LECTURE 22

Continue working our way through
[Pipatsrisawat & Darwiche '11]

Result: "A proof system giving a natural description of what CDCL solvers do polynomially simulates full resolution"

Important concepts

UNIT resolution — one of the resolved clauses always has size 1

$\mathbb{D} \vdash_1 a$ unit derivation of literal a.

CLR STATE $S = (F, G, D)$

G learned clauses
F original formula
D decisions [maybe use Δ in notes]

$\mathbb{D}$ 1 in-consistent if $\mathbb{D} \vdash_1 \bot$
1-consistent otherwise

EXTENDED BRANCHING SEQUENCE
variable decisions and restarts

E.g. $\sigma = \langle x, \overline{y}, R, \overline{x} \rangle$

$CLR(F, \sigma)$ endstate after starting with F and branching acc to σ

$KB(CLR(F, \sigma))$ Knowledge base of this state $F \land G$

DEF 6

Suppose CLR_{LS} run on F with branching sequence σ generating conflict clauses $C_1, \ldots, C_k$

Let S^i 1-inconsistent state for which C_i conflict clause.

Let Π_i trivial res deriv of C_i from $KB(S^i)$.

Then σ <u>induces</u> resolution derivation

$DER_{LS}(F, \sigma) = \Pi_1, \Pi_2, \ldots, \Pi_k$

KEY CONCEPTS

If $C = \bar{a}_1 \vee \ldots \vee \bar{a}_{m-1} \vee a_m$ $\in \mathcal{C}r$ is a clause

we can write this $\bigwedge_{i=1}^{m-1} a_i \to a_m$

(just to follow [PD11] notation

Let $T = \bigwedge_{i=1}^{m-1} a_i$ denote such terms.

DEF 7 (1-EMPOWERMENT)

The clause $C = T \to a$ is <u>1-empowering</u> wrt $\mathbb{D}$ if

1) $\mathbb{D} \models T \to a$
2) $\mathbb{D} \wedge T$ is 1-consistent
3) $\mathbb{D} \wedge T \nvdash_1 a$

a is <u>empowering literal</u> of C. If $\mathbb{D} \models C$ and C contains no empowering literals, then C is <u>absorbed</u> by F

cf [AFT11]

Clause C implied by $\mathbb{D}$ is 1-empowering if it allows unit resolution to derive new implication that would be impossible otherwise

<u>Ex 8</u> $\mathbb{D}$ =

$$\begin{array}{l} x \vee y \vee z \\ x \vee y \vee \bar{z} \\ x \vee \bar{y} \vee z \\ x \vee \bar{y} \vee \bar{z} \\ \bar{z} \vee u \\ z \vee w \end{array}$$

1) $x \vee y$ is 1-empowering, since

$\mathbb{D} \wedge \bar{y} \not\vdash_1 x$ but if we have learnt $x \vee y$, we will then get this propagation

2) $u \vee w$ is <u>not</u> 1-empowering but absorbed

$\mathbb{D} \wedge \bar{u} \vdash_1 w$

$\mathbb{D} \wedge \bar{w} \vdash_1 u$

We already get all propagation $u \vee w$ can offer

OBSERVATION 9 (cf monotonicity lemma in [AFT 11])

1) If $C \subseteq C'$ then C' is absorbed by C

2) Adding more clauses to knowledge base can make 1-empowering clause absorbed, but can never make absorbed clause 1-empowering.

3) Every asserting clause is 1-empowering wrt knowledge base at the time of its derivation with ~~its~~ asserted literal as empowering

<u>Proof</u> Exercise (juggle with the definitions).

DEF 10 (1-provability)
Clause C is <u>1-provable</u> wrt clause set $\mathcal{D}$ if $\mathcal{D} \land \neg C \vdash_1 \bot$.

Ex 11 Go back to Ex 8 and clause set $\mathcal{D}$ there.

$$\mathcal{D} \vDash x \lor y$$
$$\mathcal{D} \vDash x$$

C/ $x \lor y$ is 1-provable, but x is <u>not</u>.
1-empowering <u>and</u> 1-provable : Cannot currently propagate to get C, but making right decisions will yield conflict falsifying C!

By Def 3 every conflict clause is 1-provable from wrt the knowledgebase $F \land G$ at the time of its derivation from $S = (F, G, D)$.

LEMMA 12 (1ST KEY LEMMA)
Let F be a 1-consistent CNF formula and suppose $\pi : F \vdash \bot$ is a res ref. Then $\exists\, C \in \pi$ which is both 1-empowering and 1-provable wrt F.

Follows from two propositions

PROP 13
Suppose $F \vDash C_1 = C_1' \lor x,\ C_2 = C_2' \lor \bar{x}$ but that neither clause is 1-empowering wrt F.
Then $C = C_1' \lor C_2'$ is 1-provable wrt F

Proof Suppose C_1 not 1-empowering, i.e. some condition in Def 7 fails.

$F \nvDash C_1$ by assumption $\Rightarrow$ not cond 1.

Suppose $F \wedge \neg C_1'$ not 1-consistent, i.e.

$$F \wedge \neg C_1' \vdash_1 \bot$$

Then $F \wedge C \vdash_1 \bot$ so C 1-provable

Takes care of cond 2.

Otherwise cond 3 fails and

$$F \wedge \neg C_1' \vdash_1 x \qquad (*)$$

Same analysis for $C_2 = C_2' \vee \bar{x}$ yields that either we're already done or

$$F \wedge \neg C_2' \vdash_1 \bar{x} \qquad (**)$$

Combining $(*)$ & $(**)$ get $F \wedge \neg C \vdash_1 \bot$, so $C = C_1' \vee C_2'$ is 1-provable. QED $\boxed{\text{L}}$

<u>PROP 14</u> If $\pi : F \vdash C$ but C is not 1-provable wrt F, then π contains clause that is both 1-provable & 1-empowering

<u>Proof</u> Let $\pi = \{ C_1, C_2, \ldots, C_L \}$ $\quad C_L = C$

Let C_i first clause in π not 1-provable

Can have $C_i = C_L = C$.

C_i resolvent of C_j, C_k 1-provable.

By Prop 13 one of C_j & C_k has to be 1-empowering $\boxed{\text{L}}$

Proof of Lem 12 $F \nvdash_1 \bot$ by assumption, since F is 1-consistent. Prop 14 now provides 1-empowering & 1-provable clause $\boxed{\text{L}}$

LEMMA 15 (2ND KEY LEMMA)
Let $S = (F, G, D)$ be 1-inconsistent CLR state.
Let C conflict clause of S that is 1-empowering wrt $F \wedge G$. Then $\exists$ trivial res derivation of $C' \subseteq C$ from $F \wedge G$

Why do we need this?
To be able to follow σ and get "correct proof format"?

Proof sketch
Let $C = C' \vee a$ with empow lit a.
Let σ branching sequence on $\neg C'$ (in any order) followed by $\bar{a}$. Let DECISION be the decision learning scheme as defined in [ZMMM '01] (and as discussed in [AFT 11])
This learning scheme $\Rightarrow$ conflict clauses only over decision variables.
Following σ leads to conflict
DECISION will learn $C' \subseteq C$
Prop 4 of [BKS '04] Now implies that $\exists$ trivial res derivation of C'. $\boxed{\square}$

Are these the same clauses as the ones in [AFT 11]

Given any res ref $\pi : F \vdash \bot$, want to p-simulate it in CLR.

that we do not automatically assert a clause to a contradiction?

Focus on clauses $C \in \pi$ which are both 1-empowering and 1-provable. Show that for any such C, can force CLR to derive clauses so that unit resolution sees any implications C would allow us to derive (so although we may not have learnt C, get the same effect for all practical purposes) (cf [AFT 11])

LEMMA 16 (3RD KEY LEMMA)

F CNF formula over n variables

C 1-empowering and 1-provable wrt F.

Then for any asserting learning scheme AS exists extended branching sequence σ such that

1) $KB(CLR_{AS}(F,\sigma))$ absorbs C

2) $L(DER_{AS}(F,\sigma)) = O(n^4)$

Proof Let $C = C' \vee a$, a any empowering literal.

Let σ ext branching sequence containing $\neg C'$ in any order.

By Def 7, cond 2 (of 1-emp)

CLR state S right after σ is 1-consistent.

The state S does not derive a by unit resolution (Def 7, cond 3)

The state S does not derive $\overline{a}$ by unit res, because if it did, since C is 1-provable we would get $S \vdash_1 \bot$, i.e. 1-inconsistent state.

So we are free to pick $\overline{a}$ as our next decision.

$\Longrightarrow$ falsifies C.

$\Longrightarrow$ Since C is 1-provable, get 1-inconsistent state S^*

Suppose AS learns C^* from S^*.
If $F \wedge C^*$ absorbs $C \Rightarrow$ done
Otherwise, add C^* to knowledge base.

Unit propagate and take care of conflicts repeatedly until we reach 1-consistent state. $\leq n$ conflicts, since backtrack ≥ 1 level per conflict.
Every conflict gives trivial res deriv
$\Rightarrow$ length $O(n)$
So total length of derivation up to 1-consistent state is $O(n^2)$ by Lem 15.

Now repeat from top. We claim this can only happen $O(n)$ times.

Every asserting clause yields new implied literal (its asserted literal) under δ (or subset of δ).
So for each repetition get one more literal set already before deciding on a

Suppose C is 1-empowering (and ~~1-provable~~ obviously still)
$\Rightarrow$ decision on a still needed for conflict.
Next time we learn a clause, will have different asserting literal — every variable used at most once.

Thus, after at most n steps, a gets implied by assertion of some asserting clause. $\Rightarrow a$ no longer empowering. This has now taken
$O(n^3)$ steps

Finally, C can contain at most n empowering literals, so after repeating all of this $\leq n$ times, for a total derivation of size $O(n^4)$, C must be completely absorbed. $\boxtimes$

End of proof of Lem 16

MAIN RESULT

<u>Thm 17</u> (MAIN) CLR with any asserting clause learning scheme p-simulates resolution

<u>Cor 18</u> CLR with the 1UIP learning scheme* p-simulates resolution

*) Dominant learning scheme used in practice.

Proof [Thm 17]

Given $\pi : F \vdash \bot$ F has n variables any asserting CL scheme AS, construct ext. branching sequence σ that induces CLR_{AS} to generate $\pi' : F \vdash \bot$ with $L(\pi') = O(n^4 \, L(\pi))$ (π' CLR-proof)

Given current $\mathbb{D} = F \wedge G$ Assume $\mathbb{D}$ 1-consistent (else just $\leq n$ resolution steps to $\bot$)

By Lem 12 $\exists$ 1-empow & 1-prov clause C wrt $\mathbb{D}$

By Lem 16, absorb C in length $O(n^4)$

Repeat. In every iteration absorb 1 clause, which will never become 1-empow again

This gives weak p-sim. Can actually find refutation constructively

<u>Thm 19</u>

The extended branching sequence required for Thm 17 can be constructed in ~~tame~~ time poly$(S(F), L(\pi))$.

<u>Proof</u> Suffices to show that can find 1-empow and 1-provable clause in poly time.

1-provability: Assert $\neg C$ and do unit ~~prop~~ resolution
Time linear in $S(F)$

1-empowerment: Assert $\neg C$ except for one literal and do unit res. Repeat for all literals

Thm 17 works also for certain non-asserting CL schemes (with slightly worse parameters) See Sec 6.1 in paper for details.

Also not necessary to restart after every conflict

Let $\phi(i)$ be # conflicts between $(i-1)$st and ith conflict.

A restart policy is f-spaced, $f: N \to N$, if $\forall i \;\; \phi(i) \le f(i)$

Polynomially spaced $\Leftrightarrow f(i) = i^{O(1)}$ 22:XI

(THM 20) CDCL with asserting learning scheme* and polynomially spaced restart policy can p-simulate resolution

(* or slightly weaker as mentioned above)

E.g. Arithmetic policies
Luby policies [LSZ '93]

Restart every X conflicts

$X := X + Y$

$f(i) = X + (i-1)Y$

$$\phi(i) = \begin{cases} 2^{k+1} & \text{if } i = 2^k - 1 \\ \phi(i - 2^{k-1} + 1) & \text{if } 2^{k-1} \le i < 2^k - 1 \end{cases}$$