Want to prove

THM 1

F CNF formula over n variables

$W_R(F \vdash \bot) \leq w$

$L_R(F \vdash \bot) = L$

Then any CDCL solver (as modelled in AFT) using arbitrary asserting learning scheme learns empty clause $\bot$ after at most

$$4wL \ln\{(4wnL)n^{w+1}\}$$

conflicts and restarts with prob $\geq 1/2$

Total # clauses of width $= w$ over n vars

$$2^k \binom{n}{k} \leq 2n^k$$

So total length of refut of F over n vars if $W(F \vdash \bot) \leq w$ is at most

$$\sum_{i=0}^{w} 2^i \binom{n}{i} \leq 1 + 2\sum_{i=1}^{w} n^i$$

$$\leq 1 + 2n \sum_{i=0}^{w-1} n^i$$

$$= 1 + 2n \frac{n^w - 1}{n-1}$$

$$\leq 4n^w$$

COR 2

F CNF formula over n vars

$W(F \vdash \bot) = w$

With prob $\geq 1/2$ any CDCL solver (as modelled in AFT) using arbitrary asserting learning scheme learns empty clause $\bot$ after at most

$$16\,w\,(w+1)\,\ln(16wn)\,n^{2w+1}$$

conflicts and restarts

But we won't prove Thm 1 today

Instead consider concrete learning scheme, learning clause A_1 as defined last time

Recall general idea

Given $\pi = \{ C_1, C_2, \ldots, C_L \}$

Want to show that π is learned clause by clause.

Doesn't quite work — will show that proof is <u>absorbed</u> clause by clause

TERMINOLOGY & NOTATION

<u>Complete round</u> started with $\mathbb{D}$

Start with assignment / state $S_0 = \emptyset$

Run until conflict or all vars set.

Round represented as sequence of states

$S_0 = \emptyset, \ S_1, S_2, \ldots$

<u>Round</u> Initial segment of complete round up to state where either

a) $\perp \in \mathbb{D}\restriction_{S_t}$ CONCLUSIVE ROUND

b) No unit clause in $\mathbb{D}\restriction_{S_t}$ INCONCLUSIVE ROUND (since no clause can be learned)

R, R' rounds

A, B, C clauses

$\mathbb{C}, \mathbb{D}, \mathbb{E}$ sets of clauses

R' SUBSUMES R if R' extends R as a partial t v a

R' and R AGREE on C if assign the same values to $Vars(C)$ in $\{0, 1, *\}$

R BRANCHES in C if all decision variables $\subseteq Vars(C)$

<u>Next</u> Two important but technical lemmas

High level: "Inconclusive rounds are robust wrt order of assignments"

Any inconclusive round subsumes any other round that agrees on its decision assignments.

LEMMA 3
(LEMMA 1 in AFT)

Suppose $\mathbb{D} \subseteq \mathbb{D}'$, C any clause, R' inconclusive round (started with $\mathbb{D}'$) ~~started with D~~
Then $\forall$ R started with $\mathbb{D}$ that branches in C and agrees with R' on C, R' subsumes R.

Proof By induction over $R = (S_0, \ldots, S_t)$. Prove: Every assignment in S_i made in R'. Base case empty. If R makes decision, its on $Vars(C)$ and agrees with R' by assumption.
Implied assignment $x = b$. Then
$$\exists A \in \mathbb{D} \quad A\restriction_{S_{i-1}} = x^b$$
Then R' has had to set $x = b$ since it is inconclusive (no conflict) and a round (no further unit prop possible). $\boxed{Z}$

One can find an R as in Lem 1 (i.e. lemma not vacuous)

LEMMA 4
(LEMMA 2 in AFT) C any clause,
$\mathbb{D} \subseteq \mathbb{D}'$, ($R'$ inconclusive started with $\mathbb{D}'$
Then $\exists$ inconclusive R started with $\mathbb{D}$ that branches in C, agrees with R' on C, and is subsumed by R'!

(Skip proof to save time.)

Want to define which clauses are sort of "implicitly learned".

DEF 5
(DEF 3) (ABSORPTION)

$\mathbb{D}$ ABSORBS (non-empty) clause A at $x^b \in Lit(A)$ if every inconclusive round started with $\mathbb{D}$ that falsifies $A \setminus \{x^b\}$ sets $x = b$.

$\mathbb{D}$ absorbs A if it does so at every literal in A

Note If no inconclusive round can falsify B, then vacuously $\mathbb{D}$ absorbs $B \vee x$ ~~and~~ at x.

LEM 6
(LEMMA 4) If $\mathbb{D}$ absorbs A, then $\mathbb{D} \models A$.

Skip proof.

Converse does not hold.

So in particular, can have $C_1 \vee x$ and $C_2 \vee \bar{x}$ absorbed, but not $C_1 \vee C_2$. We will be interested in understanding how this can happen and what it says about the structure of these clauses.

One more important tech lemma

(LEMMA 5) 7 (MONOTONICITY OF ABSORPTION)

Suppose $\mathbb{D} \subseteq \mathbb{E}$ and $A \subseteq B$

1. If $A \in \mathbb{D}$ then $\mathbb{D}$ absorbs A
2. If $\mathbb{D}$ absorbs A then $\mathbb{D}$ absorbs B
3. If $\mathbb{D}$ absorbs A then $\mathbb{E}$ absorbs A

Just apply the definitions

Proof 1. Suppose inconclusive round falsifies $A \setminus \{x^b\}$, $A \in D$. Then at some point get unit clause x^b. So x will be set. Propagation on x^b gives ~~set~~ $x = b \Rightarrow$ absorption.

Else $x = 1-b \Rightarrow$ conflict $\Rightarrow$ not inconclusive.

2. Fix $x^b \in Lit(B)$ and let $B' = B \setminus \{x^b\}$

$x^b \notin Lit(A) \Rightarrow$ No way of falsifying $A \Rightarrow$ absorption by remark above

$x^b \in Lit(A)$. Let $A' = A \setminus \{x^b\}$

Any round falsifying B' now falsifies A', hence by assumption sets $x = b$. So B is absorbed

3. Follows from Lemma 2.

Fix $x^b \in Lit(A)$ Let $A' = A \setminus \{x^b\}$

Let R' inconclusive round started with E that falsifies A'. Want to argue that R' sets $x = b$.

By lem 4, $\exists$ inconclusive R started with D that falsifies A' (branches on A' and agrees with R' there) and is subsumed by R'. By assumption, A is absorbed by R, so R sets $x = b$. But R' subsumes R, so then R' also sets $x = b$ and R' absorbs A at x^b.

Suppose $C = Res(A, B)$ and solver has absorbed A, B. We want it to absorb C as well.

But $C = Res(A, B)$; A, B absorbed;
does <u>not</u> guarantee that C is absorbed,
at least not immediately. Need to understand
when this is or is not the case

LEMMA 8

$\mathbb{D}$ set of clauses
A, B resolvable clauses absorbed by $\mathbb{D}$
$C = Res(A, B)$
If $x^b \in Lit(C)$ and $\mathbb{D}$ does not absorb
C at x^b, then $x^b \in Lit(A) \cap Lit(B)$.

Proof

Suppose $C = Res(A, B, y)$
$A = A' \vee y \qquad B = B' \vee \overline{y}$
If $\mathbb{D}$ does not absorb C at x^b, then $\exists R$
(inconclusive) that
- falsifies $C \setminus \{x^b\}$
- does not set $x = b$

$C = A' \vee B'$ means $x^b \in Lit(A')$ or $x^b \in Lit(B')$
or both. If both $\Rightarrow$ done. Suppose
$x^b \in Lit(A')$, $x^b \notin Lit(B')$
Then R falsifies B'. Since B absorbed
R sets $y = 0$. But then R falsifies
$A \setminus \{x^b\}$, and since A absorbed R
sets $x = b$. But then C is absorbed at x^b
after all, contrary to assumption. $\square$

But in this case, C can be "almost absorbed" by a round that is "beneficial"

DEF 9

R inconclusive round started with $\mathbb{D}$

R is BENEFICIAL FOR a clause A at $x^b \in Lit(A)$ if

- R falsifies $A \setminus \{x^b\}$
- R branches in $A \setminus \{x^b\}$
- R does not assign x $\qquad x \stackrel{d}{=} 1-b$
- Would yield conclusive round if extended by $\cancel{x \stackrel{d}{=} b}$

Round R extended by $x \stackrel{d}{=} 1-b$ is also called beneficial

R is beneficial for A if it is so at ~~for~~ some literal in A.

LEMMA 10

A, B resolvable clauses absorbed by $\mathbb{D}$

$C = Res(A, B)$.

If C non-empty and not absorbed by $\mathbb{D}$, then $\exists$ round started with $\mathbb{D}$ that is beneficial for C.

<u>Proof</u> Let $A = A' \vee y$ $B = B' \vee \bar{y}$

$C = A' \vee B'$

By Lem 8 $\exists\ x^b$

D does not absorb C at x^b

$x^b \in Lit(A') \cap Lit(B')$

$\exists$ inconclusive R' falsifying $C \setminus \{x^b\}$ but not setting $x = b$.

R' cannot set $x = 1-b$

If so R' falsifies A' and B'

By absorption, would have to set $y = 1$ and $y = 0$

Hence R' does not assign x.

Now apply Lem 4 to get inconclusive R started with D

R branching in $C \setminus \{x^b\}$

R agrees with R' on $C \setminus \{x^b\}$, i.e., falsifies

R leaves x unassigned (subsumed by R')

<u>Claim</u> R is beneficial for C at x^b.

Only thing left to argue for this is that $x \stackrel{d}{=} 1-b$ would yield conclusive round.

But we sort of already argued this:

R' extended with $x = 1-b$ falsifies A' and B'

If this extended round were inconclusive, would absorb A and B, i.e. set $y = 1$ & $y = 0$ at the same time $\lightning$ So must be conclusive.

Main tech lemma: "If A, B have been absorbed and $C = Res(A,B)$, then C is likely to be absorbed or at least have a beneficial round"

LEMMA 11

A, B resolvable, absorbed by $\mathbb{D}$

$Res(A,B) = C \neq \perp$

$|Vars(\mathbb{D})| = n \qquad W(C) = w$

Let $R_0, R_1, \ldots, R_{t-1}$ t consecutive complete rounds, $\mathbb{D}_0, \ldots, \mathbb{D}_{t-1}$ corresponding sets of clauses.

Then prob that

none of R_i beneficial for C

and none of $\mathbb{D}_i$ absorb C

is $\leq \exp(-t/(4n^w))$

Proof $\mathcal{R}_i$ = R_i not beneficial for C

$\mathcal{D}_i$ = $\mathbb{D}_i$ doesnot absorb C

Want to bound $\Pr\left[\bigcap_{i=0}^{t-1} \mathcal{R}_i \cap \mathcal{D}_i\right] =$

$$= \prod_{j=0}^{t-1} \Pr\left[\mathcal{R}_j \cap \mathcal{D}_j \,\middle|\, \bigcap_{i=0}^{j-1} \mathcal{R}_i \cap \mathcal{D}_i\right] \qquad (\dagger)$$

$$\leq \prod_{j=0}^{t-1} \Pr\left[\mathcal{R}_j \,\middle|\, \mathcal{D}_j \cap \bigcap_{i=0}^{j-1} \mathcal{R}_i \cap \mathcal{D}_i\right] \qquad (*)$$

Fix j and look at

$$\Pr\left[\mathcal{R}_j \,\middle|\, \mathcal{D}_j \cap \bigcap_{i=0}^{j-1} \mathcal{R}_i \cap \mathcal{D}_i\right] \qquad (*)$$

By these conditions, Lem 10 says
$\exists^R$ beneficial for C at some particular x^6

So LB on (*) given by probability of choosing exactly this round

$$\Pr\left[\begin{array}{l}\text{first } w-1 \text{ decisions falsify } C\upharpoonright\{x^6\} \\ \text{and } w\text{th decision is } x = 1-6\end{array}\right] \geq$$

$$\geq \left[\left(\frac{w-1}{2n}\right)\left(\frac{w-2}{2(n-1)}\right)\cdots\left(\frac{1}{2(n-w+2)}\right)\right] \frac{1}{2(n-w+1)}$$

$$\geq \frac{(w-1)!}{2^w n^w} \geq \frac{1}{4n^w}$$

<u>Note</u> round started with D_j might not be able to make these decisions as some of the assignments may be implied.

However this is OK by Lemma L [dropping details] and can only improve probability.

Substituting this estimate for (*) into (+), we get that prob is

$$\Pr\left[\bigcap_{i=0}^{t-1} R_i \neg D_i\right] \leq \left(1-\frac{1}{4n^w}\right)^t \leq$$

$$\leq \exp\left(-t/(4n^w)\right)$$

using $1+x \leq \exp(x)$ valid for all $x \in \mathbb{R}$

Now we want to argue for
$\pi = \{ C_1, C_2, \ldots, C_L \}$
that following sequence of events likely

(1a) Beneficial round for C_1 happens
(1b) Current clause set absorbs C_1
(2a) Beneficial round for C_2 happens
(2b) Current clause set absorbs C_2
Et cetera ...

Let us assume that clause learning scheme is AFT DECISION
$\Rightarrow$ Clause learned is subset of (negations of) decisions made

Beneficial round branches on C
$\Rightarrow$ AFT learns $C' \subseteq C$
$\Rightarrow$ Next clause set absorbs C (by Lemma 7). monotonicity

So ~~step~~ from (X_a) to (X_b) immediate
(in general, requires work)

Immediate consequence of Lem 11:

<u>LEMMA 12</u>

A, B resolvable, absorbed by $\mathcal{D}$
$\mathrm{Res}(A, B) = C \neq \bot$
$|\mathrm{Vars}(\mathcal{D})| = n \qquad W(C) = w$

$$\Pr[\, C \text{ not absorbed after } t \text{ restarts} \,]$$
$$\leq \exp\left(-t / (4n^{w}) \right)$$

Now we can prove (weaker version of) Thm 1

THM 13

F CNF formula over n variables
$W_R(F \vdash \bot) = w$
$L_R(F \vdash \bot) = L$
With prob at least $1/2$, any CDCL solver (as modelled in AFT) using the AFT-decision clause learning scheme learns the empty clause after at most

$$4L \cdot \ln(4L) \cdot n^w$$

conflicts and restarts.

Proof

Last step in res ref is $\mathrm{Res}(x, \bar{x})$ for some x.
Prove that after $4L \ln(4L) n^w$ restarts, $\Pr[x^b \text{ not absorbed}] \leq 1/4$
So both x & $\bar{x}$ absorbed with prob $\geq 1/2$.

Then any round must be conclusive without even making decisions, so $\bot$ is learned.

Let $\pi_b = \{C_1^b, C_2^b, \ldots, C_m^b = x^b\}$ deriv of x^b
$r \leq L - 1$. $C_i^b \neq \bot$ $W(C_i^b) \leq w$.

Let $\mathbb{D}_0, \mathbb{D}_1, \ldots, \mathbb{D}_s$ sequence of clause sets

$s = r \cdot t \qquad t = \lceil 4 \ln(4r) n^w \rceil$

$\mathcal{E}_i$ = All $C_1, \ldots, C_i$ absorbed by $\mathbb{D}_{it}$

$\overline{\mathcal{E}_i}$ = complement

$\Pr[\mathcal{E}_0] = 1 \qquad \Pr[\overline{\mathcal{E}_0}] = 0$ vacuously.

$$\Pr[\overline{\mathcal{E}_i}] = \Pr[\overline{\mathcal{E}_i} \mid \mathcal{E}_{i-1}] \cdot \Pr[\mathcal{E}_{i-1}] + \Pr[\overline{\mathcal{E}_i} \mid \overline{\mathcal{E}_{i-1}}] \cdot \Pr[\overline{\mathcal{E}_{i-1}}]$$
$$\leq \Pr[\overline{\mathcal{E}_i} \mid \mathcal{E}_{i-1}] + \Pr[\overline{\mathcal{E}_{i-1}}]$$

$C_i \in F$ initial by monotonicity lemma $\to p_i = 0$

$p_i = \Pr[\overline{\mathcal{E}_i} \mid \mathcal{E}_{i-1}]$

C_i derived by resolution

$p_i \leq \exp(-t/4n^w)$ by Lem 12

$\leq 1/4r$ by choice of t

Summing up,

$$\Pr[\overline{\mathcal{E}_r}] \leq \sum_{i=1}^{r} p_i$$
$$\leq r/4r = 1/4.$$

Thm follows. $\boxtimes$