restriction\rho \neq 0] \leq (3/4)^{Deg(m)}$$

If we fix a ~~degree~~ threshold d, it follows that

$$Pr_\rho [Deg(m\restriction\rho) \geq d] < (3/4)^d$$

Suppose that $\pi: Peb(G)[\oplus] \vdash \bot$ is PCR-refutation in size T and space s

$\Pr[\exists\, m \in \pi\restriction\rho \text{ of degree} \geq d]$
$< T \cdot (3/4)^d$ $\quad (*)$

Set $d = \log_{4/3} T$ then

$(*) < T \cdot 1/T = 1$, so $\exists\, \rho$

s.t. $Dg(\pi\restriction\rho) \leq d = O(\log T)$

So we now have PCR-refut $\pi\restriction\rho$ of $Peb(G)$ possibly after flipping signs with

$\#$ ax downloads $\leq T$

~~monom~~ var space $\leq$ monom space $\cdot$ degree
$\leq s \cdot O(\log T)$

So if G_n has no BW-pebblings in simultaneous time T and space S, then PCR cannot refute $Peb(G)[\oplus]$ in size $O(T)$ and space $O(S/\log T)$ simultaneously.

This gives us <u>almost</u> the same set of results for PCR as for resolution (with upper bounds for res / PC), except:

- not quite right; we lose a $\log T$ factor
- no unconditional space LB (why?)

But it seems morally clear that $Sp_{PCR}(Peb(G)[\oplus] \vdash \bot) \geq \text{BW-Peb}(G)$.